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On 2-prime ideals in commutative semigroups

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Abstract We introduce and study the concepts of 2-prime and weakly 2-prime ideals in a commutative semigroup. We characterize semigroups where 2-prime ideals are prime and also characterize semigroups where every ideal is weakly 2-prime.

Keywords commutative semigroup · prime ideal · 2-prime ideal · weakly 2-prime ideal

Mathematics Subject Classification (2010) 20M12 · 20M14

1 Introduction

Throughout this paper, unless otherwise mentioned, all semigroups are commutative, in particular S will denote such a semigroup.
If S is a semigroup with unity but not a group then it has unique maximal ideal say M, which is the union of all proper ideals of S and we write the semigroup as (S, M). If A is a subset of a semigroup S, then $S - A$ denotes the complement of A in S. A nonempty subset I of S is called an ideal of S if $SI \subseteq I$. We define the residual of an ideal I of S by $a \in S$ as $(I : a) = \{s \in S : sa \in I\}$ and radical of I as $\sqrt{I} = \{s \in S : s^n \in I$ for some integer $n\}$. For an element $a \in S$, $(a) = Sa = \{sa : s \in S\}$ is an ideal of S generated by a. An ideal I of S is said to be proper if $I \neq S$. A proper ideal P of S is said to be prime if $ab \in P$ for any a,$b \in S$ implies either $a \in P$ or $b \in P$. A proper ideal I of S is said to be primary if $ab \in I$ for any a,$b \in S$ implies $a \in I$ or $b \in \sqrt{I}$ and semiprimary if $\sqrt{I}$ is a prime ideal of S. For further various definitions and terms involved, we refer ([1], [2], [6], [7], [11], [13], [14]). The concept of 2-prime (weakly 2-prime) ideals as a generalisations of prime (weakly prime) ideals in a commutative ring was introduced in [3], [8] and rings in which 2-prime ideals are prime was characterized in [10]. These observations prompted us to study 2-prime ideals, weakly 2-prime ideals in semigroup and to characterize semigroups in

which 2-prime ideals are prime.
In this paper, we define 2-prime ideals in a commutative semigroup ($cf.$ Definition 2.1) and establish its relation with prime and semiprimary ideal ($cf.$ Lemma 2.2). We prove that every maximal ideal of a commutative ring and semigroup is 2-prime ($cf.$ Theorem 2.5, 2.6) but the converses are not true. Then we characterize semigroups in which 2-prime ideals are maximal ($cf.$ Theorem 2.7, 2.8). We characterize 2-prime ideals in a commutative semigroup ($cf.$ Theorem 2.11) and also studied some properties of 2-prime ideals ($cf.$ Proposition 2.9, Theorem 2.16). In §3 we characterize semigroup in which 2-prime ideals are prime and in §4 we characterize semigroups in which every proper ideal is weakly 2-prime (resp. strong weakly 2-prime).
Before going to the main work, we discuss some preliminaries which are necessary,

Theorem 1.1 *[12] If M is a maximal ideal of a semigroup S such that $S - M$ contains either more than one element or an idempotent, then M is a prime ideal of S.*

Lemma 1.2 *[4] Let P be a prime ideal of S and I be an ideal of S with $I \subseteq P$. Then the following statements are equivalent*
1. *P is a minimal prime ideal over I;*
2. *For every element $p \in P$, there is an element $s \in S - P$ and a nonnegative integer n such that $sp^n \in I$.*

2 2-prime ideals

Definition 2.1 *A proper ideal I of a semigroup S is said to be a 2-prime ideal if $ab \in I$ for $a, b \in S$ implies $a^2 \in I$ or $b^2 \in I$.*

The following lemmas are obvious, hence we omit the proof.

Lemma 2.2 1. *Every prime ideal of a semigroup S is a 2-prime ideal of S.*
2. *Every 2-prime ideal of a semigroup S is a semiprimary ideal of S.*

The following examples show that converse of above lemmas need not true.

Example 2.1 Consider the ideal $I = \{m \in \mathbb{N} : m \geq 2\}$ in the semigroup $S = \{\mathbb{N}, +\}$, which is clearly 2-prime ideal but not a prime ideal of S.

