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A note on some fractional integral inequalities via generalized fractional integral

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Abstract In this paper, using a generalized fractional integral, we obtained new integral inequalities of the Hermite-Hadamard type for s-convex functions.

Keywords s-convex function · generalized fractional integral · Hermite-Hadamard integral inequality

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1 Introduction

One of the most fruitful notions in current Mathematics is that of the convex function, which has spread in various directions (the interested reader can consult [12], where a fairly complete overview of the generalizations and extensions of the convex function concept is presented). In our work we will use the notion of a s-convex function, given in the following definition.

An $s-convex$ function was introduced in Breckner's paper and a number of properties and connections with $s-convexity$ (see [3]).

Definition 1.1 *Let s be a real number, $s \in (0, 1]$. A function $\phi : [0, \infty) \to \mathbb{R}$ is said to be **s-convex in the first sense** if*

$$\phi(\alpha x + \beta y) \leq \alpha^s \phi(x) + \beta^s \phi(y). \tag{1.1}$$

for $x, y \in [0, \infty)$, $\alpha, \beta > 0$ and $\alpha^s + \beta^s = 1$

Definition 1.2 *Let s be a real number, $s \in (0, 1]$. A function $\phi : [0, \infty) \to [0, \infty)$ is said to be **s-convex in the second sense** if*

$$\phi(\alpha x + \beta y) \leq \alpha^s \phi(x) + \beta^s \phi(y). \tag{1.2}$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Of course, $s - convexity$ means just convexity when $s = 1$ for functions defined on $[0, \infty)$.

Some observations about this kind of functios:

– ϕ is s-convex if and only if

$$\phi(tx + (1-t)y) \le t^s \phi(x) + (1-t)^s \phi(y)$$

for each $x, y \in (0, \infty)$ and $t \in [0, 1]$.

– ϕ is convex if and only if ϕ is s-convex in any sense.

– If ϕ is s-convex in the second sense and $\phi(0) = 0$ then ϕ is s-convex in the first sense.

One of the most important inequalities, that has attracted many inequality experts in the last few decades, is the famous Hermite–Hadamard inequality for convex functions:

$$\phi\left(\frac{a_1 + a_2}{2}\right) \le \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(x)\, dx \le \frac{\phi(a_1) + \phi(a_2)}{2} \tag{1.3}$$

holds for any function ϕ convex on the interval $[a_1, a_2]$. This inequality was published by Hermite in 1883 and, independently, by Hadamard in 1893. It gives an estimation of the mean value of a convex function, and it is important to note that it also provides a refinement to the Jensen inequality. The interested reader is referred to [1,5–7,9–11,13] and references therein for more information and other extensions of the Hermite–Hadamard inequality.

In [4], we can see a variant of Hadamard's inequality wich holds for s-convex functions in the second sense.

Theorem 1.3 *Suppose that $\phi : [0, \infty) \to [0, \infty)$ is an s-convex functions in the second sense, where $s \in (0, 1)$, and let $a_1, a_2 \in [0, \infty)$, $a < b$. If $\phi \in L^1[a_1, a_2]$, then the following inequalities hold:*

$$2^{s-1} \phi\left(\frac{a_1 + a_2}{2}\right) \le \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(x)\, dx \le \frac{\phi(a_1) + \phi(a_2)}{s+1} \tag{1.4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.4).

All through the work we utilize the functions Γ (see [15,16,20,21]) and Γ_k (see [5]):

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t, \quad \Re(z) > 0, \tag{1.5}$$

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k}\, \mathrm{d}t, k > 0. \tag{1.6}$$

Clearly if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

To encourage comprehension of the subject, we present the definition of Riemann-Liouville fractional integral (with $0 \le a_1 < t < a_2 \le \infty$). The first is the classic Riemann-Liouville fractional integrals.

