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Multiplicative Lie triple higher derivations on generalized matrix algebras

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Abstract Let $\mathbb{N}$ be the set of nonnegative integers and $\mathcal{G} = (\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a 2-torsion free generalized matrix algebra over a commutative ring $\mathcal{R}$. In the present paper, under some lenient assumptions on $\mathcal{G}$, it is shown that if $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of mappings $\delta_n : \mathcal{G} \to \mathcal{G}$ (not necessarily linear) satisfying $\delta_n([[a,b],c]) = \sum_{r+s+t=n} [[\delta_r(a), \delta_s(b)], \delta_t(c)]$ for all $a, b, c \in \mathcal{G}$, then for each $n \in \mathbb{N}$, $\delta_n = d_n + \tau_n$; where $d_n : \mathcal{G} \to \mathcal{G}$ is an additive mapping satisfying $d_n(ab) = \sum_{r+s=n} d_r(a)d_s(b)$ for all $a, b \in \mathcal{G}$, i.e., $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is an additive higher derivation on $\mathcal{G}$ and $\tau_n : \mathcal{G} \to Z(\mathcal{G})$(where $Z(\mathcal{G})$ is the center of $\mathcal{G}$) is a map vanishing at every second commutator $[[a,b],c]$.

Keywords generalized matrix algebra · higher derivation · multiplicative Lie higher derivation · multiplicative Lie triple higher derivation

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1 Introduction

Let $\mathcal{R}$ be a commutative ring with unity, $\mathcal{A}$ be an algebra over $\mathcal{R}$ and $Z(\mathcal{A})$ be the center of $\mathcal{A}$. Recall that an $\mathcal{R}$-linear mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a derivation if $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in \mathcal{A}$. If the condition of linearity is replaced by additivity in the above definition, then d is said to be an additive derivation. In particular, derivation d is called an inner derivation if there exists some $a \in \mathcal{A}$ such that $d(a) = ax - xa$ for all $x \in \mathcal{A}$. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a Lie (resp. Lie triple) derivation if $d([a,b]) = [d(a), b] + [a, d(b)]$ (resp. $d([[a,b],c]) = [[d(a),b],c] + [[a,d(b)],c] + [[a,b],d(c)]$) holds for all $a, b, c \in \mathcal{A}$, where $[a,b] = ab - ba$ is the usual Lie product. If the condition of linearity is dropped from the above definition, then the corresponding Lie derivation and Lie triple derivation are called multiplicative Lie derivation and multiplicative Lie triple derivation respectively. Obviously, every derivation is a Lie derivation

and every Lie derivation is a Lie triple derivation. However, the converse statements are not true in general. For example if we consider the algebra $\mathcal{A}$ of all 3×3 strictly upper triangular matrices over $\mathbb{Z}$, the ring of integers, and define a map $d : \mathcal{A} \to \mathcal{A}$ such that $d \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$.

Then it can easily be seen that d is a Lie triple derivation on $\mathcal{A}$ which is neither a derivation nor a Lie derivation on $\mathcal{A}$.

Let us recall some basic facts related to Lie derivations, Lie higher derivations and Lie triple higher derivation of an associative algebra. Many kinds of higher derivations, which consist of a family of some additive mappings, have been widely studied in commutative and noncommutative rings. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ be a family of linear mappings $d_n : \mathcal{A} \to \mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$, the identity map on $\mathcal{A}$. Then $\mathcal{D}$ is called

(*i*) a higher derivation on $\mathcal{A}$ if for every $n \in \mathbb{N}$, $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ for all $a, b \in \mathcal{A}$.

(*ii*) a Lie higher derivation on $\mathcal{A}$ if for every $n \in \mathbb{N}$, $d_n([a, b]) = \sum_{i+j=n} [d_i(a), d_j(b)]$ for all $a, b \in \mathcal{A}$.

(*iii*) a Lie triple higher derivation on $\mathcal{A}$ if for every $n \in \mathbb{N}$, $d_n([[a, b], c]) = \sum_{i+j+k=n} [[d_i(a), d_j(b)], d_k(c)]$ for all $a, b, c \in \mathcal{A}$.

If we assume $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ be a family of mappings $d_n : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) in the above definitions then the corresponding higher derivation, Lie higher derivation and Lie triple higher derivation is said to be multiplicative higher derivation, multiplicative Lie higher derivation and multiplicative Lie triple higher derivation respectively. Moreover, if $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is assumed to be the family of additive mappings, then in the above definition higher derivation, Lie higher derivation and Lie triple higher derivation is said to be additive higher derivation, additive Lie higher derivation and additive Lie triple higher derivation respectively. Note that d_1 is always a derivation, Lie derivation and Lie triple derivation if $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is a higher derivation, Lie higher derivation and Lie triple higher derivation respectively.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras over a fixed unital 2-torsion free commutative ring $\mathcal{R}$. A Morita context consists of two $\mathcal{R}$-algebras $\mathcal{A}$ and $\mathcal{B}$, two bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$, and two bimodule homomorphisms called the pairings $f_{\mathcal{MN}} : \mathcal{M} \bigotimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$ and $\psi_{\mathcal{NM}} : \mathcal{N} \bigotimes_{\mathcal{A}} \mathcal{M} \to \mathcal{B}$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{M}\underset{\mathcal{B}}{\otimes}\mathcal{N}\underset{\mathcal{A}}{\otimes}\mathcal{M} & \xrightarrow{f_{\mathcal{M}\mathcal{N}}\otimes\mathcal{I}_{\mathcal{M}}} & \mathcal{A}\underset{\mathcal{A}}{\otimes}\mathcal{M} \\ \downarrow{\scriptstyle \mathcal{I}_{\mathcal{M}}\otimes\Psi_{\mathcal{N}\mathcal{M}}} & & \downarrow{\scriptstyle \cong} \\ \mathcal{M}\underset{\mathcal{B}}{\otimes}\mathcal{B} & \xrightarrow{\cong} & \mathcal{M} \end{array} \quad and \quad \begin{array}{ccc} \mathcal{N}\underset{\mathcal{A}}{\otimes}\mathcal{M}\underset{\mathcal{B}}{\otimes}\mathcal{N} & \xrightarrow{\Psi_{\mathcal{N}\mathcal{M}}\otimes\mathcal{I}_{\mathcal{N}}} & \mathcal{B}\underset{\mathcal{B}}{\otimes}\mathcal{N} \\ \downarrow{\scriptstyle \mathcal{I}_{\mathcal{N}}\otimes f_{\mathcal{M}\mathcal{N}}} & & \downarrow{\scriptstyle \cong} \\ \mathcal{N}\underset{\mathcal{A}}{\otimes}\mathcal{A} & \xrightarrow{\cong} & \mathcal{N}. \end{array}$$

If $(\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N},f_{\mathcal{M}\mathcal{N}},\psi_{\mathcal{N}\mathcal{M}})$ is a Morita context, then the set

$$\mathcal{G}=\begin{pmatrix}\mathcal{A} & \mathcal{M}\\ \mathcal{N} & \mathcal{B}\end{pmatrix}=\left\{\begin{pmatrix}a & m\\ n & b\end{pmatrix}\middle| a\in\mathcal{A}, m\in\mathcal{M}, n\in\mathcal{N}, b\in\mathcal{B}\right\}$$

is an algebra under the usual matrix operations. Such an $\mathcal{R}$-algebra is called a generalized matrix algebra (see [3]). We assume that $\mathcal{M}$ and $\mathcal{N}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule and $(\mathcal{B},\mathcal{A})$-bimodule respectively. Obviously, when $\mathcal{M}=0$ or $\mathcal{N}=0$, then $\mathcal{G}$ exactly degenerates to the so-called triangular algebra.

Let $1_{\mathcal{A}}$ (resp. $1_{\mathcal{B}}$) be the identity of the algebra $\mathcal{A}$ (resp. $\mathcal{B}$) and let $I=\begin{pmatrix}1_{\mathcal{A}} & 0\\ 0 & 1_{\mathcal{B}}\end{pmatrix}$ be the identity of generalized matrix algebra $\mathcal{G}$. Throughout this paper, we shall use the following notations: $p_1=\begin{pmatrix}1_{\mathcal{A}} & 0\\ 0 & 0\end{pmatrix}$ and $p_2=I-p_1=\begin{pmatrix}0 & 0\\ 0 & 1_{\mathcal{B}}\end{pmatrix}$. Set $\mathcal{G}_{11}=\left\{\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}\middle| a\in\mathcal{A}\right\}$, $\mathcal{G}_{12}=\left\{\begin{pmatrix}0 & m\\ 0 & 0\end{pmatrix}\middle| m\in\mathcal{M}\right\}$, $\mathcal{G}_{21}=\left\{\begin{pmatrix}0 & 0\\ n & 0\end{pmatrix}\middle| n\in\mathcal{N}\right\}$ and $\mathcal{G}_{22}=\left\{\begin{pmatrix}0 & 0\\ 0 & b\end{pmatrix}\middle| b\in\mathcal{B}\right\}$. Then we can write $\mathcal{G}=\mathcal{G}_{11}+\mathcal{G}_{12}+\mathcal{G}_{21}+\mathcal{G}_{22}$, where $\mathcal{G}_{11}$ is a subalgebra of $\mathcal{G}$ isomorphic to $\mathcal{A}$, $\mathcal{G}_{22}$ is a subalgebra of $\mathcal{G}$ isomorphic to $\mathcal{B}$, $\mathcal{G}_{12}$ is a $(\mathcal{G}_{11},\mathcal{G}_{22})$-bimodule isomorphic to the bimodule $\mathcal{M}$ and $\mathcal{G}_{21}$ is a $(\mathcal{G}_{22},\mathcal{G}_{11})$-bimodule isomorphic to the bimodule $\mathcal{N}$. To simplify the notation we will use the following convention: $a_{11}\in\mathcal{A}=\mathcal{G}_{11}$, $a_{22}\in\mathcal{B}=\mathcal{G}_{22}$, $a_{12}\in\mathcal{M}=\mathcal{G}_{12}$ and $a_{21}\in\mathcal{N}=\mathcal{G}_{21}$. Then each element $x\in\mathcal{G}$ can be represented in the form $x=a_{11}+a_{12}+a_{21}+a_{22}$, where $a_{11}\in\mathcal{G}_{11}, a_{22}\in\mathcal{G}_{22}$, $a_{12}\in\mathcal{G}_{12}$ and $a_{21}\in\mathcal{G}_{12}$.

