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An analytic and numerical study for two classes of differential equations of fractional order involving Caputo and Khalil derivatives

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Abstract In this paper, we study new classes of differential equations of fractional order. The first considered problem involves the derivative of Caputo, while the second one involves conformable Khalil derivative. For the first class, we prove an existence and uniqueness result, then, we discuss an example to show the applicability of the result. For the second one, we apply the exp-function method to find new traveling wave solutions for a generalised conformable fractional partial differential equation, then some examples on Ostrovsky and KdV equations are given to illustrate the efficiency and accuracy of the method. Some graphs are plotted and discussed to show more the importance of the obtained results.

Keywords Caputo derivative · existence of solution · fixed point · Ostrovsky equation · conformable derivative · Exp-function method · traveling wave

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1 Introduction

The nonlinear wave phenomena of propagation and interaction are very important, they can be well presented by a variety of nonlinear wave equations. In fact, in 1978, Ostrovsky has derived the following standard wave equation for internal waves in a rotating ocean [5, 12, 26]:

$$(u_t + c_0 u_x + puu_x + qu_{xxx})_x = \gamma u, \tag{1.1}$$

where c_0 is the velocity, p is a given coefficient, q and γ are the dispersion coefficients.
In the literature, we find also the reduced Ostrovsky equation [27]:

$$(u_t + c_0 u_x + puu_x)_x = \gamma u. \tag{1.2}$$

Based on the above problem, several authors have derived the same equation for different physical situations. For instance, Vakhnenko and Parkes [29] have proved

that the reduced Ostrovsky $Eq(1,2)$ can be given by:

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0. \tag{1.3}$$

On the other hand, the theory of differential equations of non integer order has played very important roles for modeling many physical and biological processes. For more information, we cite the research papers [3,6–8,13,14]. This fractional theory has become a new domain of research in differential equations theory. In particular, research on the existence of unique solutions for fractional differential equations is of big importance since it helps physicians to well and better understand the behaviour of several real word phenomena. For more details, see the papers [4,19,21,23]. In this sense, Y. Zhang [32] has considered the VIM method to solve the following fractional generalized KdV equation:

$${}_0^R D_t^\alpha u(x,t) + au^p(x,t)u_x(x,t) + bu_{xxx}(x,t) = 0. \tag{1.4}$$

Then, M.I. El Bahi et al. [11] have been concerned with the study of the following new KdV-Like equation:

$$D_t^\alpha u + \left(3(1-\delta)u + (\delta+1)\frac{u_{xx}}{u}\right)u_x - \delta u_{xxx} = 0, 0 < \alpha \leq 1. \tag{1.5}$$

Recently, H. Askari [2] has considered the Weyl derivative to study the problem:

$$W_+^\alpha \left[u_t - \beta u_{xxx} + \left(u^2\right)_x\right] = \gamma u, \quad \gamma > 0, \quad 0 < \alpha \leq 1. \tag{1.6}$$

After that, several methods have been proposed to obtain some of the exact solutions and traveling wave solutions to nonlinear fractional differential equations such as, the exp-function method [31], the fractional sub-equation method [30], the (G'/G)-expansion method [33] and also the first integral method [22]. In fact, there are many ways to obtain some of the solutions and traveling waves to nonlinear fractional problems, one of them is the Exp-function method (EFM for short). This method is the most recent analytical approach that has been used for solving nonlinear differential problems in physics. Also, it seems that the EFM method is more effective and simple than other methods and a lot of solutions can be obtained by using a simple computer program. The EFM method was first proposed in 2006 [15] and then investigated in [16] and [25], also it has been applied by and Navickas et al. in [24], and later in 2012, by Ebaid [10]. However, its applications to fractional calculus are rare. In 2013, the considered method was extended to fractional differential problems, and it becomes a tool for this theory, (see He [17]). In [20] C. Koroglu and T. Ozis extended the EFM method to obtain approximate and exact solutions for $Eq(1,3)$.
The aim of the first part of this paper to study the following fractional differential

problem:

$$\begin{cases} D^{\alpha}D^{\beta}D^{\gamma}u(t)=\dfrac{a_1 f(t,u(t),D^{\gamma}u(t))+a_2 g(t,D^{\gamma}u(t),D^{\rho}D^{\gamma}u(t))+a_3 h(t,u(t))}{K(u(t))}, \\ u(0)+u(1)=\int_0^{\eta} bu(s)ds, 0<\eta<1, \\ D^{\gamma}u(0)+D^{\gamma}u(1)=0, \\ D^{\mu}D^{\mu}u(0)+D^{\mu}D^{\mu}u(1)=0, \\ t\in[0,1], \\ 0<\alpha,\beta,\gamma,\rho,\mu\leq 1, \\ a_1,a_2,a_3,b\in\mathbb{R}, \end{cases} \tag{1.7}$$

where $D^{\alpha},D^{\beta},D^{\gamma},D^{\mu},D^{\rho}$ are the Caputo fractional, with $0<\alpha,\beta,\gamma,\rho\leq 1$,$\alpha+\beta\notin]0,1),\beta+\gamma\notin]0,1),\gamma+\rho\notin]0,1),I:=[0,1]$ and $f,g:I\times\mathbb{R}^2\to\mathbb{R}$, $h:I\times\mathbb{R}\to\mathbb{R}$, and $K:\mathbb{R}\to\mathbb{R}_+^*$ are given functions.
We prove an existence and uniqueness result, then we discuss an illustrative example.
In the second part of this paper, we will be concerned with traveling waves, we use the EFM method to find new traveling wave solutions of the following problem:

$$\left(T_x^{\beta}u(x,t)\right)\left(T_x^{\beta}\left(T_t^{\alpha}u(x,t)\right)\right)-\left(u(x,t)\right)\left(T_x^{2\beta}\left(T_t^{\alpha}u(x,t)\right)\right)-f\left(u,T_t^{\alpha}u\right)=0, \tag{1.8}$$

where T_x^{β},T_t^{α} are the conformable fractional derivatives [18], with $0<\alpha,\beta\leq 1$ and $f:\mathbb{R}^2\to\mathbb{R}$ is a given function.
It is important to note that there is a relationship with the above two parts, in fact, under some particular values on the parameters α,β,γ,ρ as well as on the other input data of the problem (1.7), we find some interesting classes of ordinary differential equations that will appear in the study of the second part of this paper.
This paper is organized as follows: In section 2, we present some preliminaries and lemmas. In section 3, we present in the first part our main results for the uniqueness of solutions of (1.7). Then, an example is treated to illustrate our result. In the second part, we present the EFM method to find new traveling wave solutions for some proposed conformable fractional partial equations of type Ostrovsky and KdV. Some graphs are plotted and a conclusion follows at the end.

2 Preliminaries on Caputo derivatives

To present our main results, we need to use the references [19,23]. We have:

Definition 2.1 *The Riemann-Liouville integral of order $\alpha>0$ for a function defined on I is:*

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,$$

with $\Gamma(\alpha):=\int_0^{\infty}e^{-u}u^{\alpha-1}du$.

Definition 2.2 *Suppose that $f \in C^n(I,\mathbb{R})$ and $n-1 < \alpha \leq n$, then, the Caputo derivative is given by:*

$$\begin{aligned} D^{\alpha} f(t) &= J^{n-\alpha} \frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds. \end{aligned}$$

The following lemmas are also needed.

Lemma 2.3 *Taking $n \in \mathbb{N}^*$ and $n-1 < \alpha < n$, then the general solution of $D^{\alpha} y(t) = 0$ is given by*

$$y(t) = \sum_{i=0}^{n-1} k_i t^i, k_i \in \mathbb{R}, t \in I.$$

Lemma 2.4 *When $n \in \mathbb{N}^*$ and $n-1 < \alpha < n$, in this case, we have*

$$J^{\alpha} D^{\alpha} y(t) = y(t) + k_0 + k_1 t + ... + k_n t^n, k_i \in \mathbb{R}.$$

Let us now present to the reader the lemma.