Definition 1.4 *Let $\phi \in L_1[a_1, a_2]$. Then the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ are defined by (right and left respectively):*

$$^{\alpha}I_{a_1^+}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_{a_1}^{x}(x-t)^{\alpha-1}\phi(t)\,dt, \quad x > a_1 \tag{1.7}$$

$$^{\alpha}I_{a_2^-}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{a_2}(t-x)^{\alpha-1}\phi(t)\,dt, \quad x < a_2. \tag{1.8}$$

We can define the fractional derivatives, of any order n, corresponding to the previous operator as follows.

Definition 1.5 *For $\phi(x) \in L(a_1, a_2)$, $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ and $n = [Re(\alpha)] + 1$, then the left and right sided Riemann-Liouville fractional derivatives are defined as:*

$$(D^{\alpha}_{a_1+}\phi)(x) = \left(\frac{d}{dx}\right)^{n}\left(I^{n-\alpha}_{a_1+}\phi\right)(x), \tag{1.9}$$

$$(D^{\alpha}_{a_2-}\phi)(x) = \left(-\frac{d}{dx}\right)^{n}\left(I^{n-\alpha}_{a_2-}\phi\right)(x). \tag{1.10}$$

Next we present the weighted integral operators, which will be the basis of our work.

Definition 1.6 *Let $\phi \in L_1[a_1, a_2]$ and let w be a continuous and positive function, $w : I \to \mathbb{R}$, with first and second order derivatives piecewise continuous on I and $w(0) = w(1) = 0$. Then the weighted fractional integrals are defined by (right and left respectively):*

$$^{w}I_{a_1+}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_{a_1}^{x} w''\left(\frac{x-t}{a_2-a_1}\right)\phi(t)\,dt, \quad x > a_1 \tag{1.11}$$

$$^{w}I_{a_2-}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{a_2} w''\left(\frac{t-x}{a_2-a_1}\right)\phi(t)\,dt, \quad x < a_2. \tag{1.12}$$

Remark 1.1 It is easy to notice that if in the previous definition, we put $w(t) = \frac{(a_2-a_1)^{\alpha-1}t^{\alpha+1}}{\alpha(\alpha+1)\Gamma(\alpha)}$, then we obtain the Riemann-Liouville fractional integral operators. Other known fractional integral operators can be derived, without difficulty, from the previous definition.

In the work [14], inequalities of the Hermite-Hadamard type are obtained, for functions with first derivative and their powers, in the s-convex class. In this paper, we obtain new variants of the inequality (1.3), within the framework of the weighted integral operators of the Definition 1.6 for s-convex functions, considering the second derivative.

2 Hermite-Hadamard type inequalities for fractional weighted integrals

Now let's present the following result, which we will use throughout the work.

Lemma 2.1 *Let ϕ a real function defined on some interval I, $I \subset \mathbb{R}$, twice differentiable on $\mathring{I}$, $a, b \in \mathring{I}$, $a < b$ and $w : I \to \mathbb{R}$, with first and second order derivatives piecewise continuous on I, $w(0) = w(1) = 0$. If $\phi'' \in L[a, b]$, then we have the following equality:*

$$\begin{aligned}
&\frac{-w(0)(\phi'(a_1)(x-a_1) + \phi'(a_2)(a_2-x)) + w'(0)(\phi(a_1) + \phi(a_2)) - 2w'(1)\phi(x))}{a_2-a_1}\\
&+w(1)\phi'(x) + \frac{\Gamma(\alpha)}{a_2 - a_1}\left({}^{w}I^{\alpha}_{a_1^+}\phi(x) + {}^{w}I^{\alpha}_{a_2^-}\phi(x)\right)\\
&= \frac{(x-a_1)^2}{a_2-a_1}\int_0^1 w(t)\phi''(tx+(1-t)a_1)\,dt + \frac{(a_2-x)^2}{a_2-a_1}\int_0^1 w(t)\phi''(tx+(1-t)a_2)\,dt
\end{aligned}$$