Example 2.2 Consider the semigroup $S = \{a, b, c\}$ with the following multiplication $a^2 = a$, $b^2 = a$, $ab = bc = ac = c^2 = c$. Here $I = \{c\}$ is a semiprimary ideal but not a 2-prime ideal of S, since $ab \in I$ but neither $a^2 \in I$ nor $b^2 \in I$.

Theorem 2.3 *Let S be a semigroup such that $(\sqrt{I})^2 \subseteq I$ for every semiprimary ideal I of S. Then every semiprimary ideal of S is 2-prime.*

Proof. Let $ab \in I \subseteq \sqrt{I}$ for $a,b \in S$. Then either $a \in \sqrt{I}$ or $b \in \sqrt{I}$, since I is semiprimary. Let $a \in \sqrt{I}$. Then $a^2 \in (\sqrt{I})^2 \subseteq I$, by hypothesis. Consequently, I is a 2-prime ideal of S.

□

Theorem 2.4 *Let S be a semigroup in which prime ideals are maximal and $(\sqrt{I})^2 \subseteq I$ for every semiprimary ideal I of S. Then the followings are equivalent*
1. *I is a 2-prime ideal of S.*
2. *I is a semiprimary ideal of S.*
3. *I is a primary ideal of S.*

If S is a semigroup with unity, then it has unique maximal ideal which is prime also and hence 2-prime. But if S is a semigroup without unity then maximal ideal need not be prime, for example consider the ideal $I = \{m \in \mathbb{N} : m \geq 2\}$ in the semigroup $S = \{\mathbb{N}, +\}$, which is maximal but not prime. But there is a relation between maximal ideal and 2-prime ideal of a semigroup without unity.

Theorem 2.5 *Let S be a semigroup without unity and assume maximal ideal exists. Then every maximal ideal of S is a 2-prime ideal of S.*

Proof. Let M be a maximal ideal of a semigroup S without unity and $ab \in M$ with $a^2 \notin M$ for some $a, b \in S$. Now if $b^2 \notin M$, then a, b, a^2, b^2 all belongs to $S - M$. If any of a, a^2, b, b^2 are not equal, then $S - M$ contains more than one element and if all are equal then $S - M$ contains an idempotent element. Hence M is a prime ideal of S ([12], Theorem 1a). Consequently M is a 2-prime ideal of S.

□

Remark 2.1 The converse of above theorem need not be true. Consider the ideal $I = \{m \in \mathbb{N} : m \geq 3\}$ in the semigroup $S = (\mathbb{N}, +)$, which is clearly 2-prime but not a maximal ideal of S. The theorem 2.5 is also true in the case of commutative ring, the proof is as follows.

Theorem 2.6 *Let R be a commutative ring. Then every maximal ideal of R is a 2-prime ideal of R.*

Proof. If R is a commutative ring with unity then every maximal ideal is prime and hence 2-prime.
Let us consider the case that R is a commutative ring without unity and $ab \in M$ with $a^2 \notin M$ for some $a\ b \in R$. If $b^2 \notin M$, then clearly $a, b \in R - M$. Hence we have $R = M + (b) = M + (a)$. Since $a \in R$, $a^2 = (m + r_1 b + z_1 b)(n + r_2 a + z_2 a)$ for some m, $n \in M$, r_1, $r_2 \in R$ and z_1, $z_2 \in \mathbb{Z}$. Therfore $a^2 \in M$, a contradiction. Hence $b^2 \in M$, consequently M is a 2-prime ideal of R.

□

The following is a characterization of a semigroup with unity in which 2-prime ideals are maximal, the result is obvious, hence we omit the proof.

Theorem 2.7 *Let (S, M) be a semigroup with unity. Then 2-prime ideals of S are maximal if and only if S has a unique 2-prime ideal A such that $S = A \cup H$, where H is the group of units and A is an archimedian subsemigroup of S.*

Theorem 2.8 *Let S be a semigroup without unity. Then 2-prime ideals of S are maximal if and only if complements of every 2-prime ideals contains exactly one non-idempotent element or forms a subgroup of S.*