The center of $\mathcal{G}$ is

$$Z(\mathcal{G})=\left\{a\oplus b=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\middle| a\in\mathcal{A}, b\in\mathcal{B}, am=mb, na=bn \text{ for all } m\in\mathcal{M}, n\in\mathcal{N}\right\}.$$

Define two natural projections $\pi_{\mathcal{A}}:\mathcal{G}\to\mathcal{A}$ and $\pi_{\mathcal{B}}:\mathcal{G}\to\mathcal{B}$ by

$$\pi_{\mathcal{A}}\begin{pmatrix}a & m\\ n & b\end{pmatrix}=a \quad\text{and}\quad \pi_{\mathcal{B}}\begin{pmatrix}a & m\\ n & b\end{pmatrix}=b.$$

Moreover, $\pi_{\mathcal{A}}(Z(\mathcal{G})) \subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{G})) \subseteq Z(\mathcal{B})$, and there exists a unique algebra isomorphism $\eta : \pi_{\mathcal{B}}(Z(\mathcal{G})) \to \pi_{\mathcal{A}}(Z(\mathcal{G}))$ such that $\eta(b)m = mb$ and $n\eta(b) = bn$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$. The proof of the following lemma can be seen in [16, Lemma 3.2].

Lemma 1.1 *Let $\mathcal{G} = (\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra. If $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$, then there is a unique algebra isomorphism $\eta : Z(\mathcal{B}) \to Z(\mathcal{A})$ such that $\eta(b)m = mb$ and $n\eta(b) = bn$ for all $b \in Z(\mathcal{B})$, $m \in \mathcal{M}, n \in \mathcal{N}$; or equivalently, $\eta(b) \oplus b \in Z(\mathcal{G})$ for all $b \in Z(\mathcal{B})$.*

There has been a great interest in the study of characterization of Lie derivations and Lie triple derivations for many years. The first quite surprising result is due to Martindale who proved that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive (see [12]). Miers [13] initially established that every Lie derivation d on a von Neumann algebra $\mathcal{A}$ can be uniquely written as the sum $d = \delta + \tau$ where δ is an inner derivation of $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into its center $Z(\mathcal{A})$ vanishing on each commutator. Furthermore, Miers [14] obtained an analogous decomposition for Lie triple derivations of von Neumann algebras with no abelian summands. Yu and Zhang [18] proved that every nonlinear Lie derivation of triangular algebras is the sum of an additive derivation and a map from triangular algebra into its center sending commutators to zero. Ji, Liu and Zhao [6] proved the similar result for nonlinear Lie triple derivation of triangular algebras. In addition, the characterization of Lie derivations and Lie triple derivations on various rings and algebras are considered in [4], [5],[1], [2], [7], [9], [11], [17], [15].

It is the objective of this article is to investigate multiplicative Lie triple higher derivation on generalized matrix algebras. Many researchers have made important contributions to the related topics (see [16], [15], [10],[8]). Xiao and Wei [17] proved that every multiplicative Lie higher derivation of triangular algebras is the sum of an additive higher derivation and a multiplicative functional vanishing on all commutators. Ji, Liu and Zhao [6] gave a characterization of nonlinear Lie triple derivations on triangular algebras. Motivated by the above work, we study the characterization of multiplicative Lie triple higher derivations on generalized matrix algebras.

2 Characterizations of multiplicative Lie triple higher derivations

The main result states as follows:

Theorem 2.1 *Let $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$ be a 2-torsion free generalized matrix algebra consisting of algebras $\mathcal{A}$ and $\mathcal{B}$ over a commutative ring $\mathcal{R}$ with unity $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ respectively and $\mathcal{M}$ & $\mathcal{N}$ be a faithful $(\mathcal{A}, \mathcal{B})$ & $(\mathcal{B}, \mathcal{A})$-bimodule respectively. Suppose that*

(*i*) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ *and* $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$.
(*ii*) *For any* $a \in \mathcal{A}$, *if* $[a, \mathcal{A}] \in Z(\mathcal{A})$, *then* $a \in Z(\mathcal{A})$ *or for any* $b \in \mathcal{B}$, *if* $[b, \mathcal{B}] \in Z(\mathcal{B})$, *then* $b \in Z(\mathcal{B})$.
(*iii*) *Either* $\mathcal{A}$ *or* $\mathcal{B}$ *does not contain nonzero central ideals.*
(*iv*) *If* $\mathcal{N} \neq 0$, *then for each* $M \in \mathcal{M}$, *the condition* $M\mathcal{N} = 0 = \mathcal{N}M$ *implies* $M = 0$.
(*v*) *For each* $N \in \mathcal{N}$, *the condition* $\mathcal{M}N = 0 = N\mathcal{M}$ *implies* $N = 0$.

If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ *is a sequence of maps* $\delta_n : \mathcal{G} \to \mathcal{G}$ *(not necessarily linear) such that* $\delta_n[[a, b], c] = \sum_{r+s+t=n} [[\delta_r(a), \delta_s(b)], \delta_t(c)]$ *for all* $a, b, c \in \mathcal{G}$, *then for each* $n \in \mathbb{N}$, $\delta_n = d_n + \tau_n$; *where* $d_n : \mathcal{G} \to \mathcal{G}$ *is an additive mapping satisfying* $d_n(ab) = \sum_{r+s=n} d_r(a)d_s(b)$ *for all* $a, b \in \mathcal{G}$, *i.e.,* $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ *is an additive higher derivation of* $\mathcal{G}$ *and* $\tau_n : \mathcal{G} \to Z(\mathcal{G})$ *is a map vanishing at every second commutator* $[[a, b], c]$.

The proof of the above theorem proceeds by induction on n. In fact, the proof for the case $n = 1$ can be derived directly from Theorem 1 of [15]. But in order to develop the proof of our main theorem for multiplicative Lie triple higher derivation, we need entirely different set of properties (see B_1 and B_m in the proof of Theorem 2.1). Hence, we first prove Theorem 2.1 for the case $n = 1$ and show that for every multiplicative Lie triple derivation δ on $\mathcal{G}$ there exist a derivation d on $\mathcal{G}$ and a mapping $\tau : \mathcal{G} \to Z(\mathcal{G})$ vanishing on the second commutators such that $\delta(a) = d(a) + \tau(a)$ for all $a \in \mathcal{G}$. We facilitate our discussion with the following Lemmas.

Lemma 2.2 $\delta(0) = 0$.

Proof. $\delta(0) = \delta([[0, 0], 0]) = [[\delta(0), 0], 0] + [[0, \delta(0)], 0] + [[0, 0], \delta(0)] = 0$. □

Lemma 2.3 $p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathcal{G})$.

Proof. For any $a_{12} \in \mathcal{G}_{12}$. Since $[[a_{12}, p_1], p_1] = 0$, we have

$$\begin{aligned}\delta(a_{12}) &= \delta([[a_{12}, p_1], p_1]) \\ &= [[\delta(a_{12}), p_1], p_1] + [[a_{12}, \delta(p_1)], p_1] + [[a_{12}, p_1], \delta(p_1)] \\ &= \delta(a_{12})p_1 - p_1\delta(a_{12})p_1 - p_1\delta(a_{12})p_1 + p_1\delta(a_{12}) + a_{12}\delta(p_1)p_1 \\ &\quad -a_{12}\delta(p_1) + p_1\delta(p_1)a_{12} - a_{12}\delta(p_1) + \delta(p_1)a_{12}.\end{aligned}$$

On multiplying the above equation from left by p_1 and right by p_2, we get $2\big(p_1\delta(p_1)a_{12} - a_{12}\delta(p_1)p_2\big) = 0$, this implies that $p_1\delta(p_1)a_{12} - a_{12}\delta(p_1)p_2 = 0$. That is

$$p_1\delta(p_1)p_1a_{12} - a_{12}p_2\delta(p_1)p_2 = 0 \quad \text{for all} \quad a_{12} \in \mathcal{G}_{12}. \tag{2.1}$$

Similarly, for any $a_{21} \in \mathcal{G}_{21}$. Since $[[a_{21}, p_1], p_1] = 0$, we have

$$a_{21}p_1\delta(p_1)p_1 - p_2\delta(p_1)p_2a_{21} = 0 \quad \text{for all} \quad a_{21} \in \mathcal{G}_{21}. \tag{2.2}$$

and hence from (2.1) and (2.2) together with Lemma 1.1, we get $p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathcal{G})$.

□

In the sequel, we define $f : \mathcal{G} \to \mathcal{G}$ by

$$f(a) = \delta(a) + \delta_{p_1\delta(p_1)p_2 - p_2\delta(p_1)p_1}(a) \text{ for all } a \in \mathcal{G}$$

where $\delta_{p_1\delta(p_1)p_2 - p_2\delta(p_1)p_1}$ is an inner derivation determined by $p_1\delta(p_1)p_2 - p_2\delta(p_1)p_1$. It is easy to verify that

$$f([[a,b],c]) = [[f(a),b],c] + [[a,f(b)],c] + [[a,b],f(c)]$$

holds for all $a, b, c \in \mathcal{G}$. Moreover, by Lemma 2.3, we have

$$\begin{aligned} f(p_1) &= \delta(p_1) - p_1\delta(p_1)p_2 - p_2\delta(p_1)p_1 \\ &= \delta(p_1)p_1 + \delta(p_1)p_2 - p_1\delta(p_1)p_2 - p_2\delta(p_1)p_1 \\ &= p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathcal{G}) \end{aligned}$$

Thus $f(p_1) \in Z(\mathcal{G})$.

Lemma 2.4 *$f(p_1ap_2 + p_2ap_1) = p_1f(a)p_2 + p_2f(a)p_1$ for all $a \in \mathcal{G}$.*

Proof. Since $[[a,p_1],p_1] = p_1a - 2p_1ap_1 + ap_1 = p_1ap_2 + p_2ap_1$, it follows that

$$\begin{aligned} f(p_1ap_2 + p_2ap_1) &= f([[a,p_1],p_1]) = [[f(a),p_1],p_1] \\ &= p_1f(a)p_2 + p_2f(a)p_1. \end{aligned}$$

□

Lemma 2.5 *$f(p_2) \in Z(\mathcal{G})$.*

Proof. Using the similar arguments as that used in the proof of Lemma 2.3, we get

$$p_1f(p_2)p_1 + p_2f(p_2)p_2 \in Z(\mathcal{G}).$$

Since $f(p_2) = p_1f(p_2)p_1 + p_1f(p_2)p_2 + p_2f(p_2)p_1 + p_2f(p_2)p_2$, by Lemma 2.4, we have

$$p_1f(p_2)p_2 + p_2f(p_2)p_1 = f(p_1p_2p_2 + p_2p_2p_1) = 0.$$

Consequently, we get $f(p_2) = p_1f(p_2)p_1 + p_2f(p_2)p_2 \in Z(\mathcal{G})$.

□

Lemma 2.6 *If $[a,b] \in Z(\mathcal{G})$ for any $a, b \in \mathcal{G}$, then $[f(a),b] + [a,f(b)] \in Z(\mathcal{G})$.*

Proof. For $[a,b] \in Z(\mathcal{G})$, we have $[[a,b],c] = 0$ for all $c \in \mathcal{G}$.

$$\begin{aligned} 0 = f(0) = f[[a,b],c] &= [[f(a),b],c] + [[a,f(b)],c] \\ &= [[f(a),b] + [a,f(b)],c] \end{aligned}$$

for all $c \in \mathcal{G}$. Thus $[f(a),b] + [a,f(b)] \in Z(\mathcal{G})$.

