Some $L_k$-biconservative Lorentzian hypersurfaces in Minkowski 5-space

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Abstract. A Lorentzian hypersurface $M^4_1$ of Minkowski 5-space (i.e. $E_5^1$), defined by an isometric immersion $x : M^4_1 \rightarrow E_5^1$, is said to be $L_k$-biconservative if the tangent component of $L^2_kx$ is identically zero, where $L_k$ is the $k$th extension of Laplace operator $\Delta = L_0$. The operator $L_k$ is the linearized operator arisen from the first variation of $(k + 1)$th mean curvature vector field on $M^4_1$. This subject is motivated by a well-known conjecture of Bang-Yen Chen which says that the condition $\Delta^2x = 0$ implies the minimality for submanifolds of Euclidean spaces. In this paper, we study $L_k$-biconservative Lorentzian hypersurfaces of $E_5^1$ in four different cases based on the matrix representation forms of the shape operator. We show that if such a hypersurface has constant mean curvature and at most two distinct principal curvatures, then its $(k + 1)$th mean curvature is constant.

Keywords. Lorentzian hypersurface · biharmonic · $L_k$-biconservative

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1 Introduction

Let $x : M^n \rightarrow \mathbb{E}^{n+1}$ be an isometric immersion of a hypersurface $M^n$ in the Euclidean $(n + 1)$-space with Laplace operator $\Delta$, shape operator $S$ associated to a chosen unit normal vector field $n$ and ordinary mean curvature function $H$. By definition, $M^n$ is said to be harmonic (biharmonic) if $x$ satisfies equation $\Delta x = 0$ ($\Delta^2 x = 0$, respectively). It is called biconservative if the tangent component of $\Delta^2 x$ vanishes identically. A famous equality due to Beltrami says that $\Delta x = -nHn$. Then, the conditions $\Delta x = 0$ and $\Delta^2 x = 0$ are equivalent to $H \equiv 0$ and $\Delta (Hn) = 0$, respectively. Clearly, every minimal hypersurface is biharmonic but not vice versa. In this context, a well-known conjecture of Bang-Yen Chen (in 1987) says that each biharmonic submanifold of a Euclidean space is minimal. In 1992, Dimitrić proved that every biharmonic hypersurface in a Euclidean space with at most two distinct principal curvatures is minimal [5]. Also, Hasanis and
VLACHOS [6] and DEFEVER [4] have proved the Chen conjecture on hypersurfaces of Euclidean 4-space. In 2007, biharmonic Lorentz hypersurfaces of Minkowski 4-space have been studied [3]. In 2013, Akutagawa and Maeta have affirmed the conjecture on some submanifolds of Euclidean spaces [1].

In this paper, we study $L_k$-biconservative Lorentzian hypersurfaces of pseudo-Euclidean 5-space $\mathbb{E}_5^5$. The operator $L_k$ is an extension of the Laplace operator $L_0 = \Delta$, which stands for the linearized operator of the first variation of $(k+1)$th mean curvature function (see, for instance, [2, 7, 11, 12]). $L_k$ is defined (on $M^4_1$) by $L_k(f) = \text{tr}(P_k \circ \nabla^2 f)$ for any $f \in C^\infty(M^4_1)$, where $P_k$ denotes the $k$th Newton transformation associated to the second fundamental form of the hypersurface and $\nabla^2 f$ is the hessian of $f$. The hypersurface $x: M^4_1 \to \mathbb{E}_5^5$ is said to be $L_k$-harmonic ($L_k$-biharmonic) if $x$ satisfies condition $L_kx = 0$ ($L_k^2x = 0$, respectively). It is called $L_k$-biconservative if the tangent component of $L_k^2x$ vanishes identically. We show that, every $L_k$-biconservative Lorentzian hypersurface of $\mathbb{E}_5^5$, with constant $k$th mean curvature and some additional conditions on principal curvatures, has constant $(k+1)$th mean curvature.

Here, we present the organization of paper. In section 2 we remember some notations and definitions which will be needed in paper. In section 3, we illustrate some examples of standard $L_k$-biconservative Lorentzian hypersurfaces of $\mathbb{E}_5^5$ for nonnegative integers $k$ less than 4. In section 4, we study the $L_k$-biconservative Lorentzian hypersurfaces with constant mean curvature and at most two distinct principal curvatures, separately according to four possible types $I$, $II$, $III$ and $IV$. We show that, if a hypersurface $M^4_1$ of type $I$ has constant mean curvature and at most two distinct principal curvatures, then it has constant mean curvature (see theorems 4.1, 4.2 and 4.3). In Theorem 4.4, we study $L_k$-biconservative Lorentzian hypersurfaces of type $II$, and we show that if such a hypersurface has at most two distinct principal curvatures, then it’s $(k+1)$th mean curvature is constant. Theorems 4.5 and 4.6 state the same result on $L_k$-biconservative Lorentzian hypersurfaces $M^4_1$ of types $III$ and $IV$.

2 Preliminaries

First, we recall some preliminaries from [2, 7, 8] and [9]-[13]. The 5-dimensional pseudo-Euclidean space, $\mathbb{E}_5^5$, is the Euclidean 5-space endowed with the Lorentz product defined by

$$\langle x, y \rangle := -x_1y_1 + \sum_{i=2}^{5} x_iy_i,$$

for every two vectors $x, y \in \mathbb{R}^5$. Throughout the paper, we study on every Lorentzian hypersurface of $\mathbb{E}_5^5$, defined by an isometric immersion $x: M^4_1 \to \mathbb{E}_5^5$. The symbols $\nabla$ and $\bar{\nabla}$ stand for the Levi-Civita connection on $M^4_1$ and $\mathbb{E}_5^5$, respectively. For every tangent vector fields $X$ and $Y$ on $M$, the Gauss formula is given by $\bar{\nabla}_X Y = \nabla_X Y + \langle SX, Y \rangle n$, for every $X, Y \in \chi(M)$, where, $n$ is a (locally) unit normal vector field on $M$ and $S$ is the shape
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operator of $M$ relative to $n$. For each non-zero vector $X \in \mathbb{E}_5^1$, the real value $<X, X>$ may be a negative, zero or positive number and then, the vector $X$ is said to be time-like, light-like or space-like, respectively. According to whether the induced metric on a nondegenerate hypersurface $M_4^\nu$ of index $\nu$ in $\mathbb{E}_5^1$ is positive definite or indefinite, $M_4^\nu$ is called Riemannian (when $\nu = 0$) or Lorentzian (when $\nu = 1$), and therefore every normal vector on $M_4^\nu$ is time-like or light-like, respectively.