Proof. Integrating by parts twice, we get

$$\begin{aligned}
\int_0^1 w(t)\phi''(tx+(1-t)a_1)\,dt &= \left.\frac{w(t)\phi'(tx+(1-t)a_1)}{x-a_1}\right|_0^1 - \int_0^1 \frac{w'(t)\phi'(tx+(1-t)a_1)}{x-a_1}dt\\
&= \frac{w(1)\phi'(x) - w(0)\phi'(a_1)}{x-a_1} - \left.\frac{w'(t)\phi(tx+(1-t)a_1)}{(x-a_1)^2}\right|_0^1 + \int_0^1 \frac{w''(t)\phi(tx+(1-t)a_1)}{(x-a_1)^2}dt\\
&= \frac{w(1)\phi'(x) - w(0)\phi'(a_1)}{x-a_1} + \frac{w'(0)\phi(a_1)\, w'(1)\phi(x)}{(x-a_1)^2} + \int_0^1 \frac{w''(t)\phi(tx+(1-t)a_1)}{(x-a_1)^2}dt
\end{aligned}$$

Putting $z = tx + (1-t)a_1$, so $dz = (x - a_1)dt$, with this change of variables, we obtain

$$\begin{aligned}
\int_0^1 w(t)\phi''(tx+(1-t)a_1)\,dt &= \frac{w(1)\phi'(x) - w(0)\phi'(a_1)}{x-a_1} \qquad (2.1)\\
&+ \frac{w'(0)\phi(a_1) - w'(1)\phi(x)}{(x-a_1)^2} + \frac{\Gamma(\alpha)\,{}^{w}I^{\alpha}_{a_1^+}\phi(x)}{(x-a_1)^2}
\end{aligned}$$

Analogously,

$$\int_0^1 w(t)\phi''(tx+(1-t)a_2)\,dt = \frac{w(1)\phi'(x) - w(0)\phi'(a_2)}{a_2 - x} + \frac{w'(0)\phi(a_2) - w'(1)\phi(x)}{(a_2 - x)^2} + \frac{\Gamma(\alpha)\,{}^w I^{\alpha}_{a_2^-}\phi(x)}{(a_2 - x)^2} \tag{2.2}$$

Multiplying both sides of (2.1) and (2.2) by $\frac{(x-a_1)^2}{a_2-a_1}$ and $\frac{(a_2-x)^2}{a_2-a_1}$, respectively, and adding the resulting identities we obtain

$$\frac{(x-a_1)^2}{a_2-a_1}\int_0^1 w(t)\phi''(tx+(1-t)a_1)\,dt + \frac{(a_2-x)^2}{a_2-a_1}\int_0^1 w(t)\phi''(tx+(1-t)a_2)\,dt$$
$$= \frac{x-a_1}{a_2-a_1}\big(w(1)\phi'(x) - w(0)\phi'(a_1)\big) + \frac{a_2-x}{a_2-a_1}\big(w(1)\phi'(x) - w(0)\phi'(a_2)\big)$$
$$+\frac{w'(0)\phi(a_1)-w'(1)\phi(x)}{a_2-a_1} + \frac{w'(0)\phi(a_2)-w'(1)\phi(x)}{a_2-a_1} + \frac{\Gamma(\alpha)}{a_2-a_1}$$
$$\left({}^w I^{\alpha}_{a_1^+}\phi(x) + {}^w I^{\alpha}_{a_2^-}\phi(x)\right)$$

□

Remark 2.1 If we take $w(t) = t(1-t^{\alpha})$, we obtain the Lemma 1 of [2] working with $\phi''(ta_2+(1-t)a_1)$ instead of $\phi''(tx+(1-t)a_1)$ and with $\phi''(ta_1+(1-t)a_2)$ instead of $\phi''(tx+(1-t)a_2)$.