□

Lemma 2.7 $f(a_{ij}) \subseteq \mathcal{G}_{ij},\ 1 \leq i \neq j \leq 2.$

Proof. For $a_{12} \in \mathcal{G}_{12}$, we have $a_{12} = [[a_{12},p_1],p_1]$. Thus

$$f(a_{12}) = f([[a_{12},p_1],p_1]) = [[f(a_{12}),p_1],p_1] = p_1f(a_{12})p_2 + p_2f(a_{12})p_1,$$

and hence we see that $p_1f(a_{12})p_1 = p_2f(a_{12})p_2 = 0$. Now for any $a_{12}, b_{12} \in \mathcal{G}_{12}$, by Lemma 2.6, we have

$$[f(a_{12}),b_{12}] + [a_{12},f(b_{12})] \in Z(\mathcal{G}). \tag{2.3}$$

Since $a_{12} = [p_1,a_{12}]$, we have

$$\begin{aligned} [a_{12},f(b_{12})] &= [[p_1,a_{12}],f(b_{12})] \\ &= f([[p_1,a_{12}],b_{12}]) - [[f(p_1),a_{12}],b_{12}] - [[p_1,f(a_{12})],b_{12}] \\ &= -[[p_1,f(a_{12})],b_{12}]. \end{aligned}$$

Now, we have

$$\begin{aligned} [f(a_{12}),b_{12}] + [a_{12},f(b_{12})] &= [p_2f(a_{12})p_1,b_{12}] - [[p_1,f(a_{12})],b_{12}] \\ &= 2[p_2f(a_{12})p_1,b_{12}] \in Z(\mathcal{G}). \end{aligned} \tag{2.4}$$

It follows that $[p_2f(a_{12})p_1,b_{12}] = p_2f(a_{12})p_1b_{12} - b_{12}p_2f(a_{12})p_1 \in Z(\mathcal{G})$ for all $a_{12}, b_{12} \in \mathcal{G}_{12}$. Thus, $\mathcal{M}p_2f(a_{12})p_1 \subseteq Z(\mathcal{A})$ and $p_2f(a_{12})p_1\mathcal{M} \subseteq Z(\mathcal{B})$. Since $\mathcal{M}p_2f(a_{12})p_1$ is central ideal of $\mathcal{A}$. So, by assumption (iii), $\mathcal{M}p_2f(a_{12})p_1 = 0$, so $p_2f(a_{12})p_1\mathcal{M} = 0$. Applying condition (v), we have, $p_2f(a_{12})p_1 = 0$. Thus, $f(a_{12}) = p_1f(a_{12})p_2$ and $f(a_{12}) \in \mathcal{G}_{12}$.

Similarly, using condition (iv) and by the same argument one can prove that $f(a_{21}) = p_2f(a_{21})p_1$ and $f(a_{21}) \in \mathcal{G}_{21}$.

□

Lemma 2.8 *$f(\mathcal{G}_{ii}) \subseteq \mathcal{G}_{ii} \oplus \mathcal{G}_{jj}$. There exists a mapping $\tau_i : \mathcal{G}_{ii} \to Z(\mathcal{G})$ such that $f(a_{ii}) - \tau_i(a_{ii}) \in \mathcal{G}_{ii}$ for all $a_{ii} \in \mathcal{G}_{ii}$, where $i,j = 1,2$ and $i \neq j$.*

Proof. For any $a_{ii} \in \mathcal{G}_{ii}$, where i = 1,2. First we show for $i = 1$, by Lemma 2.4, we have

$$p_1f(a_{11})p_2 = f(p_1a_{11}p_2) = f(0) = 0.$$

Similarly, we can get $p_2f(a_{11})p_1 = 0$, hence, $f(a_{11}) \in \mathcal{G}_{11} \oplus \mathcal{G}_{22}$
Similarly, we can prove the result for $i = 2$. Thus $f(\mathcal{G}_{ii}) \subseteq \mathcal{G}_{ii} \oplus \mathcal{G}_{jj}$.

Now we can write $f(a_{11})$=$p_1f(a_{11})p_1+p_2f(a_{11})p_2$. Moreover, since $a_{11}a_{22} = 0$ for any $a_{22} \in \mathcal{G}_{22}$ and $c \in \mathcal{G}$, it is easy to check that

$$\begin{aligned} 0 = f([[a_{11}, a_{22}], c]) &= [[f(a_{11}), a_{22}], c] + [[a_{11}, f(a_{22})], c] \\ &= [[f(a_{11}), a_{22}] + [a_{11}, f(a_{22})], c]. \end{aligned}$$

Multiplying by p_2 on both the sides of the above equation we get

$$\begin{aligned} 0 &= p_2[[f(a_{11}), a_{22}] + [a_{11}, f(a_{22})], c]p_2 \\ &= [[p_2f(a_{11})p_2, p_2a_{22}p_2] + [p_2a_{11}p_2, p_2f(a_{22})p_2], p_2cp_2] \\ &= [[p_2f(a_{11})p_2, a_{22}], p_2cp_2]. \end{aligned}$$

This implies that $[p_2f(a_{11})p_2, a_{22}] \in Z(\mathcal{G}_{22})$. Hence by hypothesis (ii), we find that
$p_2f(a_{11})p_2 \in Z(\mathcal{G}_{22})$.
Define $\tau_1 : \mathcal{G}_{11} \to Z(\mathcal{G})$ such that $\tau_1(a_{11}) = \eta(p_2f(a_{11})p_2) \oplus p_2f(a_{11})p_2$, where η is the map defined in Lemma 1.1. Thus we get

$$\begin{aligned} f(a_{11}) - \tau_1(a_{11}) &= p_1f(a_{11})p_1 + p_2f(a_{11})p_2 - \eta(p_2f(a_{11})p_2) - p_2f(a_{11})p_2 \\ &= p_1f(a_{11})p_1 - \eta(p_2f(a_{11})p_2) \in \mathcal{G}_{11}. \end{aligned}$$

Similarly, we can define a mapping $\tau_2 : \mathcal{G}_{22} \to Z(\mathcal{G})$ by $\tau_2(a_{22})$=$p_1f(a_{22})p_1 \oplus \eta^{-1}(p_1f(a_{22})p_1)$. Then

$$\begin{aligned} f(a_{22}) - \tau_2(a_{22}) &= p_1f(a_{22})p_1 + p_2f(a_{22})p_2 - p_1f(a_{22})p_1 - \eta^{-1}(p_1f(a_{22})p_1) \\ &= p_2f(a_{22})p_2 - \eta^{-1}(p_1f(a_{22})p_1) \in \mathcal{G}_{22}. \end{aligned}$$

□

Our next aim is to show that f is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.

Lemma 2.9 *Let $a_{ii} \in \mathcal{G}_{ii}$ and $a_{ij} \in \mathcal{G}_{ij}$, $1 \leq i \neq j \leq 2$. Then $f(a_{ii} + a_{ij}) - f(a_{ii}) - f(a_{ij}) \in Z(\mathcal{G})$.*

Proof. Let $a_{11} \in \mathcal{G}_{11}$, $a_{12} \in \mathcal{G}_{12}$. We have

$$\begin{aligned} f([[a_{11} + a_{12}, p_1], p_1]) &= [[f(a_{11} + a_{12}), p_1], p_1] + [[a_{11} + a_{12}, f(p_1)], p_1] \\ &\quad + [[a_{11} + a_{12}, f(p_1)], f(p_1)] \\ &= [[f(a_{11} + a_{12}), p_1], p_1]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} f([[a_{11} + a_{12}, p_1], p_1]) &= f([[a_{11}, p_1], p_1]) + f([[a_{12}, p_1], p_1]) \\ &= [[f(a_{11}), p_1], p_1] + [[f(a_{12}), p_1], p_1]. \end{aligned}$$

Combining the above two identities, we get
$[[f(a_{11} + a_{12}) - f(a_{11}) - f(a_{12}), p_1], p_1] = 0$, that is

$$\begin{aligned} 0 = p_1(f(a_{11} + a_{12}) - f(a_{11}) - f(a_{12}))p_2 \\ + p_2(f(a_{11} + a_{12}) - f(a_{11}) - f(a_{12}))p_1. \end{aligned} \tag{2.5}$$

Now, for any $b_{12} \in \mathcal{G}_{12}$ and by Lemma 2.4, we have

$$f([[a_{11}+a_{12}, b_{12}], p_1]) = [[f(a_{11}+a_{12}), b_{12}], p_1] + [[a_{11}+a_{12}, f(b_{12})], p_1].$$

On the other hand, we have

$$\begin{aligned} f([[a_{11}+a_{12}, b_{12}], p_1]) &= f([[a_{11}, b_{12}], p_1]) + f([[a_{12}, b_{12}], p_1]) \\ &= [[f(a_{11}), b_{12}], p_1] + [[a_{11}, f(b_{12})], p_1] \\ &\quad + [[f(a_{12}), b_{12}], p_1] + [[a_{12}, f(b_{12})], p_1]. \end{aligned}$$

Combining the above two identities, we arrive at

$$[[f(a_{11}+a_{12}) - f(a_{11}) - f(a_{12}), b_{12}], p_1] = 0.$$

In other words,

$$\begin{aligned} &(p_1(f(a_{11}+a_{12}) - f(a_{11}) - f(a_{12}))p_1)b_{12} \\ &= b_{12}(p_2(f(a_{11}+a_{12}) - f(a_{11}) - f(a_{12}))p_2). \end{aligned} \tag{2.6}$$

Similarly, for any $b_{21} \in \mathcal{G}_{21}$, one can get

$$\begin{aligned} &b_{21}(p_1(f(a_{11}+a_{12}) - f(a_{11}) - f(a_{12}))p_1) \\ &= (p_2(f(a_{11}+a_{12}) - f(a_{11}) - f(a_{12}))p_2)b_{21}. \end{aligned} \tag{2.7}$$

Equations (2.6) and (2.7), together with Lemma 1.1, gives that $f(a_{11} + a_{12}) - f(a_{11}) - f(a_{12}) \in Z(\mathcal{G})$. Similarly, one can easily prove the other part.