Lemma 2.5 *Let G be any continuous function over I. Then, writing*

$$\begin{cases} D^{\alpha} D^{\beta} D^{\gamma} u(t) = \frac{G(t)}{K(u(t))}, \\ u(0) + u(1) = \int_0^{\eta} bu(s) ds, \\ D^{\gamma} u(0) + D^{\gamma} u(1) = 0, \\ D^{\mu} D^{\mu} u(0) + D^{\mu} D^{\mu} u(1) = 0 \end{cases} \tag{2.1}$$

is equivalent to write:

$$\begin{aligned} u(t) = {} & J^{\alpha+\beta+\gamma}\left[\frac{G(t)}{K(u(t))}\right] - \frac{1}{2-b\eta} J^{\alpha+\beta+\gamma}\left[\frac{G(1)}{K(u(1))}\right] + \frac{b}{2-b\eta} J^{\alpha+\beta+\gamma+1}\left[\frac{G(\eta)}{K(u(\eta))}\right] \\ & + \frac{z_1 z_3}{(2-b\eta)[2z_2 z_3 - z_1]} J^{\alpha+\beta+\gamma-2\mu}\left[\frac{G(1)}{K(u(1))}\right]\left[\frac{(2-b\eta)t^{\gamma}-1}{z_5} + \frac{2z_2 - 2(2-b\eta)z_2 t^{\beta+\gamma}}{z_4}\right. \\ & \left. + \frac{b\eta^{\gamma+1}}{z_6} - \frac{2bz_2\eta^{\beta+\gamma+1}}{z_7}\right] + \frac{z_2}{(2-b\eta)[2z_2 z_3 - z_1]} J^{\alpha+\beta}\left[\frac{G(1)}{K(u(1))}\right]\left[\frac{(2-b\eta)z_1 t^{\beta+\gamma} - z_1}{z_4}\right. \\ & \left. + \frac{z_3 - (2-b\eta)z_3 t^{\gamma}}{z_5} + \frac{bz_1\eta^{\beta+\gamma+1}}{z_7} - \frac{bz_3\eta^{\gamma+1}}{z_6}\right], \end{aligned} \tag{2.2}$$

such that one takes into account:

$z_1 = \Gamma(\beta+\gamma-2\mu+1)$, $z_2 = \Gamma(\beta+1)$, $z_3 = \Gamma(\gamma-2\mu+1)$, $z_4 = \Gamma(\beta+\gamma+1)$, $z_5 = \Gamma(\gamma+1)$, $z_6 = \Gamma(\gamma+2)$, $z_7 = \Gamma(\beta+\gamma+2)$.

Proof. The reader who is familiarized with fractional calculus theory can see that the two identities:

$$D^{\beta} D^{\gamma} u(t) = J^{\alpha}\left[\frac{G(t)}{K(u(t))}\right] - k_0,$$

and

$$D^{\gamma}u(t) = J^{\alpha+\beta}\left[\frac{G(t)}{K(u(t))}\right] - k_0 J^{\beta}(1) - k_1,$$

are valid.

So, it yields that

$$u(t) = J^{\alpha+\beta+\gamma}\left[\frac{G(t)}{K(u(t))}\right] - k_0\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} - k_1\frac{t^{\gamma}}{\Gamma(\gamma+1)} - k_2. \tag{2.3}$$

Therefore,

$$\begin{cases} -2k_2 + J^{\alpha+\beta+\gamma}\left[\frac{G(1)}{K(u(1))}\right] - \frac{k_0}{\Gamma(\beta+\gamma+1)} - \frac{k_1}{\Gamma(\gamma+1)} = \int_0^{\eta} bu(s)ds, \\ -2k_1 + J^{\alpha+\beta}\left[\frac{G(1)}{K(u(1))}\right] - \frac{k_0}{\Gamma(\beta+1)} = 0, \\ J^{\alpha+\beta+\gamma-2\mu}\left[\frac{G(1)}{K(u(1))}\right] - \frac{k_0}{\Gamma(\beta+\gamma-2\mu+1)} - \frac{k_1}{\Gamma(\gamma-2\mu+1)} = 0. \end{cases}$$

