On equality of coset preserving subcentral automorphisms

Parisa Seifizadeh · AmirAli Farokhniaee

Abstract Let $G$ be a finite non-abelian $p$-group, where $p$ is a prime and $M$ and $N$ are two subcentral characteristic subgroups of $G$. An automorphism $\alpha$ of $G$ is called subcentral automorphism if, for all $g \in G$, $g^{-1} \alpha(g) \in M$ and for all $n \in N$, $n^{-1} \alpha(n) = 1$. Let $\text{Aut}_M^G(G)$, $C_{\text{Aut}_M^G(G)}(Z(G))$ and $\text{Aut}_N^G(G)$ denote, respectively, the group of all subcentral automorphisms of $G$, the group of all subcentral automorphisms of $G$ fixing the center of $G$, elementwise, and the group of all derival automorphisms of $G$ fixing the elements of $N$. In this study, we present necessary and sufficient conditions on a finite $p$-group, $G$, such that $\text{Aut}_M^G(G) = C_{\text{Aut}_M^G(G)}(Z(G))$ and $\text{Aut}_N^G(G) = \text{Aut}_N^G(G)$. Moreover, we investigate the necessary and sufficient conditions for the equality of inner automorphisms and the group of subcentral automorphisms that fix the center and the Frattini subgroup of, $G$, element wise.

Keywords subcentral automorphism · subcentral autocommutator subgroup · inner automorphism · finite $p$-group · torsion-free group

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1 Introduction

In this paper, $p$ denotes a prime number. Let $G$ be a finite group. We denote by $G'$, $Z(G)$, $\text{Aut}(G)$, $\text{Inn}(G)$ and $\text{exp}(G)$, respectively, the commutator subgroup, the center, the group of all automorphisms, the inner automorphisms of $G$ and the exponent of $G$. For any group $H$ and an abelian group $L$, $\text{Hom}(H,L)$ denotes the group of all homomorphisms from $H$ to $L$. For any group $G$, Hegarty [6] gave the following definitions of $Z(G)$ and $G'$:

\[
Z(G) = \{ g \in G \mid \alpha(g) = g, \forall \alpha \in \text{Inn}(G) \},
\]
\[
G' = \langle g \in G \mid \alpha(g) = g, \forall \alpha \in \text{Inn}(G) \rangle;
\]

similarly, he defined the subgroups $L(G)$ and $G^*$ of $G$ as follows:

\[
L(G) = \{ g \in G \mid \alpha(g) = g, \forall \alpha \in \text{Aut}(G) \},
\]
\[
G^* = \langle g \in G \mid \alpha(g) = g, \forall \alpha \in \text{Aut}(G) \rangle.
\]
He called $L(G)$ the absolute center of $G$ and $G^*$ the autocommutator subgroup of $G$. Note that $L(G) \leq Z(G)$ and $G^* \leq G^*$. An automorphism $\alpha$ of $G$ is called a central automorphism if $[x, \alpha] = x^{-1} \alpha(x) \in Z(G)$, for each $x \in G$. The central automorphisms of $G$, denoted by $\text{Aut}^{Z(G)}(G)$ fix $G^*$ elementwise and form a normal subgroup of the full automorphism group of $G$. Franciosi et al., [5] showed that if $Z(G)$ is torsion-free and $Z(G)/G^* \cap Z(G)$ is torsion, then $\text{Aut}^{Z(G)}(G)$ acts trivially on $Z(G)$. Let $M$ and $N$ be two subcentral characteristic subgroups of $G$. An automorphism $\alpha$ of $G$ is called a subcentral automorphism if $[x, \alpha] = x^{-1} \alpha(x) \in M$, for each $x \in G$, and $[n, \alpha] = n^{-1} \alpha(n) = 1$, for each $n \in N$. Then $\text{Aut}_N^G(G)$ is the group of all such automorphisms of $G$.