For simplicity let us denote

$$\frac{-w(0)(\phi'(a_1)(x-a_1) + \phi'(a_2)(a_2-x)) + w'(0)(\phi(a_1)+\phi(a_2)) - 2w'(1)\phi(x))}{a_2-a_1}$$
$$+w(1)\phi'(x) + \frac{\Gamma(\alpha)}{a_2-a_1}\left({}^w I^{\alpha}_{a_1^+}\phi(x) + {}^w I^{\alpha}_{a_2^-}\phi(x)\right) = L(HH)$$

We want to estimate the above functional, as a way of refining the well-known Hermite-Hadamard Inequality, for the case of the weighted integrals of the Definition 1.6.

Theorem 2.2 *Let ϕ a real function defined on some interval I, $I \subset \mathbb{R}$, twice differentiable on $\mathring{I}$, the interior of I, where $a_1, a_2 \in I$ with $a_1 < a_2$ and $w : I \to \mathbb{R}$, with first and second order derivatives piecewise continuous on I, $w(0) = w(1) = 0$. If $|\phi''|$ is a s-convex function on $[a_1, a_2]$, then for all $x \in [a_1, a_2]$ and $\alpha > 0$ we have:*

$$|L(HH)| \le \left(\frac{(x-a_1)^2}{a_2-a_1} + \frac{(a_2-x)^2}{a_2-a_1}\right)|\phi''(x)|\int_0^1 w(t)t^s dt$$
$$+\left(\frac{(x-a_1)^2}{a_2-a_1}|\phi''(a_1)| + \frac{(a_2-x)^2}{a_2-a_1}|\phi''(a_2)|\right)\int_0^1 w(t)(1-t)^s dt$$

Proof. From Lemma 2.1, property of the modulus and using the hypothesis that $|\phi''|$ is s-convex, we have:

$$\begin{aligned}
|L(HH)| &\leq \frac{(x-a_1)^2}{a_2-a_1}\int_0^1 w(t)|\phi''(tx+(1-t)a_1)|dt \\
&+ \frac{(a_2-x)^2}{a_2-a_1}\int_0^1 w(t)|\phi''(tx+(1-t)a_2)|dt \\
&\leq \frac{(x-a_1)^2}{a_2-a_1}\int_0^1 w(t)\left(t^s|\phi''(x)|+(1-t)^s|\phi''(a_1)|\right)dt \\
&+ \frac{(a_2-x)^2}{a_2-a_1}\int_0^1 w(t)\left(t^s|\phi''(x)|+(1-t)^s|\phi''(a_2)|\right)dt \\
&= \frac{(x-a_1)^2}{a_2-a_1}\left(\int_0^1 w(t)t^s|\phi''(x)|dt+\int_0^1 w(t)(1-t)^s|\phi''(a_1)|dt\right) \\
&+ \frac{(a_2-x)^2}{a_2-a_1}\left(\int_0^1 w(t)t^s|\phi''(x)|+\int_0^1 w(t)(1-t)^s|\phi''(a_2)|dt\right) \\
&= \left(\frac{(x-a_1)^2}{a_2-a_1}+\frac{(a_2-x)^2}{a_2-a_1}\right)|\phi''(x)|\int_0^1 w(t)t^s dt \\
&+ \left(\frac{(x-a_1)^2}{a_2-a_1}|\phi''(a_1)|+\frac{(a_2-x)^2}{a_2-a_1}|\phi''(a_2)|\right)\int_0^1 w(t)(1-t)^s dt
\end{aligned}$$

□

Remark 2.2 If we take $w(t) = t(1-t^\alpha)$, from this result we obtain the Theorem 2.2 of [17] where they work with $\phi''(ta_2+(1-t)a_1)$ and with $\phi''(ta_1+(1-t)a_2)$ instead of $\phi''(tx+(1-t)a_1)$ and $\phi''(tx+(1-t)a_2)$, respectively.