□

Lemma 2.10 *f is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.*

Proof. Let $a_{12}, b_{12} \in \mathcal{G}_{12}$. By Lemmas 2.4, 2.6 & 2.7, we see that

$$\begin{aligned} f(a_{12}+b_{12}) &= f([[p_1+a_{12}, p_2+b_{12}], p_2]) \\ &= [[f(p_1+a_{12}), p_2+b_{12}], p_2] + [[p_1+a_{12}, f(p_2+b_{12})], p_2] \\ &\quad + [[p_1+a_{12}, p_2+b_{12}], f(p_2)] \\ &= [[f(p_1)+f(a_{12}), p_2+b_{12}], p_2] + [[p_1+a_{12}, f(p_2)+f(b_{12})], p_2] \\ &= f(a_{12}) + f(b_{12}). \end{aligned}$$

Hence, f is additive on $\mathcal{G}_{12}$. Similarly, one can prove that f is additive on $\mathcal{G}_{21}$.

□

For any $a \in \mathcal{G}$, define $d(a) = f(p_1ap_1) + f(p_1ap_2) + f(p_2ap_1) + f(p_2ap_2) - (\tau_1(p_1ap_1) + \tau_2(p_2ap_2))$. By Lemmas 2.7 and 2.8, we have

Lemma 2.11 *Let $a_{ij} \in \mathcal{G}_{ij}$, $1 \leq i \neq j \leq 2$. Then*

(i) $d(a_{ij}) \in \mathcal{G}_{ij}$, $1 \leq i \neq j \leq 2$,
(ii) $d(a_{12}) = f(a_{12})$ *and* $d(a_{21}) = f(a_{21})$,

(iii) $d(a_{11}+a_{12}+a_{21}+a_{22}) = d(a_{11}) + d(a_{12}) + d(a_{21}) + d(a_{22})$.

Now, we shall show that d is an additive derivation. First, we shall prove the additivity of d.

By Lemma 2.10 and (ii) part of Lemma 2.11, we immediately get the following result.

Lemma 2.12 *d is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.*

Lemma 2.13 *Let $a_{ii} \in \mathcal{G}_{ii}$, $a_{ij} \in \mathcal{G}_{ij}$, $1 \leq i \neq j \leq 2$. Then*

(i) $d(a_{ii}b_{ij}) = d(a_{ii})b_{ij} + a_{ii}d(b_{ij})$,
(ii) $d(b_{ij}a_{jj}) = d(b_{ij})a_{jj} + b_{ij}d(a_{jj})$.

Proof. Since $a_{11}b_{12} = [[a_{11}, b_{12}], p_2]$, by Lemmas 2.5 & 2.11, we have

$$\begin{aligned} d(a_{11}b_{12}) &= f(a_{11}b_{12}) = f([[a_{11}, b_{12}], p_2]) \\ &= [[f(a_{11}), b_{12}], p_2] + [[a_{11}, f(b_{12})], p_2] + [[a_{11}, b_{12}], f(p_2)] \\ &= [[d(a_{11}), b_{12}], p_2] + [[a_{11}, d(b_{12})], p_2] \\ &= d(a_{11})b_{12} + a_{11}d(b_{12}). \end{aligned}$$

Similarly, it is easy to prove the other identities.

□

Lemma 2.14 *d is additive.*

Proof. We divide the proof into the following two steps.
Step 1. Let $a_{11}, b_{11} \in \mathcal{G}_{11}$. For any $b_{21} \in \mathcal{G}_{21}$, by Lemma 2.13, we have

$$d(b_{21}(a_{11}+b_{11})) = d(b_{21})(a_{11}+b_{11}) + b_{21}d(a_{11}+b_{11}).$$

On the other hand, by Lemmas 2.12 and 2.13, we have

$$\begin{aligned} d(b_{21}(a_{11}+b_{11})) &= d(b_{21}a_{11} + b_{21}b_{11}) = d(b_{21}a_{11}) + d(b_{21}b_{11}) \\ &= d(b_{21})a_{11} + b_{21}d(a_{11}) + d(b_{21})b_{11} + b_{21}d(b_{11}). \end{aligned}$$

Comparing the above two equalities, we get

$$b_{21}(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11})) = 0.$$

Since, $d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}) \in \mathcal{G}_{11}$ and $\mathcal{N}$ is a faithful $(\mathcal{B}, \mathcal{A})$-bimodule, it follows that

$$d(a_{11}+b_{11}) = d(a_{11}) + d(b_{11}).$$

Hence d is additive on $\mathcal{G}_{11}$.
Similarly, one can prove that d is additive on $\mathcal{G}_{22}$.

Step 2. Let $x = a_{11} + a_{12} + a_{21} + a_{22}$, $y = b_{11} + b_{12} + b_{21} + b_{22}$ be in $\mathcal{G}$. By Lemmas 2.11, 2.12 and Step 1, we have

$$\begin{aligned} d(x+y) &= d((a_{11} + a_{12} + a_{21} + a_{22}) + (b_{11} + b_{12} + b_{21} + b_{22})) \\ &= d(a_{11} + b_{11}) + d(a_{12} + b_{12}) + d(a_{21} + b_{21}) + d(a_{22} + b_{22}) \\ &= d(a_{11}) + d(b_{11}) + d(a_{12}) + d(b_{12}) + d(a_{21}) + d(b_{21}) + d(a_{22}) + d(b_{22}) \\ &= (d(a_{11}) + d(a_{12}) + d(a_{21}) + d(a_{22})) + (d(b_{11}) + d(b_{12}) + d(b_{21}) + d(b_{22})) \\ &= d(a_{11} + a_{12} + a_{21} + a_{22}) + d(b_{11} + b_{12} + b_{21} + b_{22}) \\ &= d(x) + d(y). \end{aligned}$$

□

Lemma 2.15 *d is a derivation.*

Proof. We divide the proof into the following two steps.
Step 1. Let $a_{11}, b_{11} \in \mathcal{G}_{11}$. For any $a_{21} \in \mathcal{G}_{21}$, by Lemma 2.13, we have

$$d(a_{21}a_{11}b_{11}) = d(a_{21})a_{11}b_{11} + a_{21}d(a_{11}b_{11}).$$

On the other hand,

$$\begin{aligned} d(a_{21}a_{11}b_{11}) &= d(a_{21}a_{11})b_{11} + a_{21}a_{11}d(b_{11}) \\ &= d(a_{21})a_{11}b_{11} + a_{21}d(a_{11})b_{11} + a_{21}a_{11}d(b_{11}). \end{aligned}$$

Comparing these two equalities, we have

$$a_{21}(d(a_{11}b_{11}) - d(a_{11})b_{11} - a_{11}d(b_{11})) = 0$$

for all $a_{21} \in \mathcal{G}_{21}$. Since $\mathcal{N}$ is a faithful $(\mathcal{B}, \mathcal{A})$-bimodule, we get

$$d(a_{11}b_{11}) = d(a_{11})b_{11} + a_{11}d(b_{11}).$$

Similarly, for any $a_{22}, b_{22} \in \mathcal{G}_{22}$, we can get

$$d(a_{22}b_{22}) = d(a_{22})b_{22} + a_{22}d(b_{22}).$$

Step 2. Let $x = a_{11} + a_{12} + a_{21} + a_{22}, y = b_{11} + b_{12} + b_{21} + b_{22}$ be in $\mathcal{G}$. On the one hand, by Lemmas 2.11, 2.12 and Step 1, we have

$$\begin{aligned} d(xy) &= d((a_{11} + a_{12} + a_{21} + a_{22})(b_{11} + b_{12} + b_{21} + b_{22})) \\ &= d(a_{11}b_{11}) + d(a_{11}b_{12}) + d(a_{12}b_{21}) + d(a_{12}b_{22}) \\ &\quad + d(a_{21}b_{11}) + d(a_{21}b_{12}) + d(a_{22}b_{21}) + d(a_{22}b_{22}) \\ &= d(a_{11})b_{11} + a_{11}d(b_{11}) + d(a_{11})b_{12} + a_{11}d(b_{12}) \\ &\quad + d(a_{12})b_{21} + a_{12}d(b_{21}) + d(a_{12})b_{22} + a_{12}d(b_{22}) \\ &\quad + d(a_{21})b_{11} + a_{21}d(b_{11}) + d(a_{21})b_{12} + a_{21}d(b_{12}) \\ &\quad + d(a_{22})b_{21} + a_{22}d(b_{21}) + d(a_{22})b_{22} + a_{22}d(b_{22}). \end{aligned}$$

On the other hand, since $d(\mathcal{G}_{ij}) \subseteq \mathcal{G}_{ij}$, we have

$$\begin{aligned} d(x)y + xd(y) &= d(a_{11}+a_{12}+a_{21}+a_{22})(b_{11}+b_{12}+b_{21}+b_{22}) \\ &\quad +(a_{11}+a_{12}+a_{21}+a_{22})d(b_{11}+b_{12}+b_{21}+b_{22}) \\ &= d(a_{11})b_{11}+a_{11}d(b_{11})+d(a_{11})b_{12}+a_{11}d(b_{12})+d(a_{12})b_{21}+a_{12}d(b_{21}) \\ &\quad +d(a_{12})b_{22}+a_{12}d(b_{22})+d(a_{21})b_{11}+a_{21}d(b_{11})+d(a_{21})a_{12} \\ &\quad +a_{21}d(a_{12})+d(a_{22})b_{21}+a_{22}d(b_{21})+d(a_{22})b_{22}+a_{22}d(b_{22}). \end{aligned}$$

Hence, $d(xy) = d(x)y + xd(y)$, i.e., d is a derivation.

□

Lemma 2.16 *For all $a \in \mathcal{G}$, there holds $p_1f(a)p_1 + p_2f(a)p_2 - f(p_1ap_1) - f(p_2ap_2) \in Z(\mathcal{G})$.*

Proof. Let $b_{12} \in \mathcal{G}_{12}$ and $a \in \mathcal{G}$. since

$$[p_1, [b_{12}, a]] = b_{12}p_2ap_2 - p_1ap_1b_{12},$$

it follows

$$\begin{aligned} f(b_{12}p_2ap_2 - p_1ap_1b_{12}) &= f([p_1, [b_{12}, a]]) \\ &= [p_1, [f(b_{12}), a]] + [p_1, [b_{12}, f(a)]] \\ &= p_1f(b_{12})p_1ap_2 - p_1ap_1f(b_{12})p_2 + p_2ap_2f(b_{12})p_1 \\ &\quad +b_{12}p_2f(a)p_2 - p_1f(a)p_1b_{12} - p_2f(b_{12})p_1ap_1. \end{aligned}$$

On the other hand, by Lemmas 2.4 and 2.10, we have

$$\begin{aligned} &f(b_{12}p_2ap_2 - p_1ap_1b_{12}) \\ &= f(b_{12}p_2ap_2) - f(p_1ap_1b_{12}) \\ &= f([[p_1, b_{12}], p_2ap_2]) - f([p_1ap_1, [p_1, b_{12}]]) \\ &= [[p_1, f(b_{12})], p_2ap_2] + [[p_1, b_{12}], f(p_2ap_2)] \\ &\quad -[f(p_1ap_1), [p_1, b_{12}]] - [p_1ap_1, [p_1, f(b_{12})]] \\ &= p_1f(b_{12})p_2ap_2 + p_2ap_2f(b_{12})p_1 + b_{12}f(p_2ap_2) - f(p_2ap_2)b_{12} \\ &\quad -f(p_1ap_1)b_{12} + b_{12}f(p_1ap_1) - p_1ap_1f(b_{12})p_2 - p_2f(b_{12})p_1ap_1. \end{aligned}$$