Definition 2.1 For a 4-dimensional Lorentzian vector space $V_4^1$, a basis $\mathcal{B} := \{e_1, \cdots, e_4\}$ is said to be orthnormal if it satisfies $<e_i, e_j> = \epsilon_i \delta^j_i$ for $i, j = 1, \cdots, 4$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, 3, 4$. As usual, $\delta^j_i$ stands for the Kronecker function. $\mathcal{B}$ is called pseudo-orthnormal if it satisfies $<e_1, e_1> = <e_2, e_2> = 0$, $<e_1, e_2> = -1$ and $<e_i, e_j> = \delta^j_i$, for $i = 1, 2, 3, 4$ and $j = 3, 4$.

The shape operator of $M_4^1$ in $\mathbb{E}_5^1$, as a self-adjoint linear map on the tangent bundle of $M_4^1$, locally can be put into one of four possible canonical matrix forms, usually denoted by $I$, $II$, $III$ and $IV$ (see for instance [8, 13]). In cases $I$ and $IV$, with respect to an orthonormal basis of the tangent space of $M_4^1$, the matrix representation of the induced metric on $M_4^1$ is

$$G_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the shape operator of $M_4^1$ can be put into matrix forms

$$B_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad \text{and} \quad B_4 = \begin{pmatrix} \kappa & \lambda & 0 & 0 \\ -\lambda & \kappa & 0 & 0 \\ 0 & 0 & \eta_1 & 0 \\ 0 & 0 & 0 & \eta_2 \end{pmatrix}, \quad (\lambda \neq 0)$$

respectively. For cases $II$ and $III$, using a pseudo-orthnormal basis of the tangent space of $M_4^1$, the induced metric on which has matrix form

$$G_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the shape operator of $M_4^1$ can be put into matrix forms

$$B_2 = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \\ 1 & 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \\ -1 & 0 & \kappa & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

respectively. In case $IV$, the matrix $B_4$ has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.
Remark 2.1 In two cases II and III, one can substitute the pseudo-orthonormal basis $B := \{e_1, e_2, e_3, e_4\}$ by a new orthonormal basis $\tilde{B} := \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ where $\tilde{e}_1 := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e}_2 := \frac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrix representations $\tilde{B}_2$ and $\tilde{B}_3$ (instead of $B_2$ and $B_3$, respectively) as

$$
\tilde{B}_2 = \begin{pmatrix}
\kappa + \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & \kappa - \frac{1}{2} & 0 & 0 \\
0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & \lambda_2
\end{pmatrix}
$$

and

$$
\tilde{B}_3 = \begin{pmatrix}
\kappa & 0 & -\sqrt{\kappa} & 0 \\
0 & \kappa & -\sqrt{\kappa} & 0 \\
-\sqrt{\kappa} & \sqrt{\kappa} & \kappa & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix}
$$

After this changes, to unify the notations we denote the orthonormal basis by $B$ in all cases.

**Notation:** According to four possible matrix representations of the shape operator of $M^4_1$, we define its principal curvatures, denoted by unified notations $\kappa_i$ for $i = 1, \cdots, 4$, as follow.

In case I, we put $\kappa_i := \lambda_i$ for $i = 1, \cdots, 4$, where $\lambda_i$’s are the eigenvalues of $B_1$.

In case II, we take $\kappa_i := \kappa$ for $i = 1, 2$, and $\kappa_i := \lambda_{i-2}$ for $i = 3, 4$.

In case III, the shape operator $S$ has matrix representation $B_3$ and we take $\kappa_i := \kappa$ for $i = 1, 2, 3$, and $\kappa_4 := \lambda$.

Finally, in case IV, $S$ is of form $B_4$ and we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_3 := \eta_{i-2}$, for $i = 3, 4$.

The characteristic polynomial of $S$ on $M^4_1$ is of the form $Q(t) = \prod_{i=1}^4 (t - \kappa_i) = \sum_{j=0}^4 (-1)^j s_j t^{4-j}$, where, $s_0 := 1$, $s_i := \sum_{1 \leq j_1 < \cdots < j_i \leq 4} \kappa_{j_1} \cdots \kappa_{j_i}$ for $i = 1, 2, 3, 4$.

For $j = 1, \cdots, 4$, the $j$th mean curvature $H_j$ of $M^4_1$ is defined by $H_j = \frac{1}{(j-1)} s_j$. When $H_j$ is identically null, $M^4_1$ is said to be $(j-1)$-minimal.

**Definition 2.2** (i) A timelike hypersurface $x : M^4_1 \rightarrow \mathbb{E}^5_1$, with diagonalizable shape operator, is said to be *isoparametric* if all of its principal curvatures are constant.

(ii) A timelike hypersurface $x : M^4_1 \rightarrow \mathbb{E}^5_1$, with non-diagonalizable shape operator, is said to be *isoparametric* if the minimal polynomial of its shape operator is constant.

**Remark 2.2** Here we remember Theorem 4.10 from [8], which assures us that there is no isoparametric timelike hypersurface of $\mathbb{E}^5_1$ with complex principal curvatures.

The well-known Newton transformations $P_j : \chi(M) \rightarrow \chi(M)$ on $M^4_1$, is defined by

$$
P_0 = I, \quad P_j = s_j I - S \circ P_{j-1}, \quad (j = 1, 2, 3, 4),
$$

where, $I$ is the identity map. Using its explicit formula, $P_j = \sum_{i=0}^j (-1)^i s_{j-i} S^i$ (where $S^0 = I$), and the well-known Cayley-Hamilton theorem (which says
that every operator is annihilated by its characteristic polynomial) we get $P_4 = 0$. Clearly, $P_j$ is self-adjoint and it commutes with $S$ (see [2,11]).

Also, we will use the following notation:

$$
\mu_{i;k} = \sum_{1 \leq j_1 < \cdots < j_k \leq 4; i \neq j_k} \kappa_{j_1} \cdots \kappa_{j_k}, \quad (i = 1, 2, 3, 4; 1 \leq k \leq 3).
$$

(2.2)

Corresponding to four possible matrix forms of $S$, the Newton transformation $P_j$ has different forms. In case I, we have $P_j = \text{diag}[\mu_1, \ldots, \mu_4]$, for $j = 1, 2, 3$.