Proof. Let 2-prime ideals of a semigroup S without unity are maximal. Let P be a 2-prime but not prime ideal of S. Hence complements of P contains exactly one non-idempotent element otherwise P is prime ([12], Theorem $1a$). Again let a 2-prime J of S is prime. Then $x, y \in S-J$ implies $xy \in S-J$. Hence $S - J$ is a subgroup of S, since complements of maximal ideals of a commutative semigroup is Green's $\mathbb{H}$-class and all x, y, $xy \in S - J$ ([5], Theorem 2.16).
Conversely, if complements of a 2-prime ideal contains exactly one element clearly it is maximal. Again let I be a 2-prime ideal whose complements in S forms a subgroup of S. Now let I is not maximal and is contained in some proper ideal P of S. Let e be the identity element of $S - I$. Since $I \neq P$, there exists $p \in P - I$ such that $pq = e$ for some $q \in S$. Hence $e \in P$. Since $P \neq S$, there exists $l \in S - P$ such that $l = le \in P$, a contradiction. Consequently, I is a maximal ideal of S.

□

Arbitrary union of 2-prime ideals of a semigroup is a 2-prime ideal but intersection of two 2-prime ideals of a semigroup need not be 2-prime. For example consider the principal ideals $I_1 = (2)$ and $I_2 = (5)$ in the semigroup $S = (\mathbb{Z}, .)$, both are 2-prime ideals of S but $I_1 \cap I_2 = (10)$ is not a 2-prime ideal of S.

Proposition 2.9 *Let I be an ideal of a semigroup S with unity.*
(1) *If I is a 2-prime ideal of S, then there is exactly one prime ideal of S that is minimal over I.*
(2). *If I is a 2-prime ideal of S, then $\sqrt{I} = P$ is a prime ideal of S, we say that I is P-2-prime.*
(3). *If I is a prime ideal of S, then I^2 is a 2-prime ideal of S.*
(4). *If I is a P-2-prime ideal of S, then $(I : a^2)$ is a 2-prime ideal, for all $a \in S$ such that $a^2 \notin I$. In particular, $(I : a^2)$ is a P-2-prime ideal of S for all $a \in S - \sqrt{I}$.*
(5). *If I is a 2-prime ideal and $(I : x) = (I : x^2)$ for all $x \in S - I$, then $(I : x)$ is a 2-prime ideal of S.*
(6). *Let I be a 2-prime ideal of S and C be a subsemigroup of S such that $C \cap I \neq \phi$. Then $C^{-1}I$ is a 2-prime ideal of $C^{-1}S$.*
(7). *An ideal I of a semigroup S is 2-prime if and only if $I[x]$ is a 2-prime ideal of $S[x]$.*

(8). *Let I_1, I_2,...,I_n be ideals of a semigroup S and I be a 2-prime ideals of S such that $\bigcap I_i \subseteq \sqrt{I}$. Then $I_i \subseteq \sqrt{I}$ for some $i \in \{1, 2, ..., n\}$.*
(9) *Let S_1 and S_2 be two semigroups with unity and $S = S_1 \times S_2$. Then I_1 (resp. I_2) is a 2-prime ideal of S_1 (resp. S_2) if and only if $I_1 \times S_2$ (resp. $S_1 \times I_2$) is a 2-prime ideals of S.*