Theorem 2.3 *Let ϕ a real function defined on some interval I, $I \subset \mathbb{R}$, twice differentiable on $\mathring{I}$, the interior of I, where $a_1, a_2 \in I$ with $a_1 < a_2$ and $w : I \to \mathbb{R}$, with first and second order derivatives piecewise continuous on I, $w(0) = w(1) = 0$. If $|\phi''|^q$ is a s-convex function on $[a_1, a_2]$, $\frac{1}{p}+\frac{1}{q} = 1$, $x \in [a_1, a_2]$ then the following inequality for fractional integrals holds:*

$$\begin{aligned}
|L(HH)| &\leq \frac{(x-a_1)^2}{a_2-a_1}\left(\int_0^1 (w(t))^p\,dt\right)^{\frac{1}{p}}\left(\frac{|\phi''(x)|^q+|\phi''(a_1)|^q}{s+1}\right)^{\frac{1}{q}} \\
&+ \frac{(a_2-x)^2}{a_2-a_1}\left(\int_0^1 (w(t))^p dt\right)^{\frac{1}{p}}\left(\frac{|\phi''(x)|^q+|\phi''(a_2)|^q}{s+1}\right)^{\frac{1}{q}} \qquad (2.3)
\end{aligned}$$

Proof. From Lemma 2.1, property of the modulus and using the Hölder inequality, we have:

$$
\begin{aligned}
|L(HH)| &\leq \frac{(x-a_1)^2}{a_2-a_1}\int_0^1 w(t)|\phi''(tx+(1-t)a_1)|dt \\
&+ \frac{(a_2-x)^2}{a_2-a_1}\int_0^1 w(t)|\phi''(tx+(1-t)a_2)|dt \\
&\leq \frac{(x-a_1)^2}{a_2-a_1}\left(\int_0^1 (w(t))^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |\phi''(tx+(1-t)a_1)|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{(a_2-x)^2}{a_2-a_1}\left(\int_0^1 (w(t))^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\phi''(tx+(1-t)a_2)|^q dt\right)^{\frac{1}{q}}
\end{aligned}
$$

Since $|\phi''|^q$ is s-convex on $[a_1, a_2]$, we get

$$
\begin{aligned}
\int_0^1 |\phi''(tx+(1-t)a_1)|^q dt &\leq \int_0^1 \left(t^s|\phi''(x)|^q + (1-t)^s|\phi''(a_1)|^q\right) dt \\
&= |\phi''(x)|^q\int_0^1 t^s dt + |\phi''(a_1)|^q\int_0^1 (1-t)^s dt \\
&= \frac{1}{s+1}|\phi''(x)|^q + \frac{1}{s+1}|\phi''(a_1)|^q \\
&= \frac{|\phi''(x)|^q + |\phi''(a_1)|^q}{s+1} \\
\int_0^1 |\phi''(tx+(1-t)a_2)|^q dt &\leq \int_0^1 \left(t^s|\phi''(x)|^q + (1-t)^s|\phi''(a_2)|^q\right) dt \\
&= |\phi''(x)|^q\int_0^1 t^s dt + |\phi''(a_2)|^q\int_0^1 (1-t)^s dt \\
&= \frac{1}{s+1}|\phi''(x)|^q + \frac{1}{s+1}|\phi''(a_2)|^q \\
&= \frac{|\phi''(x)|^q + |\phi''(a_2)|^q}{s+1}
\end{aligned}
$$

Hence we have:

$$
\begin{aligned}
|L(HH)| &\leq \frac{(x-a_1)^2}{a_2-a_1}\left(\int_0^1 (w(t))^p\,dt\right)^{\frac{1}{p}}\left(\frac{|\phi''(x)|^q + |\phi''(a_1)|^q}{s+1}\right)^{\frac{1}{q}} \\
&+ \frac{(a_2-x)^2}{a_2-a_1}\left(\int_0^1 (w(t))^p dt\right)^{\frac{1}{p}}\left(\frac{|\phi''(x)|^q + |\phi''(a_2)|^q}{s+1}\right)^{\frac{1}{q}}
\end{aligned}
$$

□

Remark 2.3 Under the same conditions as the previous Remarks, from this Theorem we obtain Theorem 2.4 of [17].