In case II, for $j = 1, 2, 3$ we have

$$
P_j(p) = \begin{pmatrix}
\frac{\mu_{1,2,j} + (\kappa - \frac{1}{2})\mu_{1,2,j-1}}{2\mu_{1,2,j-1}} & -\frac{\mu_{1,2,j-1}}{2\mu_{1,2,j-1}} & -\frac{\mu_{1,2,j-1}}{2\mu_{1,2,j-1}} & \frac{\mu_{1,2,j}}{\mu_{1,2,j-1}} \\
\frac{\mu_{1,2,j}}{\mu_{1,2,j-1}} & \frac{\mu_{1,2,j}}{\mu_{1,2,j-1}} & \frac{\mu_{1,2,j}}{\mu_{1,2,j-1}} & \mu_{3,j} \\
\end{pmatrix}.
$$

In case III, $P_j(p)$ is of form

$$
\begin{pmatrix}
u_3 & -\nu_{j-1} & -\nu_{j-1} & \nu_{j-1} \\
u_{j-1} & \nu_{j-1} & \nu_{j-1} & \nu_{j-1} \\
u_{j-1} & \nu_{j-1} & \nu_{j-1} & \nu_{j-1} \\
u_{j-1} & \nu_{j-1} & \nu_{j-1} & \nu_{j-1} \\
\end{pmatrix},
$$

where $u_3 = u_2 = 0$, $u_1 = \lambda$, $u_0 = 1$ and $u_{-1} = u_{-2} = 0$. In case IV, we have

$$
P_j = \begin{pmatrix}
\frac{\kappa\mu_{1,2,j-1} + \mu_{1,2,j}}{\lambda\mu_{1,2,j-1}} & -\lambda\mu_{1,2,j-1} \\
\lambda\mu_{1,2,j-1} & \kappa\mu_{1,2,j-1} + \mu_{1,2,j} & \mu_{3,j} \\
\end{pmatrix}.
$$

(2.3)

Fortunately, in all cases we have the following important identities, similar to those in [2,11].

(i) $s_{j+1} = \kappa_i \mu_{i;j} + \mu_{i;j+1}$, $1 \leq i \leq 4$; $1 \leq j \leq 3$

(ii) $\mu_{i;j+1} = \kappa_i \mu_{i,j} + \mu_{i;j+1}$, $1 \leq i, l \leq 4, i \neq l$

(2.4)

$$
(i) \quad \text{tr}(P_j) = (n - j)s_{j+1} = c_j H_j,
(ii) \quad \text{tr}(P_j \circ S) = (n - (n - j - 1))s_{j+1} = (j+1)s_{j+1} = c_j H_{j+1},
$$

where $c_j = (n - j)\binom{n}{j} = (j+1)\binom{n}{j+1}$.

$$
\text{tr}S^2 = 4(4H_1^2 - 3H_2), \quad \text{tr}(P_j \circ S^2) = \binom{n}{j+1}[nH_j H_{j+1} - (n - j - 1)H_{j+2}].
$$

(2.5)

The linearized operator of the $(j+1)$th mean curvature of $M$, $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula

$$
L_j(f) := \text{tr}(P_j \circ \nabla^2 f),
$$

5
where, \( <\nabla^2 f(X), Y> = <\nabla_X \nabla f, Y> \) for every \( X, Y \in \chi(M) \).

Associated to the orthonormal frame \( \{e_1, \cdots, e_4\} \) of tangent space on \( M^4_1 \), for \( k = 0, \cdots, 3 \), \( L_k(f) \) has an explicit expression as

\[
L_k(f) = \sum_{i=1}^{4} \epsilon_i \mu_{i,k}(\epsilon_i e_i f - \nabla e_i e_i f).
\]  

(2.6)

For a Lorentzian hypersurface \( x : M^4_1 \to \mathbb{E}^5_1 \), with a chosen (local) unit normal vector field \( n \), for an arbitrary vector \( a \in \mathbb{E}^5_1 \) we use the decomposition \( a = a^T + a^N \) where \( a^T \in TM \) is the tangential component of \( a \), \( a^N \perp TM \), and we have the following formulae from [2,11].

\( \nabla <x,a> = a^T \), \( \nabla <n,a> = -Sa^T \).  

(2.7)

Then, we get

\[
\begin{align*}
(i) \quad & L_1 n = -6(\nabla H_2 + 2(2H_1 H_2 - H_3)n) \\
(ii) \quad & L_2 n = -4(\nabla H_3 + (4H_1 H_3 - H_4)n) \\
(iii) \quad & L_3 n = -\nabla H_4 - 4H_1 H_4 n \\
(iv) \quad & L_1^2 x = 24[P_2 \nabla H_2 - 9H_2 \nabla H_2] + 12[L_1 H_2 - 12H_2(2H_1 H_2 - H_3)]n \\
(v) \quad & L_2^2 x = 24[P_3 \nabla H_3 - 6H_3 \nabla H_3] + 12[L_2 H_3 - 4H_3(4H_1 H_3 - H_4)]n \\
(vi) \quad & L_3^2 x = -12H_4 \nabla H_4 + 4(L_3 H_4 - 4H_1 H_3^2)n \\
\end{align*}
\]

(2.9)

Assume that a hypersurface \( x : M^4_1 \to \mathbb{E}^5_1 \) satisfies the condition \( L_k^2 x = 0 \), then it is said to be \( L_k \)-biharmonic. By equalities (2.9)(iv,v,vi), \( x : M^4_1 \to \mathbb{E}^5_1 \) is \( L_k \)-biharmonic (for \( k = 1,2,3 \)) if and only if it satisfies conditions:

\[
\begin{align*}
(i) \quad & L_k H_{k+1} = \binom{4}{k+1} H_{k+1}(4H_1 H_{k+1} - (4 - k - 1)H_{k+2}), \\
(ii) \quad & P_{k+1} \nabla H_{k+1} = 3(4 - k)H_{k+1} \nabla H_{k+1}. \\
\end{align*}
\]

(2.10)

Also, \( x : M^4_1 \to \mathbb{E}^5_1 \) is said to be \( L_k \)-biconservative, if its \( (k + 1) \)th mean curvature satisfies the condition (2.10)(ii).

The well-known structure equations on \( \mathbb{E}^5_1 \) are given by \( d\omega_i = \sum_{j=1}^{5} \omega_{ij} \wedge \omega_j \),

\( \omega_{ij} + \omega_{ji} = 0 \) and \( d\omega_{ij} = \sum_{l=1}^{5} \omega_{il} \wedge \omega_{lj} \). Restricted on \( M \), we have \( \omega_5 = 0 \).
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and then, $0 = d\omega_5 = \sum_{i=1}^4 \omega_{5,i} \wedge \omega_i$. So, by Cartan’s lemma, there exist functions $h_{ij}$ such that $\omega_{5,i} = \sum_{j=1}^4 h_{ij} \omega_j$ and $h_{ij} = h_{ji}$, which gives the second fundamental form of $M$, as $B = \sum_{i,j} h_{ij} \omega_i \omega_j e_5$. The mean curvature $H$ is given by $H = \frac{1}{4} \sum_{i=1}^4 h_{ii}$. Therefore, we obtain the structure equations on $M$ as follow.

\[ d\omega_i = \sum_{j=1}^4 \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \quad (2.11) \]

\[ d\omega_{ij} = \sum_{k=1}^4 \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^4 R_{ijkl} \omega_k \wedge \omega_l, \quad (2.12) \]

for $i, j = 1, 2, 3$, and the Gauss equations $R_{ijkl} = (h_{ik} h_{jl} - h_{il} h_{jk})$, where $R_{ijkl}$ denotes the components of the Riemannian curvature tensor of $M$. Denoting the covariant derivative of $h_{ij}$ by $h_{ijk}$, we have

\[ dh_{ij} = \sum_{k=1}^4 h_{ijk} \omega_k + \sum_{k=1}^4 h_{kj} \omega_{ik} + \sum_{k=1}^4 h_{ik} \omega_{jk}, \quad (2.13) \]

and by the Codazzi equation we get

\[ h_{ijk} = h_{ikj}. \quad (2.14) \]

3 Examples

In this section, we see some examples of $L_k$-biconservative Lorentzian hypersurfaces in $E_5^5$.