Proof. (1). If possible, let P_1 and P_2 be two distinct prime ideal that are minimal over I. Hence there exists $x_1 \in P_1 - P_2$ and $x_2 \in P_2 - P_1$. By Lemma 1.2 there is $p_1 \notin P_1$ and $p_2 \notin P_2$ such that $p_1 x_1^n \in I$ and $p_2 x_2^m \in I$ for some integer m, $n \geq 1$. Since x_1, $x_2 \notin I \subseteq P_1 \cap P_2$ and I is 2-prime , hence $p_1^2 \in I \subseteq P_1 \cap P_2$ and $p_2^2 \in I \subseteq P_1 \cap P_2$. Therefore $p_1^2 \in P_1$. Since P_1 is prime so $p_1 \in P_1$, a contradiction. Similarly if $p_2^2 \in P_2$ then $p_2 \in P_2$, a contradiction. Hence there is exactly one prime ideal which is minimal over I.
(2). Since every 2-prime ideal is semiprimary ($cf.$ Lemma 2.2(2)), it is clear.
(3). Let $ab \in I^2 \subseteq I$ for some a,$b \in S$. Then either $a \in I$ or $b \in I$, since I is a prime ideal of S. Let $a \in I$. Then $a^2 \in I^2$ implies I^2 is a 2-prime ideal of S.
(4). Let $xy \in (I : a^2)$ with $x^2 \notin (I : a^2)$ for x, $y \in S$. Then $xya^2 = (xa)(ya) \in I$. Hence $(ya)^2 = y^2a^2 \in I$, since I is a 2-prime ideal of S and $x^2a^2 \notin I$. Consequently $(I : a^2)$ is a 2-prime ideal of S.
Again let $a \in S - P$ and $x \in (I : a^2)$. Then $a^2x \in I \subseteq P$. Hence $x^2 \in I$, since $a \notin P$ and I is a 2-prime ideal of S. Thus $I \subseteq (I : a^2) \subseteq P$, which implies $P = \sqrt{I} \subseteq \sqrt{(I : a^2)} \subseteq \sqrt{P} = P$. Consequently $(I : a^2)$ is a P-2-prime ideal of S.
(5) It is clear from (4).
(6). Let $(a/s)(b/t) \in C^{-1}I$ for some a, $b \in S$ and s, $t \in C$. Then there exists $u \in C$ such that $abu \in I$. Then $a^2 \in I$ or $b^2u^2 \in I$, since I is a 2-prime ideal of S. If $a^2 \in I$, then $(a/s)^2 = (ua^2/us^2) \in C^{-1}I$ and if $b^2u^2 \in I$ then $(b/s)^2 = (b^2u^2/s^2u^2) \in C^{-1}I$. Therefore $C^{-1}I$ is a 2-prime ideal of $C^{-1}S$.
(7) Let I be a 2-prime ideal of S and $(ax^i)(bx^j) \in I[x]$ for some a, $b \in S$ and i $j \in \mathbb{N}$.Then $abx^{i+j} \in I[x]$. Hence $ab \in I$ implies $a^2 \in I$ or $b^2 \in I$, since I is a 2-prime ideal of S. Now $(ax^i)^2 = a^2x^{2i} \in I[x]$ or $(bx^j)^2 = b^2x^{2j} \in I[x]$. Hence $I[x]$ is a 2-prime ideal of $S[x]$.
Similarly we can prove the converse.
(8). Let $I_i \nsubseteq \sqrt{I}$ for all $i \in \{1, 2, ..., n\}$. Then there exists $x_i \in I_i$ but $x_i \notin \sqrt{I}$ for all $i \in \{1, 2, ..., n\}$. Let $p = x_1x_2...x_n$. Then $p \in \bigcap I_i$ but $p \notin \sqrt{I}$, since $\sqrt{I}$ is a prime ideal of S, a contradiction . Hence $I_i \subseteq \sqrt{I}$ for some $i \in \{1, 2, ..., n\}$.
(9) Let I_1 be a 2-prime ideal of S_1 and $(a, b)(c, d) \in I_1 \times S_2$ for some (a, b),$(c, d) \in S$. Then $ac \in I_1$. Hence either $a^2 \in I_1$ or $c^2 \in I_1$, since I_1 is a 2-prime ideal of S_1. Hence $(a, b)^2 \in I_1 \times S_2$ or $(c, d)^2 \in I_1 \times S_2$. Consequently, $I_1 \times S_2$ is a 2-prime ideal of S.
Conversely, let $I_1 \times S_2$ be a 2-prime ideal of S and $ab \in I_1$ for some a,$b \in S_1$.

Then $(a,1)(b,1) \in I_1 \times S_2$. Hence $a^2 \in I_1$ or $b^2 \in I_1$. Consequently, I_1 is a 2-prime ideal of S_1.

□

Lemma 2.10 *Let I be a 2-prime ideal of a semigroup S. Then if $aJ \subseteq I$ for some $a \in S$ and ideal J of S, then $a^2 \in I$ or $\{x^2 : x \in J\} \subseteq I$.*

Proof. Let $aJ \subseteq I$ but $a^2 \notin I$ and $\{x^2 : x \in J\} \nsubseteq I$. Then there exists $j \in J$ such that $j^2 \notin I$. Also we have $aj \in I$. Since I is a 2-prime ideal of S either $a^2 \in I$ or $j^2 \in I$, a contradiction. Hence $a^2 \in I$ or $\{x^2 : x \in J\} \subseteq I$.

□

The following is a characterization of a 2-prime ideal in a semigroup.