Comparing the above two equalities, we get

$$(f(p_1ap_1)+f(p_2ap_2)-p_1f(a)p_1)b_{12}=b_{12}(f(p_2ap_2)+f(p_1ap_1)-p_2f(a)p_2).$$

Since $\tau_1(p_1ap_1), \tau_2(p_2ap_2) \in Z(\mathcal{G})$, we have

$$\begin{aligned} &(p_1f(a)p_1 - (f(p_1ap_1) - \tau_1(p_1ap_1)) - (f(p_2ap_2) - \tau_2(p_2ap_2)))b_{12} \\ &= b_{12}(p_2f(a)p_2 - (f(p_1ap_1) - \tau_1(p_1ap_1)) - (f(p_2ap_2) - \tau_2(p_2ap_2))). \end{aligned}$$

Since $f(p_1ap_1) - \tau_1(p_1ap_1) \in \mathcal{G}_{11}$ and $f(p_2ap_2) - \tau_2(p_2ap_2) \in \mathcal{G}_{22}$, we get

$$(p_1f(a)p_1-(f(p_1ap_1)-\tau_1(p_1ap_1)))b_{12}=b_{12}(p_2f(a)p_2-(f(p_2ap_2)-\tau_2(p_2ap_2))). \tag{2.8}$$

Similarly, for any $b_{21} \in \mathcal{G}_{21}$ and $a \in \mathcal{G}$, one can get

$$b_{21}(p_1f(a)p_1-(f(p_1ap_1)-\tau_1(p_1ap_1)))=(p_2f(a)p_2-(f(p_2ap_2)-\tau_2(p_2ap_2)))b_{21}. \tag{2.9}$$

Equation (2.8), (2.9) together with Lemma 1.1, we have

$$p_1f(a)p_1-(f(p_1ap_1)-\tau_1(p_1ap_1))+p_2f(a)p_2-(f(p_2ap_2)-\tau_2(p_2ap_2)) \in Z(\mathcal{G}).$$

Thus, $p_1f(a)p_1 + p_2f(a)p_2 - f(p_1ap_1) - f(p_2ap_2) \in Z(\mathcal{G})$.

□

Lemma 2.17 *For all $a \in \mathcal{G}$, there holds $f(a) - f(p_1ap_1) - f(p_1ap_2) - f(p_2ap_1) - f(p_2ap_2) \in Z(\mathcal{G})$.*

Proof. By Lemmas 2.4 and 2.16, we get

$$\begin{aligned}
&f(a) - f(p_1ap_1) - f(p_1ap_2) - f(p_2ap_1) - f(p_2ap_2)\\
&= p_1f(a)p_1 + p_1f(a)p_2 + p_2f(a)p_1 + p_2f(a)p_2 - f(p_1ap_1) - f(p_1ap_2)\\
&\quad -f(p_2ap_1) - f(p_2ap_2)\\
&= p_1f(a)p_1 + p_2f(a)p_2 - f(p_1ap_1) - f(p_2ap_2) \in Z(\mathcal{G}).
\end{aligned}$$

□

Lemma 2.18 *τ vanishes at second commutator $[[a, b], c]$ for all $a, b, c \in \mathcal{G}$.*

Proof. By Lemma 2.15 and the definition of d and τ, we see that $\tau(a) \in \mathcal{G}$ for all $a \in \mathcal{G}$. For any $a, b, c \in \mathcal{G}$, since d is an additive derivation, we have

$$\begin{aligned}
\tau([[a, b], c]) &= f([[a, b], c]) - d([[a, b], c])\\
&= [[f(a), b], c] + [[a, f(b)], c] + [[a, b], f(c)] - d([[a, b], c])\\
&= [[d(a) + \tau(a), b], c] + [[a, d(b) + \tau(b)], c] + [[a, b], d(c) + \tau(c)]\\
&\quad -d([[a, b], c])\\
&= [[d(a), b], c] + [[a, d(b)], c] + [[a, b], d(c)] - d([[a, b], c])\\
&= 0.
\end{aligned}$$

This completes the proof for the case $n = 1$.

□

Proof of Theorem 2.1. For $n = 1$, the expression of $\delta_n([[a, b], c])$ reduces to $\delta_1([[a, b], c]) = [[\delta_1(a), b], c] + [[a, \delta_1(b)], c] + [[a, b], \delta_1(c)]$ for all $a, b, c \in \mathcal{G}$. We proceed by induction on $n \in \mathbb{N}$. In view of the above lemmas, it is clear that $\delta_1 = d_1 + \tau_1$, where d_1 is an additive derivation, τ_1 is a mapping from $\mathcal{G}$ into $Z(\mathcal{G})$ satisfying $\tau_1([[a, b], c]) = 0$ for all $a, b, c \in \mathcal{G}$. i.e.; Theorem 2.1 is true for $n = 1$.

By Lemmas 2.2-2.18, we have

$$\mathbf{B}_1 \begin{cases} p_1\delta_1(p_1)p_1 + p_2\delta_1(p_1)p_2 \in Z(\mathcal{G}); \\ p_1\delta_1(p_2)p_1 + p_2\delta_1(p_2)p_2 \in Z(\mathcal{G}); \\ f_1(p_1ap_2 + p_2ap_1) = p_1f_1(a)p_2 + p_2f_1(a)p_1 \text{ where} \\ f_1(a) = \delta_1(a) + \delta_{p_1\delta(p_1)p_2 - p_2\delta(p_1)p_1}(a); \\ f_1(p_1) \in Z(\mathcal{G}),\ f_1(p_2) \in Z(\mathcal{G}); \\ f_1(a_{ij}) \subseteq \mathcal{G}_{ij},\ 1 \leq i \neq j \leq 2; \\ f_1(a_{ii}) - \tau_1(a_{ii}) \in \mathcal{G}_{ii},\ i = \{1, 2\}; \\ f_1(a_{ii} + a_{ij}) - f_1(a_{ii}) - f_1(a_{ij}) \in Z(\mathcal{G}),\ 1 \leq i \neq j \leq 2. \end{cases}$$

We now assume that $\delta_m(a) = d_m(a) + \tau_m(a)$ for all $a \in \mathcal{G}$ and for all $m < n$, where $\tau_m : \mathcal{G} \to Z(\mathcal{G})$ such that $\tau_m([[a, b], c]) = 0$ for all $a, b, c \in \mathcal{G}$ and δ_m satisfies $\delta_m[[a, b], c] = \sum\limits_{i+j+k=m} [[\delta_i(a), \delta_j(b)], \delta_k(c)]$. Thus we have the following properties;

$$\mathbf{B}_m \begin{cases} p_1\delta_m(p_1)p_1 + p_2\delta_m(p_1)p_2 \in Z(\mathcal{G}); \\ p_1\delta_m(p_2)p_1 + p_2\delta_m(p_2)p_2 \in Z(\mathcal{G}); \\ f_m(p_1ap_2 + p_2ap_1) = p_1f_m(a)p_2 + p_2f_m(a)p_1 \text{ where} \\ f_m(a) = \delta_m(a) + \delta_{m(p_1\delta_m(p_1)p_2 - p_2\delta_m(p_1)p_1)}(a); \\ f_m(p_1) \in Z(\mathcal{G}),\ f_m(p_2) \in Z(\mathcal{G}); \\ f_m(a_{ij}) \subseteq \mathcal{G}_{ij},\ 1 \leq i \neq j \leq 2; \\ f_m(a_{ii}) - f_{mi}(a_{ii}) \in \mathcal{G}_{ii},\ i = \{1, 2\};; \\ f_m(a_{ii} + a_{ij}) - f_m(a_{ii}) - f_m(a_{ij}) \in Z(\mathcal{G}),\ 1 \leq i \neq j \leq 2. \end{cases}$$

Our aim is to show that δ_n also satisfies the similar properties and $\delta_n(a) = d_n(a) + \tau_n(a)$ for all $a \in \mathcal{G}$, where $\tau_n : \mathcal{G} \to Z(\mathcal{G})$ such that $\tau_n([[a, b], c]) = 0$ for all $a, b, c \in \mathcal{G}$. Now we shall prove the following Lemmas to obtain our main result.

Lemma 2.19 $p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 \in Z(\mathcal{G})$.

Proof. Let $a_{12} \in \mathcal{G}_{12}$. Then

$$\begin{aligned} \delta_n(a_{12}) = \delta_n([[a_{12}, p_1], p_1]) &= [[\delta_n(a_{12}), p_1], p_1] + [[a_{12}, \delta_n(p_1)], p_1] \\ &\quad + [[a_{12}, p_1], \delta_n(p_1)] + \sum_{\substack{r+s+t=n \\ 0 \leq r,s,t \leq n-1}} [[\delta_r(a_{12}), \delta_s(p_1)], \delta_t(p_1)] \\ &= p_2\delta_n(a_{12})p_1 + p_1\delta_n(a_{12})p_2 - a_{12}\delta_n(p_1)p_2 + p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1) \\ &\quad + \delta_n(p_1)a_{12} + \sum_{\substack{r+s+t=n \\ 0 \leq r,s,t \leq n-1}} [[\delta_r(a_{12}), \delta_s(p_1)], \delta_t(p_1)]. \end{aligned}$$

On multiplying by p_1 from the left and by p_2 from the right in the above equation, we get $2(p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1)p_2) = 0$. This gives that

$p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1)p_2 = 0$, that is, $p_1\delta_n(p_1)p_1a_{12} = a_{12}p_2\delta_n(p_1)p_2$. Similarly, for any $b_{21}\in\mathcal{G}_{21}$ one can get easily $a_{21}p_1\delta_n(p_1)p_1{=}p_2\delta_n(p_1)p_2a_{21}$. Hence, we get $p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 \in Z(\mathcal{G})$.

□

Lemma 2.20 *$f_n(p_1) \in Z(\mathcal{G})$.*

Proof. Using Lemma 2.19, we have

$$\begin{aligned} f_n(p_1) &= \delta_n(p_1) - p_1\delta_n(p_1)p_2 - p_2\delta_n(p_1)p_1 \\ &= \delta_n(p_1)p_1 + \delta_n(p_1)p_2 - p_1\delta_n(p_1)p_2 - p_2\delta_n(p_1)p_1 \\ &= p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2. \end{aligned}$$

Thus $f_n(p_1) \in Z(\mathcal{G})$.