Example 3.1 Consider the hypersurface \( \{ y = (y_1, ..., y_5) \in E_5^5 | -y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2 \} \) representing $S^5_1(r) \times E^1 \subset E^5_1$ (for $r > 0$) with the Gauss map $n(y) = -\frac{1}{r}(y_1, y_2, y_3, y_4, 0)$. Clearly, it has two distinct constant principal curvatures $\kappa_1 = \kappa_2 = \frac{1}{r}$ and $\kappa_3 = \kappa_4 = 0$ and constant higher order mean curvatures $H_1 = \frac{3}{4}r^{-1}$, $H_2 = \frac{1}{2}r^{-2}$, $H_3 = \frac{1}{4}r^{-3}$ and $H_4 = 0$.

Example 3.2 Consider \( \{ y = (y_1, ..., y_5) \in E_5^5 | -y_1^2 + y_2^2 + y_3^2 = r^2 \} \) representing the hypersurface $S^2_1(r) \times E^2 \subset E^5_1$ (for $r > 0$), with the Gauss map $n(y) = -\frac{1}{r}(y_1, y_2, y_3, 0, 0)$. It has two distinct principal curvatures $\kappa_1 = \kappa_2 = \frac{1}{r}$ and $\kappa_3 = \kappa_4 = 0$ and constant higher order mean curvatures $H_1 = \frac{1}{2}r^{-1}$, $H_2 = \frac{1}{6}r^{-2}$ and $H_3 = H_4 = 0$. 

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Example 3.3 Consider the submanifold \( \{ y = (y_1, \ldots, y_5) \in \mathbb{L}^5 | -y_1^2 + y_2^2 = r^2 \} \) representing the hypersurface \( S_1^4(r) \times \mathbb{E}^3 \subset \mathbb{E}_1^5 \) (for \( r > 0 \)), with the Gauss map \( n(y) = -\frac{1}{r}(y_1, y_2, 0, 0, 0) \). Clearly, it has two distinct principal curvatures \( \kappa_1 = \frac{1}{r}, \kappa_2 = \kappa_3 = \kappa_4 = 0 \), and constant higher order mean curvatures \( H_1 = \frac{1}{4} r^{-1} \), and \( H_2 = H_3 = H_4 = 0 \).

Example 3.4 Consider the submanifold \( \{ y = (y_1, \ldots, y_5) \in \mathbb{E}_1^5[y_3^2 + y_4^2 + y_5^2 = r^2] \} \) denoting the hypersurface \( \mathbb{E}_1^3 \times S^1(r) \subset \mathbb{L}^5 \) (for \( r > 0 \)) with the Gauss map \( n(y) = -\frac{1}{r}(0, 0, 0, y_4, y_5) \). It has two distinct principal curvatures \( \kappa_1 = \kappa_2 = \kappa_3 = 0 \) and \( \kappa_4 = \frac{1}{r} \) and constant higher order mean curvatures \( H_1 = \frac{1}{4r} \), and \( H_k = 0 \) for \( k = 2, 3, 4 \).

Example 3.5 Consider \( \{ y = (y_1, \ldots, y_5) \in \mathbb{E}_1^5[y_3^2 + y_4^2 + y_5^2 = r^2] \} \) defining the product \( \mathbb{E}_1^3 \times S^2(r) \subset \mathbb{E}_1^5 \) (for \( r > 0 \)) with the Gauss map \( n(y) = -\frac{1}{r}(0, 0, y_3, y_4, y_5) \). It has two distinct principal curvatures \( \kappa_1 = \kappa_2 = 0 \) and \( \kappa_3 = \kappa_4 = \frac{1}{r} \) and constant higher order mean curvatures \( H_1 = \frac{1}{2r} \), \( H_2 = \frac{1}{6} r^{-2} \) and \( H_k = 0 \) for \( k = 3, 4 \).

4 Results

In this section, we prove six theorems on \( L_k \)-biconservative connected orientable Lorentzian hypersurface in the Minkowski 5-space with constant ordinary mean curvature. Theorems 4.1, 4.2 and 4.3 state the case that the shape operator on hypersurface is diagonalizable. Theorems 4.4, 4.5 and 4.6 are related to the cases that the shape operator on hypersurface is of type \( II \), \( III \) and \( IV \), respectively.

Theorem 4.1 Let \( x : M_1^4 \to \mathbb{E}_1^5 \) be isometric immersion of a \( L_k \)-biconservative orientable Lorentzian hypersurface (for a positive integer number \( k < 4 \)) in the Lorentz-Minkowski 5-space \( \mathbb{E}_1^5 \) having diagonalizable shape operator. If \( M_1^4 \) has a principal curvature of multiplicity four, then it has constant \((k+1)\)th mean curvature.

Proof. Considering the open subset \( \mathcal{U} \) of \( M \) as \( \mathcal{U} := \{ p \in M_1^4 : \nabla h_{k+1}^2(p) \neq 0 \} \), we prove that \( \mathcal{U} \) is empty. Assuming \( \mathcal{U} \neq \emptyset \), we consider \( \{ e_1, e_2, e_3, e_4 \} \) as a local orthonormal frame of principal directions of \( A \) on \( \mathcal{U} \) such that for \( i = 1, 2, 3, 4 \) we have \( Se_i = \lambda e_i \) and then, for \( j = 1, 2, 3 \), we have

\[
\mu_{i,j} = \binom{3}{j} \lambda^2, \quad H_j = \lambda j. \tag{4.1}
\]

By condition (2.10)(ii), we have to consider two different cases based on the choice of the value of \( k \).