In this paper, we state some new definitions and show that if $M$ is torsion-free and $M/M \cap Z(G)$ is torsion, then $\text{Aut}_N^M(G)$ acts trivially on $M$ and $\text{Aut}_N^M(G)$ is a torsion-free abelian group, presented in section 2. Shabani Attar [7] gave necessary and sufficient conditions for any non-abelian finite $p$-group such that $\text{Aut}^{Z(G)}(G) = C_{\text{Aut}^{Z(G)}(G)}(Z(G))$, where $C_{\text{Aut}^{Z(G)}(G)}(Z(G))$ is the set of all central automorphisms $\alpha$ of $G$ that fix the center of $G$, elementwise. As one of the main results of this study, we find necessary and sufficient conditions on $G$, such that $\text{Aut}_N^M(G)$ is equal to $C_{\text{Aut}_N^M(G)}(Z(G))$, where $C_{\text{Aut}_N^M(G)}(Z(G))$ is the group of all subcentral automorphisms of $G$ that fix the center of $G$, elementwise. We also find necessary and sufficient conditions on the finite $p$-group, $G$, such that the subcentral automorphisms of $G$ is equal to the derival automorphisms of $G$ fixing the elements of $N$, where $\text{Aut}_N^G(G)$ is the group of all such automorphisms. These results are presented in section 3. According to the definition of the subcentral autocommutator subgroup that we present in Section 2, we denote the group of all subcentral autocommutator automorphisms $\alpha$ of $G$, by $\text{IK}(G)$. Here, $\alpha$ is called an IK-automorphism if $[g, \alpha] = g^{-1} \alpha(g) \in K$, where $K$ is subcentral autocommutator subgroup of $G$. Continuing on section 3, as another main result, we present necessary and sufficient conditions on the finite $p$-group, $G$, such that the subcentral autocommutator automorphisms of $G$ fixing the center and the Frattini subgroup of $G$, elementwise, are equal to the inner automorphisms of $G$. In other words, we find the conditions under which $\text{IK}(G)_{Z(G)} = \text{Inn}(G)$ and $\text{IK}(G)_{\Phi(G)} = \text{Inn}(G)$. Our results also include the relationship between the order of $\text{IK}_{Z(G)}(G)$ and the order of $K$.

Throughout this paper, we utilize the following well-known lemmas:

Lemma 1.1 ([1, Lemma 2.2]) Let $A$, $B$ and $C$ be abelian groups.

(i) $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$,

(ii) $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$,

(iii) $\text{Hom}(C_m, C_n) \cong C_d$, where $d = \gcd(m, n)$.
A non-abelian group $G$ is called purely non-abelian if it has no nontrivial abelian direct factor.

**Lemma 1.2** ([3, Lemma 1.2]) Let $G$ be a purely non-abelian finite group such that $M \leq Z(G)$ and $N$ are two normal subgroups of $G$, then $\text{Aut}_M^N(G) \cong \text{Hom}(G/N, M)$.

**Lemma 1.3** (See [2, Lemma 3]) Let $G$ be any group, and let $Y$ be a central subgroup of $G$ contained in a normal subgroup $X$ of $G$. Then the group of all automorphisms of $G$ that induce the identity on both $X$ and $G/X$ is isomorphic to $\text{Hom}(G/X, Y)$.

**Lemma 1.4** ([4, Lemma E]) Suppose $H$ is an abelian $p$-group of exponent $p^c$, and $L$ is cyclic group of order divisible by $p^c$. Then $\text{Hom}(H, L)$ is isomorphic to $H$.

### 2 Preliminaries

Let $M$ and $N$ be two normal subgroups of $G$. By $\text{Aut}^M(G)$, we mean the subgroup of $\text{Aut}(G)$ consisting of all the automorphisms which centralize $G/M$, and by $\text{Aut}_N(G)$, we mean the subgroup of $\text{Aut}(G)$ consisting of all the automorphisms which centralize $N$. From now on, $M$ and $N$ will be two subcentral characteristic subgroups of $G$. We denote $\text{Aut}_M^N(G) \cap \text{Aut}_N(G)$ by $\text{Aut}_M^N(G)$ and $\text{Aut}_M^N(G)$ will be called subcentral automorphisms of $G$ which preserve the elements of $N$. It can be seen that $\text{Aut}_M^N(G)$ is a normal subgroup of $\text{Aut}_Z^G(G)$.

**Definition 2.1** Let $M$ and $N$ be two subcentral characteristic subgroups of $G$, then we define

$$\text{Aut}^*(G) = \{ \alpha \in \text{Aut}_M^N(G) : \alpha \beta = \beta \alpha, \forall \beta \in \text{Aut}_M^N(G) \}.$$  

Clearly $\text{Aut}^*(G)$ is a normal subgroup of $\text{Aut}(G)$.