□

Lemma 2.21 *For $a\in\mathcal{G}$, we have $f_n(p_1ap_2 + p_2ap_1){=}p_1f_n(a)p_2{+}p_2f_n(a)p_1$.*

Proof. For any $a \in \mathcal{G}$, we have

$$\begin{aligned} f_n(p_1ap_2 + p_2ap_1) &= f_n([[a, p_1], p_1]) \\ &= [[f_n(a), p_1], p_1] + [[a, f_n(p_1)], p_1] + [[a, p_1], f_n(p_1)] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[f_r(a), f_s(p_1)], f_t(p_1)] \\ &= p_1f_n(a)p_2 + p_2f_n(a)p_1 + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[f_r(a), f_s(p_1)], f_t(p_1)]. \end{aligned}$$

By Lemma 2.20, $f_n(p_1) \in Z(\mathcal{G})$ for all $n \in \mathbb{N}$, we see that

$$f_n(p_1ap_2 + p_2ap_1) = p_1f_n(a)p_2 + p_2f_n(a)p_1.$$

□

Lemma 2.22 *$f_n(p_2) \in Z(\mathcal{G})$.*

Proof. The proof is same as that of Lemma 2.20.

□

Lemma 2.23 *$f_n(a_{ij}) \subseteq \mathcal{G}_{ij},\ 1 \le i \ne j \le 2$.*

Proof. For any $a_{12} \in \mathcal{G}_{12}$ using Lemma 2.20 and $\mathbf{B}_m$, we have

$$\begin{aligned} f_n(a_{12}) &= f_n([[a_{12}, p_1], p_1]) = [[f_n(a_{12}), p_1], p_1] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[f_r(a_{12}), f_s(p_1)], f_t(p_1)] \\ &= p_1f_n(a_{12})p_2 + p_2f_n(a_{12})p_1, \end{aligned} \tag{2.10}$$

from this we see that $p_1 f_n(a_{12})p_1 = p_2 f_n(a_{12})p_2 = 0$. Now if $a_{12}, b_{12} \in \mathcal{G}_{12}$, then $[a_{12}, b_{12}] = 0$. Thus for any $c \in \mathcal{G}$, we have

$$\begin{aligned} 0 = f_n([[a_{12}, b_{12}], c]) &= [[f_n(a_{12}), b_{12}], c] + [[a_{12}, f_n(b_{12})], c] \\ &+ \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{12}), f_s(b_{12})], f_t(c)]. \end{aligned}$$

By applying $\mathbf{B}_m$, i.e.; $f_m(U_{ij}) \subseteq \mathcal{G}_{ij},\ 1 \leq i \neq j \leq 2,\ m < n$, we have

$$[[f_n(a_{12}), b_{12}], c] + [[a_{12}, f_n(b_{12})], c] = 0 \text{ for all } c \in \mathcal{G}.$$

Thus, we get

$$[f_n(a_{12}), b_{12}] + [a_{12}, f_n(b_{12})] \in Z(\mathcal{G}). \tag{2.11}$$

Since $a_{12} = [p_1, a_{12}]$, we have

$$\begin{aligned} [a_{12}, f_n(b_{12})] &= [[p_1, a_{12}], f_n(b_{12})] \\ &= f_n([[p_1, a_{12}], b_{12}]) - [[f_n(p_1), a_{12}], b_{12}] - [[p_1, f_n(a_{12})], b_{12}] \\ &\quad - \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(p_1), f_r(a_{12})], f_s(b_{12})] \\ &= -[[p_1, f_n(a_{12})], b_{12}]. \end{aligned}$$

Now, we have

$$\begin{aligned} [f_n(a_{12}), b_{12}] + [a_{12}, f_n(b_{12})] &= [p_2 f_n(a_{12})p_1, b_{12}] - [[p_1, f_n(a_{12})], b_{12}] \\ &= 2[p_2 f_n(a_{12})p_1, b_{12}] \in Z(\mathcal{G}). \end{aligned} \tag{2.12}$$

Applying the same arguments as used in Lemma 2.7, we obtain $f_n(\mathcal{G}_{12}) \subseteq \mathcal{G}_{12}$.
Similarly, $f_n(a_{21}) = p_2 f_n(a_{21})p_1 \in \mathcal{G}_{21}$ for each $a_{21} \in \mathcal{G}_{21}$ and therefore $f_n(\mathcal{G}_{21}) \subseteq \mathcal{G}_{21}$.

□

Lemma 2.24 *$f_n(\mathcal{G}_{ii}) \subseteq \mathcal{G}_{ii} \oplus \mathcal{G}_{jj}$. There exists a mapping $\tau_{ni} : \mathcal{G}_{ii} \to Z(\mathcal{G})$ such that $f_n(a_{ii}) - \tau_{ni}(a_{ii}) \in \mathcal{G}_{ii}$ for all $a_{ii} \in \mathcal{G}_{ii}$, where $i, j = 1, 2$ and $i \neq j$.*

Proof. First we show the result for $i = 1$. Since $a_{11}p_2 = 0$ for any $a_{11} \in \mathcal{G}_{11}$ and $f_n(p_2) \in Z(\mathcal{G})$, using induction hypothesis it follows that

$$\begin{aligned} 0 &= f_n([[a_{11}, p_2], p_2]) \\ &= [[f_n(a_{11}), p_2], p_2] + [[a_{11}, f_n(p_2)], p_2] + [[a_{11}, p_2], f_n(p_2)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\leq i,j,k<n}} [[f_i(a_{11}), f_j(p_2)], f_k(p_2)] \\ &= f_n(a_{11})p_2 - p_2 f_n(a_{11})p_2 - p_2 f_n(a_{11})p_2 + p_2 f_n(a_{11}). \end{aligned}$$

Multiply by p_1 from the left to get $p_1 f_n(a_{11})p_2 = 0$. So, $f_n(a_{11}) \in \mathcal{G}_{11} \oplus \mathcal{G}_{22}$. Similarly we can find the result for $i = 2$. Thus $f_n(\mathcal{G}_{ii}) \subseteq \mathcal{G}_{ii} \oplus \mathcal{G}_{jj}$.
Now we can write $f_n(a_{11}) = p_1 f_n(a_{11})p_1 + p_2 f_n(a_{11})p_2$. Moreover, since $a_{11}a_{22} = 0$ for any $a_{22} \in \mathcal{G}_{22}$ and $c \in \mathcal{G}$, it is easy to observe that

$$\begin{aligned} 0 = f_n([[a_{11}, a_{22}], c]) &= [[f_n(a_{11}), a_{22}], c] + [[a_{11}, f_n(a_{22})], c] + [[a_{11}, a_{22}], f_n(c)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0 \le i,j,k < n}} [[f_i(a_{11}), f_j(a_{22})], f_k(c)] \\ &= [[f_n(a_{11}), a_{22}], c] + [[a_{11}, f_n(a_{22})], c] \\ &= [[f_n(a_{11}), a_{22}] + [a_{11}, f_n(a_{22})], c]. \end{aligned}$$

Multiplying by p_2 on both the sides we get

$$\begin{aligned} 0 &= p_2[[f_n(a_{11}), a_{22}] + [a_{11}, f_n(a_{22})], c]p_2 \\ &= [[p_2 f_n(a_{11})p_2, p_2 a_{22} p_2] + [p_2 a_{11} p_2, p_2 f_n(a_{22})p_2], p_2 c p_2] \\ &= [[p_2 f_n(a_{11})p_2, a_{22}], p_2 c p_2]. \end{aligned}$$

This implies that $[p_2 f_n(a_{11})p_2, a_{22}] \in Z(\mathcal{G}_{22})$. Hence by assumption (ii), we get $p_2 f_n(a_{11})p_2 \in Z(\mathcal{G}_{22})$. Define $\tau_{n1} : \mathcal{G}_{11} \to Z(\mathcal{G})$ by $\tau_{n1}(a_{11}) = \eta(p_2 f_n(a_{11})p_2) \oplus p_2 f_n(a_{11})p_2$, where η is the map defined in Lemma 1.1. Thus we get

$$\begin{aligned} f_n(a_{11}) - \tau_{n1}(a_{11}) &= p_1 f_n(a_{11})p_1 + p_2 f_n(a_{11})p_2 - \eta(p_2 f_n(a_{11})p_2) - p_2 f_n(a_{11})p_2 \\ &= p_1 f_n(a_{11})p_1 - \eta(p_2 f_n(a_{11})p_2) \in \mathcal{G}_{11}. \end{aligned}$$

Similarly, we can define $\tau_{n2} : \mathcal{G}_{22} \to Z(\mathcal{G})$ by $\tau_{n2}(a_{22}) = p_1 f_n(a_{22})p_1 \oplus \eta^{-1}(p_1 f_n(a_{22})p_1)$. Then

$$\begin{aligned} f_n(a_{22}) - \tau_{n2}(a_{22}) &= p_1 f_n(a_{22})p_1 + p_2 f_n(a_{22})p_2 - p_1 f_n(a_{22})p_1 - \eta^{-1}(p_1 f_n(a_{22})p_1) \\ &= p_2 f_n(a_{22})p_2 - \eta^{-1}(p_1 f_n(a_{22})p_1) \in \mathcal{G}_{22}. \end{aligned}$$

□

Our next goal is to show that f_n is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.

Lemma 2.25 *Let $a_{ii} \in \mathcal{G}_{ii}$ and $a_{ij} \in \mathcal{G}_{ij}$, $1 \le i \ne j \le 2$. Then $f_n(a_{ii} + a_{ij}) - f_n(a_{ii}) - f_n(a_{ij}) \in Z(\mathcal{G})$.*

Proof. For $a_{11} \in \mathcal{G}_{11}$ and $a_{12} \in \mathcal{G}_{12}$, by Lemma 2.20, we find that

$$\begin{aligned} f_n([[a_{11} + a_{12}, p_1], p_1]) &= [[f_n(a_{11} + a_{12}), p_1], p_1] + [[a_{11} + a_{12}, f_n(p_1)], p_1] \\ &\quad + [[a_{11} + a_{12}, p_1], f_n(p_1)] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0 \le r,s,t \le n-1}} [[f_r(a_{11} + a_{12}), f_s(p_1)], f_t(p_1)] \\ &= [[f_n(a_{11} + a_{12}), p_1], p_1] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0 \le r,s,t \le n-1}} [[f_r(a_{11}) + f_r(a_{12}), f_s(p_1)], f_t(p_1)]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} f_n([[a_{11}+a_{12},p_1],p_1]) &= f_n([[a_{11},p_1],p_1]) + f_n([[a_{12},p_1],p_1]) \\ &= [[f_n(a_{11}),p_1],p_1] + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{11}),f_s(p_1)],f_t(p_1)] \\ &\quad +[[f_n(a_{12}),p_1],p_1] + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{12}),f_s(p_1)],f_t(p_1)]. \end{aligned}$$

Combining the above two identities, we get

$$[[f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}),p_1],p_1]=0,$$

that is

$$\begin{aligned} 0 &= p_1(f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}))p_2 \\ &\quad +p_2(f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}))p_1. \end{aligned} \tag{2.13}$$