Consider the cases \( k = 1, 2 \). Applying condition (2.10)(ii) on the both sides of the polar decomposition \( \nabla H_{k+1} = \sum_{i=1}^4 e_i < \nabla H_{k+1}, e_i > e_i \), we
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get $\epsilon_i < \nabla H_{k+1}, e_i > (\mu_{i,k+1} - 3(4 - k)H_{k+1}) = 0$ on $\mathcal{U}$ for $i = 1, 2, 3, 4$. If $< \nabla H_2, e_i > \neq 0$ on $\mathcal{U}$ for some $i$, then we get $\mu_{i,k+1} = 3(4 - k)H_{k+1}$, which, using equalities (4.1), gives $\lambda^{k+1} = 0$ and then $H_{k+1} = 0$ on $\mathcal{U}$, which is a contradiction. Hence $\mathcal{U}$ is empty and $H_{k+1}$ is constant on $M$.

If $k = 3$, then we have $\nabla H_2^3 = 0$ which, gives $H_4$ is constant on $M$.

\[ \square \]

**Theorem 4.2** Let $x : M^4 \rightarrow E^5_1$ be isometric immersion of a $L_k$-biconservative orientable Lorentzian hypersurface (for a positive integer number $k < 4$) in the Lorentz-Minkowski 5-space $E^5_1$ having diagonalizable shape operator with constant ordinary mean curvature. If $M^4$ has exactly two principal curvatures $\lambda$ and $\eta$ of multiplicities 3 and 1 (respectively), then it’s $(k+1)$th mean curvature is constant.

**Proof.** Taking the open set $\mathcal{U} := \{ p \in M^4 : \nabla H^2_{k+1}(p) \neq 0 \}$, we prove that it is empty. Assuming $\mathcal{U} \neq \emptyset$, we consider $\{e_1, e_2, e_3, e_4\}$ a local orthonormal frame of principal directions of the shape operator $A$ on $\mathcal{U}$ such that $S e_i = \lambda e_i$ for $i = 1, 2, 3$ and $S e_4 = \eta e_4$. Therefore, we obtain

$$\mu_{1,2} = \mu_{2,2} = \mu_{3,2} = \lambda^2 + 2\lambda \eta, \mu_{4,2} = 3\lambda^2, \mu_{1,3} = \mu_{2,3} = \mu_{3,3} = \mu_{4,3} = 3\lambda^3,$$

$$4H_1 = 3\lambda + \eta, 6H_2 = 3\lambda(\lambda + \eta), 4H_3 = \lambda^2(\lambda + 3\eta), H_4 = \lambda^3 \eta.$$

(4.2)

We have to consider three different cases based on the choice of the value of $k$.

**Case 1:** $k = 1$. In this case, by conditions (2.10)(ii) we have $P_2 \nabla H_2 = 9H_2 \nabla H_2$, which, using the polar decomposition $\nabla H_2 = \sum_{i=1}^4 e_i < \nabla H_2, e_i > e_i$, gives $e_i < \nabla H_2, e_i > (\mu_{ij} - 9H_2) = 0$ on $\mathcal{U}$ for $i = 1, 2, 3, 4$. Hence, if for some $i$ we have $< \nabla H_2, e_i > \neq 0$ on $\mathcal{U}$, then we get

$$\mu_{i,2} = 9H_2.$$

(4.3)

By assumption we have $\nabla H_2 \neq 0$ on $\mathcal{U}$, which gives one or both of the following states.

**State 1.** $< \nabla H_2, e_i > \neq 0, \text{ for some } i \in \{1, 2, 3\}$. Using equalities (4.2), from (4.3) we obtain $\lambda(5\eta + 7\lambda) = 0$. If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\lambda = \frac{5}{7}H_1, \eta = -\frac{7}{5}H_1$ and $H_2 = -\frac{3}{5}H_1^2$.

**State 2.** $< \nabla H_2, e_i > = 0, \text{ for } i \in \{1, 2, 3\}$ and $< \nabla H_2, e_4 > \neq 0$. By equalities (4.2) and (4.3), we obtain $3\lambda^2 = \frac{9}{2}(\lambda \eta + \lambda^2)$, which gives $\lambda(3\eta + \lambda) = 0$. If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\lambda = \frac{3}{2}H_1, \eta = -\frac{1}{2}H_1$ and $H_2 = \frac{3}{4}H_1^2$.

Hence, $H_2$ is constant on $M$. **Case 2:** $k = 2$. In this case we have $P_3 \nabla H_3 = 6H_3 \nabla H_3$, which, using the polar decomposition $\nabla H_3 = \sum_{i=1}^4 e_i < \nabla H_3, e_i > e_i$, gives $e_i < \nabla H_3, e_i > (\mu_{i,3} - 6H_3) = 0$ on $\mathcal{U}$ for $i = 1, 2, 3, 4$. Hence, if for some $i$ we have $< \nabla H_{k+1}, e_i > \neq 0$ on $\mathcal{U}$, then we get

$$\mu_{i,3} = 6H_3.$$

(4.4)
Now, by definition we have $\nabla H_3 \neq 0$ on $U$, which gives one or both of the following states.

**State 1.** $\langle \nabla H_3, e_i \rangle \neq 0$, for some $i \in \{1, 2, 3\}$. Using equalities (4.2), from (4.4) we obtain $\lambda^2(7\eta + 3\lambda) = 0$. If $\lambda = 0$ then $H_3 = 0$. Otherwise, we get $\lambda = -\frac{14}{7}H_1$, $\eta = -\frac{2}{7}H_1$ and $H_3 = -\frac{196}{7^2}H_1^3$.

**State 2.** $\langle \nabla H_{k+1}, e_i \rangle = 0$, for $i \in \{1, 2, 3\}$ and $\langle \nabla H_{k+1}, e_4 \rangle \neq 0$. By equalities (4.2) and (4.4), we obtain $3\lambda^2 = \frac{9}{2}(\lambda\eta + \lambda^2)$, which gives $\lambda^2(3\eta - \lambda) = 0$. If $\lambda = 0$ then $H_3 = 0$. Otherwise, we have $\lambda = -\frac{6}{5}H_1$, $\eta = \frac{2}{5}H_1$ and $H_3 = \frac{108}{125}H_1^3$.

Hence, $H_3$ is constant on $M$.

**Case 3:** $k = 3$. In this case by (2.10)(ii) we have $\nabla H_4^2 = 0$, which, means that $H_4$ is constant on $M$.

\[\Box\]

**Theorem 4.3** Let $x : M_1^4 \to \mathbb{E}_5^5$ be isometric immersion of a $L_k$-biconservative orientable Lorentzian hypersurface (for a positive integer number $k < 4$) in the Lorentz-Minkowski 5-space $\mathbb{E}_5^5$ having constant ordinary mean curvature and diagonalizable shape operator. If $M_1^4$ has two principal curvatures $\lambda$ and $\eta$ both of multiplicity 2, then it is $(k+1)$th mean curvature is constant.