Now if $M = Z(G)$ and $N = G$, we have $\text{Aut}^*(G) = \{ \alpha \in \text{Aut}_Z^G(G) : \alpha \beta = \beta \alpha, \forall \beta \in \text{Aut}_Z^G(G) \}$. So $\text{Aut}_Z^G(G)$, $\text{Aut}_Z^G(G)$ and $\text{Inn}(G)$ are normal subgroups of $\text{Aut}^*(G)$. Also since $\text{Aut}_Z^G(G)$ is an abelian group, then $\text{Aut}_Z^G(G) \leq \text{Aut}^*(G)$.

**Definition 2.2** Suppose $G$ be a group. Then we define

$$K^* = \langle [g, \alpha] : g \in G, \alpha \in \text{Aut}^*(G) \rangle,$$

where $[g, \alpha] = g^{-1}g^\alpha = g^{-1}\alpha(g)$.

It is easy to check that $K^*$ is a characteristic subgroup of $G$.

The following proposition shows that each element $K^*$ is invariant under the natural action of $\text{Aut}_Z^G(G)$.

**Proposition 2.3** $\text{Aut}_Z^G(G)$ acts trivially on $K^*$.
Proof. Take an automorphism $\alpha \in \text{Aut}_N^M(G)$. This implies $x^{-1}\alpha(x) \in M$ for all $x \in G$. Therefore $\alpha(x) = xm$ for some $m \in M$. Let $\beta \in \text{Aut}^*(G)$. By definition of $\text{Aut}^*(G)$ we have

$$\alpha([x, \beta]) = \alpha(x^{-1}\beta(x)) = (\alpha(x))^{-1}\beta(\alpha(x))$$

$$= m^{-1}x^{-1}\beta(xm)$$

$$= m^{-1}x^{-1}\beta(x)m = x^{-1}\beta(x) = [x, \beta].$$

Hence the result follows.

\[ \Box \]

**Definition 2.4** Let $E^*$ be a normal subgroup of $\text{Aut}(G)$ contained in $\text{Aut}^*(G)$ then we define

$$K = \langle [g, \alpha] \mid g \in G, \alpha \in E^* \rangle.$$  

Clearly $K \leq N \leq Z(G)$ and $K \leq G^*$. $K$ will be called subcentral autocommutator subgroup of $G$ (with respect to subcentral subgroup $N$). In particular, when $E^* = \text{Inn}(G)$, we have $K = G^*$. $K$ is a subgroup of $K^*$ which is obtained in the above manner for a corresponding $E^*$. It is easy to see that $K$ is a characteristic subgroup of $G$ and hence it is a normal subgroup of $G$.

**Proposition 2.5** Let $G$ be a group with $M$ torsion-free and $M/M \cap N$ is torsion. Then $\text{Aut}_N^M(G)$ is a torsion-free abelian group which acts trivially on $M$.

Proof. Let $\alpha \in \text{Aut}_N^M(G)$. If $x$ is an element of $M$, then by the hypothesis, $x^n \in N$ for some positive integer $n$, we have $x^n = \alpha(x^n) = (\alpha(x))^n$, and hence $x^{-n}(\alpha(x))^n = 1$. Since $x^{-1}\alpha(x) \in M \leq Z(G)$, this implies $(x^{-1}\alpha(x))^n = 1$. As $M$ is torsion-free, this implies that $x^{-1}\alpha(x) = 1$ i.e., $\alpha(x) = x$. Therefore, $\text{Aut}_N^M(G)$ acts trivially on $M$.

Also since $\text{Aut}_N^M(G)$ acts trivially on $M$, we have $\text{Aut}_N^M(G)$ is abelian, let $\alpha, \beta \in \text{Aut}_N^M(G)$ and $x \in G$. Thus

$$\alpha\beta(x) = \alpha(\beta(x)) = \alpha(x^{-1}\beta(x)) = \alpha(x)\alpha(x^{-1}\beta(x))$$

$$= xx^{-1}\alpha(x)x^{-1}\beta(x)$$

$$= \beta(x)x^{-1}\alpha(x).$$

Now, consider $\alpha \in \text{Aut}_N^M(G)$ and suppose there exists positive integer $m$ such that $\alpha^m = \{1\}$. Since $x^{-1}\alpha(x) \in M$ for all $x \in G$, there exists $g \in M$ such that $\alpha(x) = xg$. Further, $\alpha^2(x) = \alpha(xg) = \alpha(x)\alpha(g) = xg^2$ (because $\alpha$ acts trivially on $M$). Hence, by induction, $\alpha^n(x) = xg^n$. But $\alpha^m = \{1\}$ implies $x = xg^m$, i.e., $g^m = 1$. As $M$ is torsion-free, we must have $g = 1$. Thus $\alpha(x) = x$, for each $x$, i.e., $\alpha = \{1\}$. Therefore, $\text{Aut}_N^M(G)$ is torsion-free, and so the theorem follows.