Now, for any $b_{12}\in\mathcal{G}_{12}$, by Lemma 2.20 and using $\mathbf{B}_m$, we have

$$\begin{aligned} f_n([[a_{11}+a_{12},b_{12}],p_1]) &= [[f_n(a_{11}+a_{12}),b_{12}],p_1] + [[a_{11}+a_{12},f_n(b_{12})],p_1] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{11}+a_{12}),f_s(b_{12})],f_t(p_1)] \\ &= [[f_n(a_{11}+a_{12}),b_{12}],p_1] + [[a_{11}+a_{12},f_n(b_{12})],p_1] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{11})+f_r(a_{12}),f_s(b_{12})],f_t(p_1)]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &f_n([[a_{11}+a_{12},b_{12}],p_1]) \\ &= f_n([[a_{11},b_{12}],p_1]) + f_n([[a_{12},b_{12}],p_1]) \\ &= [[f_n(a_{11}),b_{12}],p_1] + [[a_{11},f_n(b_{12})],p_1] + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{11}),f_s(b_{12})],f_t(p_1)] \\ &\quad +[[f_n(a_{12}),b_{12}],p_1] + [[a_{12},f_n(b_{12})],p_1] + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{12}),f_s(b_{12})],f_t(p_1)]. \end{aligned}$$

Combining the above two identities, we arrive at

$$[[f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}),b_{12}],p_1]=0.$$

In other words,

$$\begin{aligned} &(p_1(f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}))p_1)b_{12} \\ &= b_{12}(p_2(f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}))p_2). \end{aligned} \tag{2.14}$$

Similarly, for any $b_{21} \in \mathcal{G}_{21}$ one can find that

$$\begin{aligned} & b_{21}(p_1(f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}))p_1) \\ & = (p_2(f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}))p_2)b_{21} \end{aligned} \tag{2.15}$$

Equations (2.14) and (2.15), together with Lemma 1.1, gives that $f_n(a_{11}+a_{12})-f_n(a_{11})-f_n(a_{12}) \in Z(\mathcal{G})$. Similarly, one can easily prove the other part.

□

Lemma 2.26 *f_n is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.*

Proof. Let $a_{12}, b_{12} \in \mathcal{G}_{12}$. By Lemmas 2.20, 2.22 & 2.25, we have

$$\begin{aligned} f_n(a_{12}+b_{12}) &= f_n([[p_1+a_{12}, p_2+b_{12}], p_2]) \\ &= [[f_n(p_1+a_{12}), p_2+b_{12}], p_2] + [[p_1+a_{12}, f_n(p_2+b_{12})], p_2] \\ &\quad +[[p_1+a_{12}, p_2+b_{12}], f_n(p_2)] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(p_1+a_{12}), f_s(p_2+b_{12})], f_t(p_2)] \\ &= [[f_n(p_1)+f_n(a_{12}), p_2+b_{12}], p_2]+[[p_1+a_{12}, f_n(p_2)+f_n(b_{12})], p_2] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(p_1)+f_r(a_{12}), f_s(p_2)+f_s(b_{12})], f_t(p_2)]. \end{aligned}$$

Since $\sum\limits_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(p_1)+f_r(a_{12}), f_s(p_2)+f_s(b_{12})], f_t(p_2)] = 0$, the above identity gives that $f_n(a_{12}+b_{12}) = f_n(a_{12})+f_n(b_{12})$. Hence f_n is additive on $\mathcal{G}_{12}$. Similarly f_n is additive on $\mathcal{G}_{21}$.

□

Now for any $a \in \mathcal{G}$, define $d_n(a) = f_n(p_1ap_1)+f_n(p_1ap_2)+f_n(p_2ap_1)+f_n(p_2ap_2)-\tau_{n1}(p_1ap_1)-\tau_{n2}(p_2ap_2)$. By Lemmas 2.23 and 2.24, we have

Lemma 2.27 *For $a_{ij} \in \mathcal{G}_{ij}$, $1 \leq i \neq j \leq 2$, we have*

(i) $d_n(a_{ij}) \in \mathcal{G}_{ij}$, $1 \leq i \neq j \leq 2$,
(ii) $d_n(a_{12}) = f_n(a_{12})$ *and* $d_n(a_{21}) = f_n(a_{21})$,
(iii) $d_n(a_{11}+a_{12}+a_{21}+a_{22}) = d_n(a_{11})+d_n(a_{12})+d_n(a_{21})+d_n(a_{22})$.

The following lemma immediately follows from Lemma 2.26 and Lemma 2.27.

Lemma 2.28 *d_n is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.*

Lemma 2.29 *Let $a_{ii} \in \mathcal{G}_{ii}$, $a_{ij} \in \mathcal{G}_{ij}$, $1 \leq i \neq j \leq 2$. Then*

(i) $d_n(a_{ii}b_{ij}) = \sum\limits_{r+s=n} d_r(a_{ii})d_s(b_{ij})$,

(*ii*) $d_n(b_{ij}a_{jj}) = \sum_{r+s=n} d_r(a_{ij})d_s(b_{jj})$.

Proof. By Lemmas 2.22 & 2.27, we have

$$\begin{aligned} d_n(a_{11}b_{12}) &= f_n(a_{11}b_{12}) = f_n([[a_{11}, b_{12}], p_2]) \\ &= [[f_n(a_{11}), b_{12}], p_2] + [[a_{11}, f_n(b_{12})], p_2] + [[a_{11}, b_{12}], f_n(p_2)] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{11}), f_s(b_{12})], f_t(p_2)] \\ &= [[f_n(a_{11}), b_{12}], p_2] + [[a_{11}, f_n(b_{12})], p_2] \\ &\quad + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [[f_r(a_{11}), f_s(b_{12})], f_t(p_2)]. \end{aligned}$$

Here it is to be noted that t runs from 0 and $n \in \mathbb{N}$, so using $f_0(p_2) = p_2$ and induction hypothesis, we have

$$\begin{aligned} d_n(a_{11}b_{12}) &= [[d_n(a_{11}), b_{12}], p_2] + [[a_{11}, d_n(b_{12})], p_2] + \sum_{\substack{r+s=n \\ 0\leq r,s\leq n-1}} [[d_r(a_{11}), d_s(b_{12})], p_2] \\ &= d_n(a_{11})b_{12} + a_{11}d_n(b_{12}) + \sum_{\substack{r+s=n \\ 0\leq r,s\leq n-1}} d_r(a_{11})d_s(b_{12}) \\ &= \sum_{r+s=n} d_r(a_{11})d_s(b_{12}). \end{aligned}$$

Similarly, it is easy to prove the other three identities.

□

Lemma 2.30 *d_n is additive.*

Proof. We divide the proof into the following two steps.
Step 1. Let $a_{11}, b_{11} \in \mathcal{G}_{11}$. For any $b_{12} \in \mathcal{G}_{12}$, by Lemma 2.29, we have

$$d_n((a_{11}+b_{11})b_{12}) = d_n(a_{11}+b_{11})b_{12} + (a_{11}+b_{11})d_n(b_{12}) + \sum_{\substack{r+s=n \\ 0\leq r,s\leq n-1}} d_r(a_{11}+b_{11})d_s(b_{12}).$$

On the other hand, by Lemmas 2.28 and 2.29, we have

$$\begin{aligned} d_n((a_{11}+b_{11})b_{12}) &= d_n(a_{11}b_{12} + b_{11}b_{12}) = d_n(a_{11}b_{12}) + d_n(b_{11}b_{12}) \\ &= d_n(a_{11})b_{12} + a_{11}d_n(b_{12}) + \sum_{\substack{r+s=n \\ 0\leq r,s\leq n-1}} d_r(a_{11})d_s(b_{12}) \\ &\quad + d_n(b_{11})b_{12} + b_{11}d_n(b_{12}) + \sum_{\substack{r+s=n \\ 0\leq r,s\leq n-1}} d_r(b_{11})d_s(b_{12}). \end{aligned}$$

Comparing the above two equalities, we get

$$d_n(a_{11}+b_{11})b_{12} = d_n(a_{11})b_{12} + d_n(b_{11})b_{12}.$$

This is equivalent to

$$(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))b_{12} = 0.$$

Since, $d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}) \in \mathcal{G}_{11}$ and $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, it follows that

$$d_n(a_{11}+b_{11}) = d_n(a_{11}) + d_n(b_{11}).$$

Hence d_n is additive on $\mathcal{G}_{11}$.
Similarly, one can prove that d_n is additive on $\mathcal{G}_{22}$.

Step 2. Let $x = a_{11}+a_{12}+a_{21}+a_{22}$, $y = b_{11}+b_{12}+b_{21}+b_{22}$ be in $\mathcal{G}$. By Lemmas 2.27, 2.28 and Step 1, we have

$$\begin{aligned}
d_n(x+y) &= d_n((a_{11}+a_{12}+a_{21}+a_{22})+(b_{11}+b_{12}+b_{21}+b_{22})) \\
&= d_n(a_{11}+b_{11}) + d_n(a_{12}+b_{12}) + d_n(a_{21}+b_{21}) + d_n(a_{22}+b_{22}) \\
&= d_n(a_{11}) + d_n(b_{11}) + d_n(a_{12}) + d_n(b_{12}) + d_n(a_{21}) + d_n(b_{21}) \\
&\quad + d_n(a_{22}) + d_n(b_{22}) \\
&= (d_n(a_{11}) + d_n(a_{12}) + d_n(a_{21}) + d_n(a_{22})) + (d_n(b_{11}) + d_n(b_{12}) \\
&\quad + d_n(b_{21}) + d_n(b_{22})) \\
&= d_n(a_{11}+a_{12}+a_{21}+a_{22}) + d_n(b_{11}+b_{12}+b_{21}+b_{22}) \\
&= d_n(x) + d_n(y).
\end{aligned}$$

□

Lemma 2.31 *d_n is a derivation.*

Proof. We divide the proof into the following two steps.
Step 1. Let $a_{11}, b_{11} \in \mathcal{G}_{11}$. For any $a_{12} \in \mathcal{G}_{12}$, by Lemma 2.29, we have

$$\begin{aligned}
d_n(a_{11}b_{11}a_{12}) &= \sum_{k+t=n} d_k(a_{11}b_{11})d_t(a_{12}) \\
&= d_n(a_{11}b_{11})a_{12} + \sum_{\substack{k+t=n \\ t\neq 0}} d_k(a_{11}b_{11})d_t(a_{12}) \\
&= d_n(a_{11}b_{11})a_{12} + \sum_{\substack{r+s+t=n \\ t\neq 0}} d_r(a_{11})d_s(b_{11})d_t(a_{12}).
\end{aligned}$$