**Proof.** Taking the open subset $U$ of $M$ as $U := \{p \in M_1^4 : \nabla H_{k+1}^2(p) \neq 0\}$, we prove that $U$ is empty. Assuming $U \neq \emptyset$, we use the local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ of principal directions of $A$ on $U$ such that $Se_i = \lambda e_i$ for $i = 1, 2$, and $Se_i = \eta e_i$ for $i = 3, 4$. Therefore, we obtain

\[
\begin{align*}
\mu_{1,2} &= \mu_{2,2} = \eta^2 + 2\lambda \eta, \\
\mu_{3,2} &= \mu_{4,2} = \lambda^2 + 2\lambda \eta, \\
\mu_{1,3} &= \mu_{2,3} = \lambda \eta^2, \\
\mu_{3,3} &= \mu_{4,3} = \lambda^2 \eta, \\
2H_3 &= \lambda^2 \eta + \lambda \eta^2, \\
H_4 &= \lambda^2 \eta^2.
\end{align*}
\]

(4.5)

We consider three distinct cases based on the possible values of $k$.

**Case 1:** $k = 1$. By condition (2.10)(ii), we have $P_2(\nabla H_2) = 9H_2 \nabla H_2$, which, using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i < \nabla H_2, e_i > e_i$, gives $e_i < \nabla H_2, e_i > (\mu_{i,2} - 9H_2) = 0$ on $U$ for $i = 1, 2, 3, 4$. Hence, for some $i$ such that $< \nabla H_2, e_i > \neq 0$ on $U$, we get

\[
\mu_{i,2} = 9H_2.
\]

(4.6)

By definition, we have $\nabla H_2 \neq 0$ on $U$, which gives one or both of the following states.

**State 1.** $\langle \nabla H_2, e_i \rangle \neq 0$, for some $i \in \{1, 2\}$. Using equalities (4.5), from (4.6) we obtain $3\lambda^2 + \eta^2 + 8\lambda \eta = 0$, which gives $\eta = \frac{2c_0}{x_1 + c_0}H_1$, $H_2 = \frac{2}{3}(1 + \frac{2c_0}{(1 + c_0)^2})H^2$.

Then, we have

\[
\lambda = \frac{2}{1 + c_0}H_1, \\
\eta = \frac{2c_0}{1 + c_0}H_1, \\
H_2 = \frac{2}{3}(1 + \frac{2c_0}{(1 + c_0)^2})H^2.
\]

(4.7)
State 2. \(<\nabla H_2, e_i> = 0\) for \(i \in \{1, 2\}\), and \(<\nabla H_2, e_j> \neq 0\) for some \(j \in \{3, 4\}\). By equalities (4.5) and (4.6), we obtain \(3\lambda^2 = \frac{9}{2}(\lambda \mu + \lambda^2)\), which gives \(\lambda = c_0 \eta\) where \(c_0 := -4 \pm \sqrt{3}\). Then, we have:

\[
\lambda = \frac{2c_0}{1 + c_0} H_1, \quad \eta = \frac{2}{1 + c_0} H_1, \quad H_2 = \frac{2}{3}(1 + \frac{2c_0}{(1 + c_0)^2}) H^2. \tag{4.8}
\]

Hence, \(H_2\) is constant on \(M\).

**Case 2:** \(k = 2\). By condition (2.10)(ii), we have \(P_3(\nabla H_2) = 6H_3 \nabla H_3\), which, using the polar decomposition \(\nabla H_3 = \sum_{i=1}^{4} \epsilon_i <\nabla H_3, e_i> e_i\), gives \(\epsilon_i <\nabla H_3, e_i> (\mu_{i,3} - 9H_3) = 0\) on \(U\) for \(i = 1, 2, 3, 4\). Hence, for some \(i\) such that \(<\nabla H_3, e_i> \neq 0\) on \(U\), we get

\[
\mu_{i,3} = 6H_3. \tag{4.9}
\]

By definition, we have \(\nabla H_3 \neq 0\) on \(U\), which gives one or both of the following states.

**State 1.** \(<\nabla H_3, e_i> \neq 0, \text{ for some } i \in \{1, 2\}\). Using equalities (4.5), from (4.9) we obtain \(\eta \lambda (3\lambda + 2\eta) = 0\). If \(\eta \lambda = 0\), then \(H_3 = 0\) and it remains nothing to prove. If \(\eta \lambda \neq 0\), then we get \(\eta = -\frac{3}{2}\lambda\), which gives

\[
\lambda = -4H_1, \quad \eta = 6H_1, \quad H_3 = -24H_1^3. \tag{4.10}
\]

**State 2.** \(<\nabla H_3, e_i> = 0\) for \(i \in \{1, 2\}\), and \(<\nabla H_3, e_j> \neq 0\) for some \(j \in \{3, 4\}\). By equalities (4.5) and (4.9), we obtain \(\eta \lambda (3\eta + 2\lambda) = 0\). If \(\lambda \eta = 0\), then \(H_3 = 0\) and it remains nothing to prove. If \(\lambda \eta \neq 0\), then we get \(\lambda = -\frac{3}{2}\eta\), which gives

\[
\lambda = 6H_1, \quad \eta = -4H_1, \quad H_3 = -24H_1^3. \tag{4.11}
\]

Hence, \(H_3\) is constant on \(M\).

**Case 3:** \(k = 3\). In this case by (2.10)(ii) we have \(\nabla H_4^2 = 0\), which, means that \(H_4\) is constant on \(M\).

\(\square\)

**Theorem 4.4** Let \(x : M^4_1 \to \mathbb{E}^5_1\) be isometric immersion of a \(L_k\)-biconservative orientable Lorentzian hypersurface (for a positive integer number \(k < 4\)) in the Lorentz-Minkowski 5-space \(\mathbb{E}^5_1\) having shape operator of type \(II\). If \(M^4_1\) has at most two distinct principal curvatures and constant ordinary mean curvature, then it’s \((k + 1)\)th mean curvature is constant.

**Proof.** We suppose that, a Lorentzian hypersurface \(x : M^4_1 \to \mathbb{E}^5_1\) be \(L_k\)-biharmonic with shape operator of type \(II\), which has constant \(k\)th mean curvature and at most two distinct principal curvatures. First, we show that \(H_{k+1}\) is constant on \(M^4_1\). Defining the open subset \(U = \{p \in M : \nabla H^2(p) \neq 0\}\), we assume \(U \neq \emptyset\). By the assumption, with respect to a suitable (local) orthonormal tangent frame \(\{e_1, \cdots, e_4\}\) on \(M\), the shape operator \(S\) has the matrix form \(B_2\), such that \(Se_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2\),
$Se_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Se_3 = \lambda e_3$ and $Se_4 = \lambda e_4$, and then, we have
$P_2e_1 = (\lambda^2 + 2(\kappa - \frac{1}{2})\lambda)e_1 + 2\lambda e_2$, $P_2e_2 = -\lambda e_1 + (\lambda^2 + 2(\kappa + \frac{1}{2})\lambda)e_2$, and
$P_2e_3 = (\kappa^2 + 2\kappa\lambda)e_3$ and $P_2e_4 = (\kappa^2 + 2\kappa\lambda)e_4$.