\[ \Box \]
Remark 2.1 Let $G$ be a non-abelian finite $p$-group of class 2 and let $\alpha \in \text{Aut}_N^M(G)$ and $p^n = \exp(M)$. Since $g^{-1} \alpha(g) \in M$, $\alpha(g) = gm$ for $m \in M$, thus

$$\alpha(g^m) = g^m m^{p^n}[g, m]^{(p^n)^2}.$$ 

Now since $M \leq Z(G)$, $[g, m] = 1$. Also $m^{p^n} = 1$. Therefore, $\alpha(g^m) = g^m$ for every $g \in G$.

3 Main results

In this section we study the equalities of subcentral automorphisms of the group $G$, and the results follow. First, we find necessary and sufficient conditions on $G$ such that $\text{Aut}_N^M(G) = C_{\text{Aut}_N^M(G)}(Z(G))$, where $C_{\text{Aut}_N^M(G)}(Z(G))$ is the group of subcentral automorphisms of $G$ that fix $Z(G)$, elementwise.

Theorem 3.1 Let $G$ be a non-abelian finite $p$-group such that $M \leq N$. If $G/K$ is abelian, then $\text{Aut}_N^M(G) = C_{\text{Aut}_N^M(G)}(Z(G))$ if and only if $Z(G) \subseteq NG^{p^n}$, where $p^n = \exp(M)$.

Proof. Suppose $Z(G) \subseteq NG^{p^n}$, where $p^n = \exp(M)$. We know

$$C_{\text{Aut}_N^M(G)}(Z(G)) \leq \text{Aut}_N^M(G).$$

Now assume that $\sigma \in \text{Aut}_N^M(G)$, and $x \in Z(G)$. We can write $x = bg^{p^n}$ for some $b \in N$, and $g \in G$. One can easily check that $G$ is nilpotent of class 2, as $G/K$ is abelian, it implies that $G' \leq K \leq N \leq Z(G)$. Now according to the Remark 2.1, $\sigma(g^{p^n}) = g^{p^n}$ and $\sigma(b) = b$. Hence, $\sigma(x) = x$ and so $\sigma \in C_{\text{Aut}_N^M(G)}(Z(G))$. This shows that $\text{Aut}_N^M(G) \leq C_{\text{Aut}_N^M(G)}(Z(G))$ and whence $\text{Aut}_N^M(G) = C_{\text{Aut}_N^M(G)}(Z(G))$.

To prove the converse, assume that $\text{Aut}_N^M(G) = C_{\text{Aut}_N^M(G)}(Z(G))$ and $Z(G)$ is not a subset of $NG^{p^n}$. Thus exists $x \in Z(G)$, which is not in $NG^{p^n}$. Since $G/N$ is an abelian group. Let

$$G/N = \langle x_1N \rangle \times \cdots \times \langle x_kN \rangle,$$

where $x_1, x_2, \ldots, x_k \in G$. Therefore, $xN = x_1^{t_1}N \cdots x_k^{t_k}N$ for some $t_1, t_2, \ldots, t_k$. Since $x \notin NG^{p^n}$, then $x_i^{t_i} \notin G^{p^n}$, and so $p^{t_i} < p^n$ for some $i$. Now select $m \in M$ where, $o(m) = \min(p^n, o(x_iN))$, and define $f : G/N \to M$ by $x_iN \mapsto m$ and $x_jN \mapsto 1$, for $i \neq j$. Then $f$ can be considered as a homomorphism. Now, consider the map $\sigma_f : G \to G$ defined by $\sigma_f(a) = af(aN)$. Clearly, $\sigma_f$ is an endomorphism of $G$.