On the other hand,

$$\begin{aligned} d_n(a_{11}b_{11}a_{12}) &= \sum_{r+k=n} d_r(a_{11})d_k(b_{11}a_{12}) \\ &= \sum_{r+s+t=n} d_r(a_{11})d_s(b_{11})d_t(a_{12}) \\ &= \sum_{r+s=n} d_r(a_{11})d_s(b_{11})a_{12} + \sum_{\substack{r+s+t=n \\ t\neq 0}} d_r(a_{11})d_s(b_{11})d_t(a_{12}). \end{aligned}$$

Comparing the above two equalities, we have

$$\big(d_n(a_{11}b_{11}) - \sum_{r+s=n} d_r(a_{11})d_s(b_{11})\big)a_{12} = 0.$$

Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we get

$$d_n(a_{11}b_{11}) = \sum_{r+s=n} d_r(a_{11})d_s(b_{11}).$$

Similarly, we can calculate

$$d_n(a_{22}b_{22}) = \sum_{r+s=n} d_r(a_{22})d_s(b_{22}).$$

Step 2. Let $x = a_{11} + a_{12} + a_{21} + a_{22}, y = b_{11} + b_{12} + b_{21} + b_{22}$ be in $\mathcal{G}$. On the one hand, by Lemmas 2.27, 2.29 and Step 1, we have

$$\begin{aligned} d_n(xy) &= d_n((a_{11} + a_{12} + a_{21} + a_{22})(b_{11} + b_{12} + b_{21} + b_{22})) \\ &= d_n(a_{11}b_{11}) + d_n(a_{11}b_{12}) + d_n(a_{12}b_{21}) + d_n(a_{12}b_{22}) \\ &\quad + d_n(a_{21}b_{11}) + d_n(a_{21}b_{12}) + d_n(a_{22}b_{21}) + d_n(a_{22}b_{22}) \\ &= d_n(a_{11})b_{11} + a_{11}d_n(b_{11}) + d_n(a_{11})b_{12} + a_{11}d_n(b_{12}) \\ &\quad + d_n(a_{12})b_{21} + a_{12}d_n(b_{21}) + d_n(a_{12})b_{22} + a_{12}d_n(b_{22}) \\ &\quad + d_n(a_{21})b_{11} + a_{21}d_n(b_{11}) + d_n(a_{21})b_{12} + a_{21}d_n(b_{12}) \\ &\quad + d_n(a_{22})b_{21} + a_{22}d_n(b_{21}) + d_n(a_{22})b_{22} + a_{22}d_n(b_{22}). \end{aligned}$$

On the other hand, since $d(\mathcal{G}_{ij}) \subseteq \mathcal{G}_{ij}$, we have

$$\begin{aligned} &d_n(x)y + xd_n(y) \\ &= d_n(a_{11} + a_{12} + a_{21} + a_{22})(b_{11} + b_{12} + b_{21} + b_{22}) \\ &\quad + (a_{11} + a_{12} + a_{21} + a_{22})d_n(b_{11} + b_{12} + b_{21} + b_{22}) \\ &= d_n(a_{11})b_{11} + a_{11}d_n(b_{11}) + d_n(a_{11})b_{12} + a_{11}d_n(b_{12}) + d_n(a_{12})b_{21} + a_{12}d_n(b_{21}) \\ &\quad + d_n(a_{12})b_{22} + a_{12}d_n(b_{22}) + d_n(a_{21})b_{11} + a_{21}d_n(b_{11}) + d_n(a_{21})a_{12} + a_{21}d_n(a_{12}) \\ &\quad + d_n(a_{22})b_{21} + a_{22}d_n(b_{21}) + d_n(a_{22})b_{22} + a_{22}d_n(b_{22}). \end{aligned}$$

Hence, $d_n(xy) = d_n(x)y + xd_n(y)$, i.e., d_n is a derivation.

□

Lemma 2.32 *For all $a \in \mathcal{G}$, there holds $p_1 f_n(a)p_1 + p_2 f_n(a)p_2 - f_n(p_1 a p_1) - f_n(p_2 a p_2) \in Z(\mathcal{G})$.*

Proof. Let $b_{12} \in \mathcal{G}_{12}$ and $a \in \mathcal{G}$. Since $[p_1, [b_{12}, a]] = b_{12}p_2 a p_2 - p_1 a p_1 b_{12}$, using Lemmas 2.20, 2.26 and induction hypothesis, we get

$$\begin{aligned}
&f_n(b_{12}p_2 a p_2 - p_1 a p_1 b_{12}) \\
&= f_n([p_1, [b_{12}, a]]) \\
&= [p_1, [f_n(b_{12}), a]] + [p_1, [b_{12}, f_n(a)]] + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[f_r(p_1), f_s(b_{12})], f_t(a)] \\
&= p_1 f_n(b_{12}) p_1 a p_2 - p_1 a p_1 f_n(b_{12}) p_2 + p_2 a p_2 f_n(b_{12}) p_1 + b_{12} p_2 f_n(a) p_2 \\
&\quad - p_1 f_n(a) p_1 b_{12} - p_2 f_n(b_{12}) p_1 a p_1.
\end{aligned}$$

On the other hand, by Lemmas 2.20, 2.26 and using induction hypothesis, we have

$$\begin{aligned}
&f_n(b_{12}p_2 a p_2 - p_1 a p_1 b_{12}) \\
&= f_n(b_{12}p_2 a p_2) - f_n(p_1 a p_1 b_{12}) \\
&= f_n([[p_1, b_{12}], p_2 a p_2]) - f_n([p_1 a p_1, [p_1, b_{12}]]) \\
&= [[p_1, f_n(b_{12})], p_2 a p_2] + [[p_1, b_{12}], f_n(p_2 a p_2)] + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[f_r(p_1), f_s(b_{12})], f_t(p_2 a p_2)] \\
&\quad - [f_n(p_1 a p_1), [p_1, b_{12}]] - [p_1 a p_1, [p_1, f_n(b_{12})]] + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [[f_r(p_1 a p_1), f_s(p_1)], f_t(b_{12})] \\
&= p_1 f_n(b_{12}) p_2 a p_2 + p_2 a p_2 f_n(b_{12}) p_1 + b_{12} f_n(p_2 a p_2) - f_n(p_2 a p_2) b_{12} \\
&\quad - f_n(p_1 a p_1) b_{12} + b_{12} f_n(p_1 a p_1) - p_1 a p_1 f_n(b_{12}) p_2 - p_2 f_n(b_{12}) p_1 a p_1.
\end{aligned}$$

Comparing the above two equalities, we get

$$(f_n(p_1 a p_1) + f_n(p_2 a p_2) - p_1 f_n(a) p_1) b_{12} = b_{12}(f_n(p_2 a p_2) + f_n(p_1 a p_1) - p_2 f_n(a) p_2).$$

Since $\tau_{n1}(p_1 a p_1), \tau_{n2}(p_2 a p_2) \in Z(\mathcal{G})$, we have

$$\begin{aligned}
&(p_1 f_n(a) p_1 - (f_n(p_1 a p_1) - \tau_{n1}(p_1 a p_1)) - (f_n(p_2 a p_2) - \tau_{n2}(p_2 a p_2)))b_{12} \\
&= b_{12}(p_2 f_n(a) p_2 - (f_n(p_1 a p_1) - \tau_{n1}(p_1 a p_1)) - (f_n(p_2 a p_2) - \tau_{n2}(p_2 a p_2))).
\end{aligned}$$

Since $f_n(p_1 a p_1) - \tau_{n1}(p_1 a p_1) \in \mathcal{G}_{11}$ and $f_n(p_2 a p_2) - \tau_{n2}(p_2 a p_2) \in \mathcal{G}_{22}$, we get

$$(p_1 f_n(a) p_1 - (f_n(p_1 a p_1) - \tau_{n1}(p_1 a p_1)))b_{12} = b_{12}(p_2 f_n(a) p_2 - (f_n(p_2 a p_2) - \tau_{n2}(p_2 a p_2))). \tag{2.16}$$

Similarly, for any $b_{21} \in \mathcal{G}_{21}$ and $a \in \mathcal{G}$, one can get

$$b_{21}(p_1 f_n(a) p_1 - (f_n(p_1 a p_1) - \tau_{n1}(p_1 a p_1))) = (p_2 f_n(a) p_2 - (f_n(p_2 a p_2) - \tau_{n2}(p_2 a p_2)))b_{21}. \tag{2.17}$$

From (2.16), (2.17) and using Lemma 1.1, we have

$$p_1f_n(a)p_1-(f_n(p_1ap_1)-\tau_{n1}(p_1ap_1))+p_2f_n(a)p_2-(f_n(p_2ap_2)-\tau_{n2}(p_2ap_2))\in Z(\mathcal{G}).$$

Thus, $p_1f_n(a)p_1 + p_2f_n(a)p_2 - f_n(p_1ap_1) - f_n(p_2ap_2) \in Z(\mathcal{G})$.

□

Lemma 2.33 *For all $a \in \mathcal{G}$, there holds $f_n(a) - f_n(p_1ap_1) - f_n(p_1ap_2) - f_n(p_2ap_1) - f_n(p_2ap_2) \in Z(\mathcal{G})$.*

Proof. By Lemmas 2.21 and 2.32, we get

$$\begin{aligned}
&f_n(a) - f_n(p_1ap_1) - f_n(p_1ap_2) - f_n(p_2ap_1) - f_n(p_2ap_2)\\
&= p_1f_n(a)p_1 + p_1f_n(a)p_2 + p_2f_n(a)p_1 + p_2f_n(a)p_2\\
&\quad -f_n(p_1ap_1) - f_n(p_1ap_2) - f_n(p_2ap_1) - f_n(p_2ap_2)\\
&= p_1f_n(a)p_1 + p_2f_n(a)p_2 - f_n(p_1ap_1) - f_n(p_2ap_2) \in Z(\mathcal{G}).
\end{aligned}$$

□

Lemma 2.34 *τ vanishes at second commutator $[[a, b], c]$ for all $a, b, c \in \mathcal{G}$.*

Proof. By Lemma 2.33 and the definition of d_n and τ_n, we see that $\tau_n(a) \in Z(\mathcal{G})$ for all $a \in \mathcal{G}$. For any $a, b, c \in \mathcal{G}$, since d_n is an additive Lie triple higher derivation, we have

$$\begin{aligned}
\tau_n([[a, b], c]) &= f_n([[a, b], c]) - d_n([[a, b], c])\\
&= \sum_{r+s+t=n} ([[f_r(a), f_s(b)], f_t(c)]) - d_n([[a, b], c])\\
&= \sum_{r+s+t=n} ([[d_r(a) + \tau_r(a), d_s(b) + \tau_s(b)], d_t(c) + \tau_t(c)])\\
&\quad -d_n([[a, b], c])\\
&= \sum_{r+s+t=n} [[d_r(a), d_s(b)], d_t(c)] - d_n([[a, b], c])\\
&= 0.
\end{aligned}$$

This completes the proof of the theorem.

□

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