Using the polar decomposition $\nabla H_{k+1} = \sum_{i=1}^{4} e_i (H_{k+1}) e_i$, from condition (2.10) we have $P_{k+1}\nabla H_{k+1} = 3(4 - k)H_{k+1} \nabla H_{k+1}$ for $k = 1, 2, 3$, which gives

\begin{align*}
(i) & \ [\mu_{1,2;k+1} + (\kappa - \frac{1}{2})\mu_{1,2;k} - 3(4 - k)H_{k+1}]e_1 e_1 (H_{k+1}) = \frac{1}{2} \mu_{1,2;k} e_2 e_2 (H_{k+1}) \\
(ii) & \ [\mu_{1,2;k+1} + (\kappa + \frac{1}{2})\mu_{1,2;k} - 3(4 - k)H_{k+1}]e_2 e_2 (H_{k+1}) = -\frac{1}{2} \mu_{1,2;k} e_1 e_1 (H_{k+1}), \\
(iii) & \ [\mu_{3;k+1} - 3(4 - k)H_{k+1}]e_3 e_3 (H_{k+1}) = 0, \\
(iv) & \ [\mu_{4;k+1} - 3(4 - k)H_{k+1}]e_4 e_4 (H_{k+1}) = 0.
\end{align*}

(4.12)

Now, we prove some simple claims.

Claim: $e_1 (H_{k+1}) = e_2 (H_{k+1}) = e_3 (H_{k+1}) = e_4 (H_{k+1}) = 0$.

If $e_1 (H_{k+1}) \neq 0$, then by dividing both sides of equalities (4.12)(i) and (4.12)(ii) by $e_1 (H_{k+1})$ we get

\begin{align*}
(i) & \quad \mu_{1,2;k+1} + (\kappa - \frac{1}{2})\mu_{1,2;k} - 3(4 - k)H_{k+1} = \frac{1}{2} \mu_{1,2;k} u, \\
(ii) & \quad [\mu_{1,2;k+1} + (\kappa + \frac{1}{2})\mu_{1,2;k} - 3(4 - k)H_{k+1}]u = -\frac{1}{2} \mu_{1,2;k},
\end{align*}

(4.13)

where $u := \frac{e_2 e_2 (H_{k+1})}{e_1 (H_{k+1})}$. By substituting (i) in (ii), we obtain $\mu_{1,2;k}(1 + u)^2 = 0$, then $\mu_{1,2;k} = 0$ or $u = -1$. If $\mu_{1,2;k} = 0$, then, we have $\binom{2}{k}\lambda^k = 0$ which gives $\binom{k}{k+1}\lambda^{k+1} = 0$, so $H_{k+1} = 0$. If $\mu_{1,2;k} \neq 0$, we get $u = -1$, which gives $\mu_{1,2;k+1} + \kappa \mu_{1,2;k} = 3(4 - k)H_{k+1}$. We continue the proof separately in three distinct cases based on the possible values of $k$.

Case 1: $k = 1$. In this case we get $3\kappa^2 + \lambda^2 + 8\kappa\lambda = 0$. Since $2H_1 = \kappa + \lambda$ is assumed to be constant on $M$, by substituting which in the last equality, we get $\lambda^2 - H_1 \lambda - 3H_1^2 = 0$, which means that, $\lambda$, $\kappa$ and the $j$th mean curvatures (for $j = 2, 3, 4$) are constant on $M$. So, we got a contradiction and therefore, the first part of the claim is proved.

Case 2: $k = 2$. In this case we get $\kappa\lambda(3\kappa + 2\lambda) = 0$. Since $2H_1 = \kappa + \lambda$ is assumed to be constant on $M$, by substituting which in the last equality, we get $\kappa(4H_1 + \kappa)(2H_1 - \kappa) = 0$, which means that, $\lambda$, $\kappa$ and the $j$th mean curvatures (for $j = 2, 3, 4$) are constant on $M$. So, we got a contradiction and therefore, the first part of the claim is proved.

Case 3: $k = 3$. In this case, since $P_4 = 0$, we get $\nabla H_4^2 = 0$. So $H_4$ is constant on $M$. Hence, the first part of the claim is affirmed.
By a similar manner, each of assumptions $e_i(H_{k+1}) \neq 0$ for $i = 2, 3, 4$, gives the equality $\mu_{1,2,k+1} + \kappa \mu_{1,2,k} = 3(4 - k)H_{k+1}$, which implies a contradiction. So, the claim is confirmed. By assumption $H_1$ is assumed to be constant and by the first stage, it is proved that $H_{k+1}$ is constant.

\[\blacksquare\]

**Theorem 4.5** Let $x : M^4_1 \to E^5_1$ be isometric immersion of a $L_k$-biconservative orientable Lorentzian hypersurface (for a positive integer number $k < 4$) in the Lorentz-Minkowski 5-space $E^5_1$ having shape operator of type $\text{III}$. If $M^4_1$ has constant ordinary mean curvature, then its $(k+1)$th mean curvature is constant.

**Proof.** Similar to proof of Theorem 4.4, we assume that $H_{k+1}$ is non-constant and considering the open subset $\mathcal{U}$, we prove that $\mathcal{U} = \emptyset$. Similarly, we get the conditions

\begin{align*}
(i) \quad & (u_{k+1} + 2\kappa u_k + (\kappa^2 - \frac{1}{2})u_{k-1} - 3(4 - k)H_{k+1})e_1e_1(H_{k+1}) - \frac{1}{2}u_{k-1}e_2e_2(H_{k+1}) \\
& = \frac{\sqrt{2}}{2}(u_k + \kappa u_{k-1})e_3e_3(H_{k+1}), \\
(ii) \quad & \frac{1}{2}u_{k-1}e_1e_1(H_{k+1}) + (u_{k+1} + 2\kappa u_k + (\kappa^2 - \frac{1}{2})u_{k-1} - 3(4 - k)H_{k+1})e_2e_2(H_{k+1}) \\
& = -\frac{\sqrt{2}}{2}(u_k + \kappa u_{k-1})e_3e_3(H_{k+1}), \\
(iii) \quad & \frac{\sqrt{2}}{2}(u_k + \kappa u_{k-1})(e_1e_1(H_{k+1}) + e_2e_2(H_{k+1})) \\
& = -(u_{k+1} + 2\kappa u_k + \kappa^2 u_{k-1} - 3(4 - k)H_{k+1})e_3e_3(H_{k+1}), \\
(iv) \quad & (\mu_{4,k+1} - 3(4 - k)H_{k+1})e_4e_4(H_{k+1}) = 0. \\
& \quad \text{(4.14)}
\end{align*}

Now, we prove that $H_{k+1}$ is constant.