Theorem 2.4 *Let ϕ a real function defined on some interval I, $I \subset \mathbb{R}$, twice differentiable on $\mathring{I}$, the interior of I, where $a_1, a_2 \in I$ with $a_1 < a_2$ and $w : I \to \mathbb{R}$, with first and second order derivatives piecewise continuous on I, $w(0) = w(1) = 0$. If $|\phi''|^q$ is a s-convex function on $[a_1, a_2]$, $q \geq 1$, $x \in [a_1, a_2]$ then the following inequality for fractional integrals holds:*

$$\begin{aligned}|L(HH)| \leq &\frac{(x-a_1)^2}{a_2-a_1}\left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}\left(|\phi''(x)|^q\int_0^1 w(t)t^s dt+|\phi''(a_1)|^q\int_0^1 w(t)(1-t)^s dt\right)^{\frac{1}{q}}\\ &+\frac{(a_2-x)^2}{a_2-a_1}\left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}\left(|\phi''(x)|^q\int_0^1 w(t)t^s dt+|\phi''(a_2)|^q\int_0^1 w(t)(1-t)^s dt\right)^{\frac{1}{q}}\end{aligned}$$

Proof. From Lemma 2.1, property of the modulus and using the power-mean inequality we have

$$\begin{aligned}|L(HH)| \leq &\frac{(x-a_1)^2}{a_2-a_1}\int_0^1 w(t)|\phi''(tx+(1-t)a_1)|dt\\ &+\frac{(a_2-x)^2}{a_2-a_1}\int_0^1 w(t)|\phi''(tx+(1-t)a_2)|dt\\ \leq &\frac{(x-a_1)^2}{a_2-a_1}\left\{\left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 w(t)|\phi''(tx+(1-t)a_1)|^q dt\right)^{\frac{1}{q}}\right\}\\ &+\frac{(a_2-x)^2}{a_2-a_1}\left\{\left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 w(t)|\phi''(tx+(1-t)a_2)|^q dt\right)^{\frac{1}{q}}\right\}\end{aligned}$$

Since $|\phi''|^q$ is s-convex on $[a_1, a_2]$, we get:

$$\begin{aligned}\int_0^1 w(t)|\phi''(tx+(1-t)a_1)|^q dt &\leq \int_0^1 w(t)\left(t^s|\phi''(x)|^q+(1-t)^s|\phi''(a_1)|^q\right)dt\\ &= |\phi''(x)|^q\int_0^1 w(t)t^s dt+|\phi''(a_1)|^q\int_0^1 w(t)(1-t)^s dt\\ \int_0^1 w(t)|\phi''(tx+(1-t)a_2)|^q dt &\leq \int_0^1 w(t)\left(t^s|\phi''(x)|^q+(1-t)^s|\phi''(a_2)|^q\right)dt\\ &= |\phi''(x)|^q\int_0^1 w(t)t^s dt+|\phi''(a_2)|^q\int_0^1 w(t)(1-t)^s dt\end{aligned}$$

Then we have:

$$\begin{aligned}|L(HH)| \leq &\frac{(x-a_1)^2}{a_2-a_1}\left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}\left(|\phi''(x)|^q\int_0^1 w(t)t^s dt+|\phi''(a_1)|^q\int_0^1 w(t)(1-t)^s dt\right)^{\frac{1}{q}}\\ &+\frac{(a_2-x)^2}{a_2-a_1}\left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}\left(|\phi''(x)|^q\int_0^1 w(t)t^s dt+|\phi''(a_2)|^q\int_0^1 w(t)(1-t)^s dt\right)^{\frac{1}{q}}\end{aligned}$$

□

Remark 2.4 With the same conditions of the previous Remarks, it is not difficult to obtain Theorem 2.6 of [17] from this result.