Theorem 2.11 *Let I be a proper ideal of a semigroup S with identity. Then I is a 2-prime ideal of S if and only if whenever $AB \subseteq I$ for some ideals A, B of S, then either $\{x^2 : x \in A\} \subseteq I$ or $\{y^2 : y \in B\} \subseteq I$.*

Proof. Let I be a 2-prime ideal of a semigroup S. Also assume $AB \subseteq I$ for some ideal A, B of S and $\{x^2 : x \in A\} \nsubseteq I$. Then there exists an element $a \in A$ such that $a^2 \notin I$. Since $aB \subseteq I$ and $a^2 \notin I$, we conclude $\{y^2 : y \in B\} \subseteq I$ (*cf.* Lemma 2.10).
Conversely, let $ab \in I$ for some a,$b \in S$ and $a^2 \notin I$. Let $A = (a)$ and $B = (b)$. Then $AB \subseteq I$ and $\{x^2 : x \in A\} \nsubseteq I$, otherwise $a^2 \in I$. Hence $\{y^2 : y \in B\} \subseteq I$ implies $b^2 \in I$. Consequently, I is a 2-prime ideal of S.

□

Theorem 2.12 *Let $S(R)$ be the multiplicative semigroup with zero and identity of a ring R. For a proper ideal I of R the following statements are equivalent:*
1. *I is a 2-prime ideal of R.*
2. *I is a 2-prime ideal of $S(R)$;*
3. *If $AB \subseteq I$ for some ideals A, B of $S(R)$, then $\{x^2 : x \in A\} \subseteq I$ or $\{x^2 : x \in B\} \subseteq I$;*
4. *If $JK \subseteq I$ for some ideals J, K of R, then $\{x^2 : x \in J\} \subseteq I$ or $\{x^2 : x \in K\} \subseteq I$.*

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (3) It follows from Theorem 2.11.
(3) $\Rightarrow$ (4) Since J, K are ideals of R so ideals of $S(R)$, hence $\{x^2 : x \in J\} \subseteq I$ or $\{x^2 : x \in K\} \subseteq I$.
(4) $\Rightarrow$ (1) Let $ab \in I$ for a,$b \in R$ with $a^2 \notin I$. Let $J = (a)$ and $K = (b)$. Then $JK \subseteq I$ and $\{x^2 : x \in J\} \nsubseteq I$, otherwise $a^2 \in I$. Hence $\{y^2 : y \in K\} \subseteq I$ implies $b^2 \in I$. Consequently, I is a 2-prime ideal of R.

□

Definition 2.13 *A semigroup S is said to be a 2-prime semigroup if every proper ideal of S is a 2-prime ideal of S.*

Example 2.3 Clearly every proper ideal of the semigroup $S = (\mathbb{N}, +)$ is 2-prime, hence S is a 2-prime semigroup.

Since every 2-prime ideal is semiprimary ($cf.$ Lemma 2.2), we have the following result by [9],

Theorem 2.14 *Let S be a 2-prime semigroup. Then the following statements about S are true*
(1) *Prime ideals of S are linearly ordered under set inclusion.*
(2) *Idempotents of S are linearly ordered under natural ordering.*

Theorem 2.15 *Let S be a regular semigroup. Then the following statements are equivalent.*
(1) *S is a 2-prime semigroup.*
(2) *Every ideal of S is prime.*
(3) *Every ideal of S is primary.*
(4) *Every ideal of S is semiprimary.*
(5) *Prime ideals of S are totally ordered.*
(6) *Idempotents of S are linearly ordered.*
(7) *All ideals of S are linearly ordered.*

Proof. (1) $\Rightarrow$ (2) Let I be a proper ideal of a 2-prime semigroup S and $ab \in I$ for some $a, b \in S$. Then $a^2 \in I$ or $b^2 \in I$. Hence $a \in I$ or $b \in I$, since S is regular. Hence every ideal of S is prime.
(2) $\Rightarrow$ (1) It is clear.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (2) follows from ([9], Theorem 2). □

Theorem 2.16 *Let S and T be multiplicative semigroup with 1 and 0 and $f : S \to T$ be a semigroup homomorphism.*
(i) If K is a 2-prime ideal of T, then $f^{-1}(K)$ is a 2-prime ideal of S.
(ii) Let I be a proper ideal of S such that $\{(x, y) \in kerf : x \neq y\} \subseteq I \times I$.
(a) If $f(I)$ is a 2-prime ideal of T, then I is a 2-prime ideal of S.
(b) If f is onto and I is a 2-prime ideal of S, then $f(I)$ is a 2-prime ideal of S.