Claim: $e_1(H_{k+1}) = e_2(H_{k+1}) = e_3(H_{k+1}) = e_4(H_{k+1}) = 0$.

If $e_1(H_{k+1}) \neq 0$, then by dividing both sides of equalities (4.14)$(i, ii, iii)$ by $e_1e_1(H_{k+1})$, and using the identity $2H_2 = \kappa^2 + \kappa \lambda$ in type $\text{III}$, putting
\( w_1 := \frac{e_2 e_2(H_{k+1})}{e_1 e_1(H_{k+1})} \) and \( w_2 := \frac{e_3 e_3(H_{k+1})}{e_1 e_1(H_{k+1})} \), we get

(i) \( u_{k+1} + 2\kappa u_k + (\kappa^2 - \frac{1}{2})u_{k-1} - 3(4-k)H_{k+1} - \frac{1}{2}u_{k-1}w_1 \)

\[ = \frac{\sqrt{2}}{2}(u_k + \kappa u_{k-1})w_2, \]

(ii) \( \frac{1}{2}u_{k-1} + (u_{k+1} + 2\kappa u_k + (\kappa^2 + \frac{1}{2})u_{k-1} - 3(4-k)H_{k+1})w_1 \)

\[ = -\frac{\sqrt{2}}{2}(u_k + \kappa u_{k-1})w_2, \]

(iii) \( \frac{\sqrt{2}}{2}(u_k + \kappa u_{k-1})(1 + w_1) + (u_{k+1} + 2\kappa u_k + \kappa^2 u_{k-1} - 3(4-k)H_{k+1})w_2 = 0, \)

(4.15)

which, by comparing (i) and (ii), gives

\[ [u_{k+1} + 2\kappa u_k + \kappa^2 u_{k-1} - 3(4-k)H_{k+1}(1 + w_1) = 0. \]  

(4.16)

We continue the proof of the first part of the claim (i.e., \( e_1(H_{k+1}) = 0 \)) separately in three distinct cases based on the possible values of \( k \).

**Case 1:** \( k = 1 \). In this case, from (4.16), we get \( \kappa(5\lambda + 7\kappa)(1 + w_1) = 0 \).

If \( \kappa = 0 \), then \( H_2 = 0 \). Assuming \( \kappa \neq 0 \), we get \( w_1 = -1 \) or \( \lambda = -\frac{7}{5}\kappa \). If \( w_1 \neq -1 \), then \( \lambda = -\frac{7}{5}\kappa \) and then by (4.16) and (4.15)(iii) we obtain \( w_1 = -1 \), which is a contradiction. Hence we have \( w_1 = -1 \), which by (4.15)(i, iii) implies \( w_2 = 0 \) and then \( \lambda = -\frac{7}{5}\kappa \). Since \( 4H_1 = 3\kappa + \lambda \) is assumed to be constant on \( M \), then \( 4H_1 = \frac{8}{7}\kappa \) is constant, which means that, \( \lambda, \kappa \) and the \( j \)th mean curvatures (for \( j = 2, 3, 4 \)) are constant on \( M \). So, we got a contradiction and therefore, the first part of the claim is proved.

**Case 2:** \( k = 2 \). In this case, from (4.16), we get \( \kappa^2(7\lambda + 3\kappa)(1 + w_1) = 0 \).

If \( \kappa = 0 \), then \( H_2 = 0 \). Assuming \( \kappa \neq 0 \), we get \( w_1 = -1 \) or \( \lambda = -\frac{3}{5}\kappa \). If \( w_1 \neq -1 \), then \( \lambda = -\frac{3}{5}\kappa \) and then by (4.16) and (4.15)(iii) we obtain \( w_1 = -1 \), which is a contradiction. Hence we have \( w_1 = -1 \), which by (4.15)(i, iii) implies \( w_2 = 0 \) and then \( \lambda = -\frac{3}{5}\kappa \). Since \( 4H_1 = 3\kappa + \lambda \) is assumed to be constant on \( M \), then \( 4H_1 = \frac{18}{5}\kappa \) is constant, which means that, \( \lambda, \kappa \) and the \( j \)th mean curvatures (for \( j = 2, 3, 4 \)) are constant on \( M \). So, we got a contradiction and therefore, the first part of the claim is proved.

**Case 3:** \( k = 3 \). In this case, since \( P_4 = 0 \), we get \( \nabla H_4^2 = 0 \). So \( H_4 \) is constant on \( M \). Hence, the first part of the claim is affirmed.

By a similar manner, each of assumptions \( e_i(H_{k+1}) \neq 0 \) for \( i = 2, 3, 4 \), gives a contradiction. So, the claim is confirmed. By assumption \( H_1 \) is assumed to be constant and by the first stage, it is proved that \( H_{k+1} \) is constant.

\( \square \)
Theorem 4.6 Let $x: M^4 \to \mathbb{E}^5$ be isometric immersion of a $L_k$-biconservative orientable Lorentzian hypersurface (for a positive integer number $k < 4$) in the Lorentz-Minkowski 5-space $\mathbb{E}^5$ having shape operator of type IV. If $M^4$ has at most two distinct non-zero principal curvatures, then it’s $(k+1)$th mean curvature is constant and also, $M^4$ is isoparametric.

Proof. Suppose that, $H_{k+1}$ be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_{k+1}(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By assumption, the shape operator $S$ of $M^4$ is of type IV with at most two distinct nonzero eigenvalue functions, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \ldots, e_4\}$ on $M$, the shape operator $S$ has the matrix form $B_4$, such that $Se_1 = -\lambda e_2$, $Se_2 = \lambda e_1$, $Se_3 = Se_4 = 0$ and then, we have $P_2e_1 = P_2e_2 = 0$, $P_2e_3 = \lambda^2 e_3$ and $P_2e_4 = \lambda^2 e_4$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2)e_i$, from condition (2.10(ii)) we get

\begin{align}
(i) \quad 9H_2e_1e_1(H_2) &= 0, \\
(ii) \quad 9H_2e_2e_2(H_2) &= 0, \\
(iii) \quad (\lambda^2 - 9H_2)e_3e_3(H_2) &= 0, \\
(iv) \quad (\lambda^2 - 9H_2)e_4e_4(H_2) &= 0, \\
\end{align}

which gives $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.

Then, $6H_2 = \lambda^2$ is constant and $M^4$ is a timelike isoparametric hypersurface of $\mathbb{E}^5$.

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References


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