Theorem 2.5 *Let ϕ a real function defined on some interval I, $I \subset \mathbb{R}$, twice differentiable on $\mathring{I}$, the interior of I, where $a_1, a_2 \in I$ with $a_1 < a_2$ and $w : I \to \mathbb{R}$, with first and second order derivatives piecewise continuous on I, $w(0) = w(1) = 0$. If $|\phi''|^q$ is a s-concave function on $[a_1, a_2]$, $q > 1$, $x \in [a_1, a_2]$ then the following inequality for fractional integrals holds:*

$$|L(HH)| \leq \frac{2^{\frac{s-1}{q}}}{a_2 - a_1} \left(\int_0^1 (w(t))^p dt \right)^{\frac{1}{p}} \left((x - a_1)^2 \left| \phi'' \left(\frac{x + a_1}{2} \right) \right| + (a_2 - x)^2 \left| \phi'' \left(\frac{x + a_2}{2} \right) \right| \right)$$

Proof. From Lemma 2.1, property of the modulus and using Hölder inequality, we have:

$$\begin{aligned}
|L(HH)| &\leq \frac{(x - a_1)^2}{a_2 - a_1} \int_0^1 w(t) |\phi''(tx + (1 - t)a_1)| dt \\
&+ \frac{(a_2 - x)^2}{a_2 - a_1} \int_0^1 w(t) |\phi''(tx + (1 - t)a_2)| dt \\
&\leq \frac{(x - a_1)^2}{a_2 - a_1} \left(\int_0^1 (w(t))^p \, dt \right)^{\frac{1}{p}} \left(\int_0^1 |\phi''(tx + (1 - t)a_1)|^q dt \right)^{\frac{1}{q}} \\
&+ \frac{(a_2 - x)^2}{a_2 - a_1} \left(\int_0^1 (w(t))^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\phi''(tx + (1 - t)a_2)|^q dt \right)^{\frac{1}{q}}
\end{aligned}$$

Since $|\phi''|^q$ is s-concave on $[a_1, a_2]$, using the inequality (1.4), we have

$$\int_0^1 |\phi''(tx + (1 - t)a_1)|^q dt \leq 2^{s-1} \left| \phi'' \left(\frac{x + a_1}{2} \right) \right|^q$$
$$\int_0^1 |\phi''(tx + (1 - t)a_2)|^q dt \leq 2^{s-1} \left| \phi'' \left(\frac{x + a_2}{2} \right) \right|^q$$

Then we have

$$\begin{aligned}
|L(HH)| &\leq \frac{(x - a_1)^2}{a_2 - a_1} \left(\int_0^1 (w(t))^p \, dt \right)^{\frac{1}{p}} 2^{\frac{s-1}{q}} \left| \phi'' \left(\frac{x + a_1}{2} \right) \right| \\
&+ \frac{(a_2 - x)^2}{a_2 - a_1} \left(\int_0^1 (w(t))^p dt \right)^{\frac{1}{p}} 2^{\frac{s-1}{q}} \left| \phi'' \left(\frac{x + a_2}{2} \right) \right|
\end{aligned}$$

□

Remark 2.5 It can be verified that this Theorem contains, under the same previous conditions, Theorem 2.8 of [17].

3 Conclusions

In this work we have obtained new inequalities of the Hermite-Hadamard type for functions with second derivative s-convex, in the framework of heavy fractional integrals. We have seen that some results reported in the literature can be obtained as particular cases of ours.

To point out the strengths and scope of the method used, the consideration of a certain function $w(t)$ subject to certain conditions, we will point out that with this same idea and taking $w(0) = 0$ and $w(1) \neq 0$ we can obtain Lemma 2.1 (from our Lemma 2.1), Theorem 2.2 (from Theorem 2.5) and Theorem 2.4 (from Theorem 2.4) from [8]; as well as Lemma 1, Theorem 4, Theorem 7 and Theorem 10 of [18], Lemma 2.1, Theorem 2.3, Theorem 2.5 and Theorem 2.4, respectively.

On the other hand, these results can be extended to derivatives of higher orders, following our ideas. It is enough to consider a more general $w(t)$ function, that is, such that $w(0) \neq w(1) \neq 0$ thus, the work [19] can be generalized.

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