Now suppose that $g \in \text{Ker}(\sigma_f)$. Then $f(gN) = g^{-1}$. Also $\sigma_f$ acts trivially on elements of $N$, so we can write $g^{-1} = \sigma_f(g^{-1}) = g^{-1}f(g^{-1}N) = g^{-1}g = 1$. Therefore, $g = 1$. This shows that $\sigma_f$ is one-to-one, and since $G$ is finite, one can see that the homomorphism $\sigma_f$ is a bijection. Hence, $\sigma_f \in \text{Aut}_N^M(G)$. Moreover,
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\[ f(xN) = f(x_1^{p_1} N \ldots x_k^{p_k} N) \text{ and so } f(xN) = f(x_1^{p_1} N) = m^{p_1}. \] Since, \( p_1 < p_k \), therefore, \( m^{p_1} \) is a non-trivial element of \( M \). Hence, \( \sigma_f \notin C_{\Aut^M_{\N}}(Z(G)) \), which is a contradiction.

\[ \square \]

Here, we investigate the conditions when the subcentral automorphism and the group of derivations which fix \( N \), elementwise, are equal.

Let \( G \) be a non-abelian finite \( p \)-group. If \( G/K \) is abelian, then \( G' \leq K \leq N \), so we can write

\[ G/N = C_{p^{a_1}} \times C_{p^{a_2}} \times \ldots \times C_{p^{a_l}}, \]

where \( C_{p^{a_i}} \) is a cyclic group of order \( p^{a_i} \), \( 1 \leq i \leq l \), and \( a_i \geq a_{i+1} \geq 1 \). Let \( G' \leq M \) and let

\[ G' = C_{p^{b_1}} \times C_{p^{b_2}} \times \ldots \times C_{p^{b_k}} \]

and

\[ M = C_{p^{c_1}} \times C_{p^{c_2}} \times \ldots \times C_{p^{c_m}} \]

be the cyclic decompositions of the corresponding abelian group, where \( b_1 \geq b_{i+1} \geq 1 \) and \( c_i \geq c_{i+1} \geq 1 \). Since \( G' \leq M \), we have \( k \leq m \) and \( b_j \leq c_j \), for all \( 1 \leq j \leq k \). Keeping fixed the above notation, we prove the following theorem:

**Theorem 3.2** Let \( G \) be a purely non-abelian finite \( p \)-group such that \( G/K \) is abelian, then \( \Aut^M_{\N}(G) = \Aut^G_{\N}(G) \) if and only if \( G' = M \) or \( G' < M \), \( k = m \), and \( a_1 = b_s \), where \( s \) is the largest integer between \( 1 \) and \( k \) such that \( b_s < c_s \).

**Proof.** Observe that if \( \Aut^M_{\N}(G) = \Aut^G_{\N}(G) \). Then, for any commutator \([x, y] \in G'\), we have \([x, y] = x^{-1}L_y(x) \in M\), where \( L_y(x) \) is the inner automorphism of \( G \) induced by \( y \) (as every inner automorphism of \( G \) fixes the center of \( G \) elementwise and \( N \leq Z(G) \)). Thus \( G' \leq M \). We claim that \( k = m \), and \( a_1 = b_s \), where \( s \) is the largest integer between \( 1 \) and \( k \) such that \( b_s < c_s \). As \( G/K \) is abelian, we have \( G' \leq K \leq N \). Now using Lemma 1.2, \( \Aut^G_{\N}(G) \cong \text{Hom}(G/N, G') \) and \( \Aut^M_{\N}(G) \cong \text{Hom}(G/N, M) \).

Therefore \( |\Aut^G_{\N}(G)| = |\text{Hom}(G/N, G')| \) and \( |\Aut^M_{\N}(G)| = |\text{Hom}(G/N, M)| \).

By Lemma 1.1 we have

\[ |\text{Hom}(G/N, G')| = \prod_{1 \leq i \leq l, 1 \leq j \leq k} p^{\min\{a_i, b_j\}} \]

and

\[ |\text{Hom}(G/N, M)| = \prod_{1 \leq i \leq l, 1 \leq j \leq m} p^{\min\{a_i, c_j\}}. \]

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First, suppose that $\text{Aut}^M_N(G) = \text{Aut}^M_N(G')$ and $G' < M$. Then
\[ |\text{Hom}(G/N, G')| = |\text{Hom}(G/Z(G), M)|, \]
so we have $k \leq m$, and $b_j \leq c_j$, for every $j$, $1 \leq j \leq k$, therefore $\min\{a_i, b_j\} \leq \min\{a_i, c_j\}$ for all $i$, $1 \leq i \leq l$ and for all $j$, $1 \leq j \leq k$, if $k < m$, then
\[ |\text{Hom}(G/Z(G), G')| < |\text{Hom}(G/Z(G), M)|, \]
which is not true.
Thus $k = m$ and $\min\{a_i, b_j\} = \min\{a_i, c_j\}$, for all $i$, $1 \leq i \leq l$ and for all $j$, $1 \leq j \leq k$.