Proof. *(i)* Let $ab \in f^{-1}(K)$ for some $a, b \in S$. Then $f(ab) = f(a)f(b) \in K$. Hence $f(a^2) \in K$ or $f(b^2) \in K$, since K is a 2-prime ideal of T. Therefore $a^2 \in f^{-1}(K)$ or $b^2 \in f^{-1}(K)$. Consequently, $f^{-1}(K)$ is a 2-prime ideal of S.
(ii)(a) Let $f(I)$ is a 2-prime ideal of T. Clearly $f^{-1}(f(I)) = I$, since $\{(x, y) \in kerf : x \neq y\} \subseteq I \times I$. Hence I is a 2-prime ideal of S by *(i)*.
(b) Let $xy \in f(I)$ for some $x, y \in T$. Since f is surjective, there exists a, $b \in S$ such that $f(a) = x$ and $f(b) = y$. Hence $f(a)f(b) \in f(I)$ implies $f(ab) \in f(I)$, since f is a homomorphism. Again $\{(x, y) \in kerf : x \neq y\} \subseteq I \times I$ implies $ab \in I$. Hence $a^2 \in I$ or $b^2 \in I$, since I is a 2-prime ideal of S. Therefore $x^2 = (f(a))^2 = f(a^2) \in f(I)$ or $y^2 = (f(b))^2 = f(b^2) \in f(I)$. Consequently, $f(I)$ is a 2-prime ideal of T. □

Corollary 2.17 *Let S and T be commutative multiplicative semigroup with* 0 *and* 1.
(1) *If $S \subseteq T$ be an extension of semigroups and K is a* 2*-prime ideal of T, then $K \cap S$ is a* 2*-prime ideal of S.*
(2) *Let $A \subseteq B$ be two ideals of S. Then B is a* 2*-prime ideal of S if and only if B/A is a* 2*-prime ideal of S/A.*

3 2-P semigroup

Definition 3.1 *A semigroup S is said to be* 2*-P semigroup if every* 2*-prime ideal of S is a prime ideal of S.*

Example 3.1 Clearly every commutative band is a 2-P semigroup.

The following is a characterization of a 2-P semigroup.

Theorem 3.2 *A semigroup S is* 2*-P semigroup if and only if prime ideals are idempotent and every* 2*-prime ideals of S is of the form A^2, where A is a prime ideal of S.*

Proof. Let S be a 2-P semigroup and $ab \in A^2 \subseteq A$ for some $a, b \in S$, where A is a prime ideal of S. Since A is prime either $a \in A$ or $b \in A$. Let $a \in A$. Then $a^2 \in A^2$ implies A^2 is a 2-prime ideal of S and hence prime ideal of S. Clearly $A^2 \subseteq A$. Let $x \in A$. Then $x^2 \in A^2$ implies $x \in A^2$, since A^2 is a prime ideal of S. Therefore $A = A^2$, as desired.
Conversely, Let I be a 2-prime ideal of a semigroup S. Then $I = A^2 = A$, for some prime ideal A of S. Consequently, S is a 2-P semigroup.
□

The following is a characterization of a prime ideals of a semigroup in terms of 2-prime ideals.

Theorem 3.3 *A proper ideal I of a semigroup S is prime if and only if it is both* 2*-prime and semiprime.*

Proof. Let I be a prime ideal of a semigroup S. Then clearly it is both semiprime and 2-prime ideals of S.
Conversely, Let I be a proper ideal of a semigroup which is both semiprime and 2-prime ideal and $ab \in I$ for some a,$b \in S$. Then either $a^2 \in I$ or $b^2 \in I$, since I is a 2-prime ideal of S, which implies either $a \in I$ or $b \in I$, since I is a semiprime ideal of S. Consequently, I is a prime ideal of S.
□

The following is an obvious consequence of the above theorem.

Corollary 3.4 *A semigroup S is a* 2*-P semigroup if and only if* 2*-prime ideals of S are semiprime.*

Lemma 3.5 *Let (S, M) be a semigroup with unity. Then for every prime ideal P of S, PM is a 2-prime ideal of S.*
In particular, PM is prime if and only if $PM = P$.