Since $G' < M$, there exists some $j$ between 1 and $k$ such that $b_j < c_j$. Let $s$ be the largest integer between 1 and $k$ such that $b_s < c_s$. Now suppose, for a contradiction, that $b_s < a_1$. Thus $b_s = \min\{a_i, b_s\} = \min\{a_i, c_s\}$, which is impossible. Therefore, $a_1 \leq b_s$. Since $G' \leq Z(G)$, so $G$ is nilpotent of class 2, thus $b_1 = \exp(G') = \exp(G/Z(G)) = a_1$. Hence $a_1 \leq b_s \leq b_{s-1} \leq \ldots \leq b_1 = a_1$, thus $a_1 = b_s$.

Conversely, if $G' = M$, then $\text{Aut}^M_N(G) = \text{Aut}^M_N(G')$. Suppose that $G' < M$, $k = m$ and $a_1 = b_s$, where $s$ is the largest integer between 1 and $k$ such that $b_s < c_s$.

Since $G$ is purely non-abelian by applying Lemma 1.2 and Lemma 1.1 we have
\[ |\text{Aut}^M_N(G)| = |\text{Hom}(G/N, M)| = \prod_{1 \leq i \leq l, 1 \leq j \leq m} p^{\min\{a_i, c_j\}} \]
and
\[ |\text{Aut}^M_N(G')| = |\text{Hom}(G/N, G')| = \prod_{1 \leq i \leq l, 1 \leq j \leq k} p^{\min\{a_i, b_j\}}. \]

Since $a_1 = b_s$, thus
\[ 1 \leq a_1 \leq \ldots \leq a_2 \leq a_1 = b_s \leq b_{s-1} \leq \ldots \leq b_2 \leq b_1 \]
Therefore $a_i \leq b_j \leq c_j$, for all $1 \leq i \leq l$ and $1 \leq j \leq s$, whence $\min\{a_i, b_j\} = a_i = \min\{a_i, c_j\}$, for all $1 \leq i \leq l$ and $1 \leq j \leq s$. On the other hand $c_j = b_j$, for all $j > s$, so we have $\min\{a_i, b_j\} = a_i = \min\{a_i, c_j\}$, for all $1 \leq i \leq l$ and $s + 1 \leq j \leq k$.

Thus $|\text{Aut}^M_N(G)| = |\text{Aut}^M_N(G')|$. Hence $\text{Aut}^M_N(G) = \text{Aut}^M_N(G')$, because $\text{Aut}^M_N(G)$ is a subgroup of $\text{Aut}^M_N(G)$. The proof of the theorem is complete.

\[ \square \]

Suppose that $G$ be a finite non-abelian $p$-group, $M = L(G)$, $N = Z(G)$ and $E^* = \text{Inn}(G)$. Then $G' = K$, so we have $G \leq Z(G)$. Since $L(G) \leq Z(G)$ and $G' \leq Z(G)$, applying Lemma 1.3 and the above theorem, we can conclude the following corollary.

**Corollary 3.3** Let $G$ be a finite non-abelian $p$-group and $\exp(L(G)) = p$. Then $\text{Aut}^L_{Z(G)}(G) = \text{Aut}^{L(G)}_{Z(G)}(G)$ if and only if $G' = L(G)$.
We recall that Shabani Attar [8] has found necessary and sufficient conditions for any non-abelian finite $p$-group of class 2 such that $\mathsf{IA}(G) = \mathsf{Inn}(G)$, where $\mathsf{IA}(G)$ is the set all automorphisms $\alpha$ of $G$ which $[x, \alpha] \in G'$. Let $M=K$. An automorphism $\alpha$ of the group $G$ is an $\mathsf{IK}$-automorphism if $x^{-1} \alpha(x) = [x, \alpha] \in K$, for any $x \in G$. We denote by $\mathsf{IK}(G)$ the set of all such automorphisms of $G$. We know that $K \leq N \leq Z(G)$, so by using Lemma 1.2, we have $\mathsf{IK}(G) \cong \mathsf{Hom}(G/N, K)$. If $N = Z(G)$, then $\mathsf{IK}_G(G)$ is the group of all $\mathsf{IK}$-automorphisms that fix the center of $G$, elementwise. Now we obtain necessary and sufficient conditions on the finite $p$-group $G$ such that $\mathsf{IK}_G(G) = \mathsf{Inn}(G)$.

**Theorem 3.4** Let $G$ be a finite non-abelian $p$-group. If $G/K$ is abelian, then $\mathsf{IK}_G(G) = \mathsf{Inn}(G)$ if and only if $K$ is cyclic.

**Proof.** Suppose that $\mathsf{IK}_G(G) = \mathsf{Inn}(G) = G/Z(G)$, using Lemma 1.3, we have $|\mathsf{Hom}(G/Z(G), K)| = |G/Z(G)|$. Now we show that $K$ is cyclic. Assume contrarily that $K$ is not cyclic and $\exp(K) = p^\ell$, then $K = C_p \times N$ where $C_p$ is cyclic subgroup of $K$ and $H$ is a nontrivial proper subgroup of $K$. We have

$$
|G/Z(G)| = |\mathsf{Hom}(G/Z(G), K)| = |\mathsf{Hom}(G/Z(G), C_p \times H)| = |\mathsf{Hom}(G/Z(G), C_p)| |\mathsf{Hom}(G/Z(G), H)|.
$$

As $G' \leq K \leq Z(G)$, so $G$ is nilpotent of class 2, and $\exp(G/Z(G)) = \exp(G')$. Now by Lemma 1.4:

$$
|\mathsf{Hom}(G/Z(G), C_p)| = |G/Z(G)|.
$$

Therefore

$$
|G/Z(G)| = |\mathsf{Hom}(G/Z(G), K)| = |G/Z(G)| |\mathsf{Hom}(G/Z(G), H)|,
$$

which is a contradiction. Hence $K$ is cyclic. Conversely suppose that $K$ is cyclic. We have

$$
|\mathsf{IK}_G(G)| = |\mathsf{Hom}(G/Z(G), K)|.
$$

Since $G' \leq K \leq Z(G)$, so $G$ is nilpotent of class 2, thus $\exp(G/Z(G)) = \exp(G')|K|$, using Lemma 1.4, it follows that $|\mathsf{IK}_G(G)| = |G/Z(G)| = |\mathsf{Inn}(G)|$, on the other hand $G' \leq K$, whence $\mathsf{Inn}(G) \leq \mathsf{Aut}_G(G) \leq \mathsf{IK}_G(G)$. Therefore $\mathsf{IK}_G(G) = \mathsf{Inn}(G)$. 

Let $E' = \mathsf{Inn}(G)$, then $K = G'$. The following example shows that every automorphism of $\mathsf{Aut}_G(G)$ is not necessarily inner automorphism.

**Example 3.1** Consider the group

$$
G = \langle a, b, c | a^4 = b^4 = c^2 = 1, \ bab^{-1} = cac = a^{-1}, \ cbc = b^{-1} \rangle.
$$
Clearly, \(|G|=32\) and \(G\) is nilpotent of class 2, \(G'=Z(G)=C_2^2=<a^2,b^2>\) and \(G/Z(G)\cong C_2\cong \text{Inn}(G)\). We consider the automorphism \(\alpha\) given by \(\alpha(a)=ab^2, \alpha(b)=ba^2\) and \(\alpha(c)=ca^2\). One can easily check that \(\alpha \in \text{Aut}G'\) and \([z,\alpha]=1\), for every \(z \in Z(G)\). It is obvious that \(\alpha\) is a non-inner automorphism.

An immediate of Theorem 3.4, is presented in the next corollary as the necessary and sufficient conditions for finite non-abelian \(p\)-groups \(G\), where \(\text{Aut}_{Z(G)}^{G'}(G)=\text{Inn}(G)\).

**Corollary 3.5** Let \(G\) be a finite non-abelian \(p\)-group which that \(E'=\text{Inn}(G)\). Then \(G\) is nilpotent of class 2 and \(\text{Aut}_{Z(G)}^{G'}(G)=\text{Inn}(G)\) if and only if \(G'\) is cyclic.

**Proof.** Since \(K=G'\), we have \(G' \leq Z(G)\), thus \(G\) is nilpotent of class 2 and \(\text{IK}_{Z(G)}(G)=\text{Aut}_{Z(G)}^{G'}(G)\). Hence by Theorem 3.4, \(\text{Aut}_{Z(G)}^{G'}(G)=\text{Inn}(G)\) if and only if \(G'\) is cyclic.