Proof. Let $ab \in PM \subseteq P$. Then either $a \in P$ or $b \in P$, since P is a prime ideal of S. Let $a \in P$ implies $a^2 \in PM$, since $P \subseteq M$. Hence PM is a 2-prime ideal of S.
Again, let PM is prime and $x \in P$. Then $x^2 \in PM$, since $P \subseteq M$. Hence $x \in PM$, since PM is a prime ideal of S. Therefore $P \subseteq PM$ and clearly $PM \subseteq P$. Hence $PM = P$.

□

Theorem 3.6 *A semigroup (S, M) with unity is 2-P if and only $IM = P$, where P is the minimal prime ideal over a 2-prime ideal I.*

Proof. Let I be a 2-prime ideal of a semigroup (S, M) with unity. Then clearly $IM = I$ (Lemma 3.5).
Conversely, let I be a 2-prime ideal of a semigroup (S, M) and P is the minimal prime ideal over I such that $IM = P$. Then $I \subseteq P = IM \subseteq I \cap M = I$ implies $I = P$, as desired.

□

4 Weakly 2-prime

Throughout this section all semigroups are commutative with zero and identity, in particular S will denote such a semigroup.

Definition 4.1 *A proper ideal I of a semigroup S is said to be a weakly 2-prime ideal of S if $0 \neq ab \in I$ for some a,$b \in S$ implies $a^2 \in I$ or $b^2 \in I$.*

The following lemmas are obvious, hence we omit the proof.

Lemma 4.2 1. *Every 2-prime ideal of S is a weakly 2-prime ideal of S.*
2. *Every weakly prime ideal of S is a weakly 2-prime ideal of S.*

The following is a characterization of a semigroup in which every proper ideal is weakly 2-prime.

Theorem 4.3 *Every proper ideal of a semigroup S is weakly 2-prime if and only if $(a^2) \subseteq (ab)$ or $(b^2) \subseteq (ab)$ or $ab = 0$, for any a,$b \in S$ such that $(ab) \neq S$.*

Proof. Let every proper ideal of semigroup S is weakly 2-prime and $a, b \in S$ such that $(ab) \neq S$. Then (ab) is weakly 2-prime ideal of S. If $ab \neq 0$, then $0 \neq ab \in (ab)$ implies $a^2 \in (ab)$ or $b^2 \in (ab)$. Consequently $(a^2) \subseteq (ab)$ or $(b^2) \subseteq (ab)$.
Conversely, Let I be a proper ideal of S such that $0 \neq ab \in I$ for some $a, b \in S$. Then $0 \neq ab \in (ab) \subseteq I$ implies $a^2 \in (a^2) \subseteq (ab) \subseteq I$ or $b^2 \in (b^2) \subseteq (ab) \subseteq I$, as desired.

□

Definition 4.4 *A proper ideal I of a semigroup S is said to be a strong weakly 2-prime ideal if $0 \neq AB \subseteq I$ implies $\{x^2 : x \in A\} \subseteq I$ or $\{y^2 : y \in B\} \subseteq I$ for any ideals A, B of S.*

Theorem 4.5 *Every proper ideal of a semigroup S is strong weakly 2-prime ideal if and only if $\{x^2 : x \in A\} \subseteq AB$ or $\{y^2 : y \in B\} \subseteq AB$ or $AB = 0$ for any proper ideals A, B of S.*

Proof. Let every proper ideal of S is strong weakly 2-prime ideal and A, B be any two proper ideals of S. Then $AB \neq S$ and AB is a strong weakly 2-prime ideal of S. If $0 \neq AB \subseteq AB$, then $\{x^2 : x \in A\} \subseteq AB$ or $\{y^2 : y \in B\} \subseteq AB$. Conversely, Let I be a proper ideals of S and $0 \neq AB \subseteq I$ for any two proper ideals A, B of S. Then by hypothesis $\{x^2 : x \in A\} \subseteq AB \subseteq I$ or $\{y^2 : y \in B\} \subseteq AB \subseteq I$. Consequently every proper ideal of S is strong weakly 2-prime.

□

Remarks

We end this note with the following natural questions could be of one interest:
(1) Is the concept of weakly 2-prime and strong weakly 2-prime are equivalent in a semigroup with zero and identity?
(2) In which class of semigroups every weakly 2-prime ideals are 2-prime?
(3) In which class of rings 2-prime ideals are maximal?

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