\(\square\)

Another result obtained from Theorem 3.4, is asserted in the following corollary.

**Corollary 3.6** Let \(G\) be a finite non-abelian \(p\)-group such that \(G/K\) is abelian and \(\text{IK}_{Z(G)}(G)=\text{Inn}(G)\). If \(|K|=p^n\), then \(KG^{p^n} \leq Z(G)\). In particular when \(|K|=p\), then \(\Phi(G) \leq Z(G)\).

**Proof.** If \(\text{IK}_{Z(G)}(G)=\text{Inn}(G)\), applying Theorem 3.4, \(K\) is cyclic. Obviously \(K \leq Z(G)\). Since \(G/K\) is abelian, thus \(G' \leq K \leq Z(G)\), on the other hand \(|K|=p^n\), so \([a,b]^{p^n}=1\) for all \(a, b \in G\), thus \([a^{p^n}, b]=1\), whence \(a^{p^n} \in Z(G)\) and we have \(G^{p^n} \leq Z(G)\). Therefore \(KG^{p^n} \leq Z(G)\). If \(|K|=p\), then \(\Phi(G)=G^{p} \leq KG^{p} \leq Z(G)\).

\(\square\)

Now if \(N=\Phi(G)\), hypothesize that \(N\) is a subcentral characteristic subgroup of a group \(G\), so that \(\Phi(G) \leq Z(G)\). In the following proposition we give necessary and sufficient conditions on a finite \(p\)-group \(G\) under which \(\text{IK}_{\Phi(G)}(G)=\text{Inn}(G)\).

**Proposition 3.7** Let \(G\) be a finite non-abelian \(p\)-group such that \(G/K\) is abelian, then \(\text{IK}_{\Phi(G)}(G)=\text{Inn}(G)\) if and only if \(K\) is cyclic and \(Z(G)=\Phi(G)\).

**Proof.** Let \(\text{IK}_{\Phi(G)}(G)=\text{Inn}(G)\). By applying Lemma 1.3, we have

\[\text{IK}_{\Phi(G)}(G) \cong \text{Hom}(G/\Phi(G), K)\]

But \(G/\Phi(G)\) is an elementary abelian group and hence we can write

\[\text{Hom}(G/\Phi(G), K) \cong (G/\Phi(G))^d, \text{ where } d=d(K)\]
therefore $IK_{\Phi(G)}(G) = \text{Inn}(G)$ if and only if $G/Z(G) \cong (G/\Phi(G))^d$ is an elementary abelian group or equivalently $K$ is cyclic and $Z(G) = \Phi(G)$.

\[\square\]

Sury [9] showed that if $G'$ is finite and $G/Z(G)$ is generated by $d$ elements, then $|\text{Inn}(G)| \leq |G|^d$, where $G^* = \langle [g, \alpha] \mid g \in G, \alpha \in \text{Aut}(G) \rangle$ is the automcommutator subgroup of the group $G$. Here, as the last result of this paper, we find the relationship between the order of $IK_{Z(G)}(G)$ and the order of $K$.

**Proposition 3.8** Let $G$ be any group with finite derived subgroup and $N = Z(G)$. If $d$ is the minimal number of generators of the central factor group of $G$, then $|IK_{Z(G)}(G)| \leq |K|^d$.

**Proof.** Assume that $G/Z(G)$ has a minimal set of generators

$$g_1Z(G), g_2Z(G), \ldots, g_dZ(G),$$

and $\alpha \in IK_{Z(G)}(G)$, which fixes $Z(G)$ elementwise. Consider the following map

$$\phi : IK_{Z(G)}(G) \rightarrow K \times K \times \ldots \times K \quad d - \text{times}$$

defined by $\phi(\beta) = [g_1, \beta], [g_2, \beta], \ldots, [g_d, \beta]$, for all $\beta \in IK_{Z(G)}(G)$. It is obvious that $\phi$ is injective, as $\phi(\beta) = \phi(\gamma)$ implies that $[g_i, \beta] = [g_i, \gamma]$, for all $1 \leq i \leq d$ and $\beta, \gamma \in IK_{Z(G)}(G)$. Therefore $|IK_{Z(G)}(G)| \leq |K|^d$.

\[\square\]

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