We solve the above system to write

$$\begin{aligned} k_0 &= \frac{2\Gamma(\beta+\gamma-2\mu+1)\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} J^{\alpha+\beta+\gamma-2\mu}\left[\frac{G(1)}{K(u(1))}\right] \\ &\quad - \frac{\Gamma(\beta+\gamma-2\mu+1)\Gamma(\beta+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} J^{\alpha+\beta}\left[\frac{G(1)}{K(u(1))}\right], \\ k_1 &= -\frac{\Gamma(\beta+\gamma-2\mu+1)\Gamma(\gamma-2\mu+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} J^{\alpha+\beta+\gamma-2\mu}\left[\frac{G(1)}{K(u(1))}\right] \\ &\quad + \frac{\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} J^{\alpha+\beta}\left[\frac{G(1)}{K(u(1))}\right], \\ k_2 &= \frac{1}{2-b\eta} J^{\alpha+\beta+\gamma}\left[\frac{G(1)}{K(u(1))}\right] - \frac{b}{2-b\eta} J^{\alpha+\beta+\gamma+1}\left[\frac{G(\eta)}{K(u(\eta))}\right] \\ &\quad + \frac{\Gamma(\beta+\gamma-2\mu+1)\Gamma(\gamma-2\mu+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)}\left[\frac{1}{(2-b\eta)\Gamma(\gamma+1)} - \frac{2\Gamma(\beta+1)}{(2-b\eta)\Gamma(\beta+\gamma+1)}\right. \\ &\quad \left. + \frac{2b\Gamma(\beta+1)\eta^{\beta+\gamma+1}}{(2-b\eta)\Gamma(\beta+\gamma+2)} - \frac{b\eta^{\gamma+1}}{(2-b\eta)\Gamma(\gamma+2)}\right] J^{\alpha+\beta+\gamma-2\mu}\left[\frac{G(1)}{K(u(1))}\right] \\ &\quad + \frac{\Gamma(\beta+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)}\left[\frac{\Gamma(\beta+\gamma-2\mu+1)}{(2-b\eta)\Gamma(\beta+\gamma+1)} - \frac{\Gamma(\gamma-2\mu+1)}{(2-b\eta)\Gamma(\gamma+1)}\right. \\ &\quad \left. + \frac{b\Gamma(\beta+\gamma-2\mu+1)\eta^{\beta+\gamma+1}}{(2-b\eta)\Gamma(\beta+\gamma+2)}\right] J^{\alpha+\beta}\left[\frac{G(1)}{K(u(1))}\right]. \end{aligned}$$

Inserting k_0, k_1 and k_2 in (2.3), we end the proof.

□

We will use fixed point theory to study the above problem. Hence, we need the space:

$$X := \{x \in C(J,\mathbb{R}), D^{\gamma}x \in C(J,\mathbb{R}), D^{\rho}D^{\gamma}x \in C(J,\mathbb{R})\},$$

and

$$\|x\|_X = \|x\|_{\infty} + \|D^{\gamma}x\|_{\infty} + \|D^{\rho}D^{\gamma}x\|_{\infty},$$

where,

$$\|x\|_{\infty} = \sup_{t\in I}|x(t)|\ , \|D^{\gamma}x\|_{\infty} = \sup_{t\in I}|D^{\gamma}x(t)|, \|D^{\rho}D^{\gamma}x\|_{\infty} = \sup_{t\in I}|D^{\rho}D^{\gamma}x(t)|.$$

Then, we consider the application $H : X \to X$,

$$\begin{aligned}
Hu(t) \ &= \tfrac{1}{\Gamma(\alpha+\beta+\gamma)}\int_0^t (t-\tau)^{\alpha+\beta+\gamma-1}\left[\tfrac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))+a_2 g(\tau,D^{\gamma}u(\tau),D^{\rho}D^{\gamma}u(\tau))}{K(u(\tau))}\right.\\
&\left.+\tfrac{a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau - \tfrac{1}{(2-b\eta)\Gamma(\alpha+\beta+\gamma)}\int_0^1 (1-\tau)^{\alpha+\beta+\gamma-1}\left[\tfrac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))}{K(u(\tau))}\right.\\
&\left.+\tfrac{a_2 g(\tau,D^{\gamma}u(\tau),D^{\rho}D^{\gamma}u(\tau))+a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau - \tfrac{b}{(2-b\eta)\Gamma(\alpha+\beta+\gamma+1)}\\
&\times\int_0^{\eta}(\eta-\tau)^{\alpha+\beta+\gamma}\left[\tfrac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))+a_2 g(\tau,D^{\gamma}u(\tau),D^{\rho}D^{\gamma}u(\tau))+a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau\\
&+\tfrac{z_1 z_3}{(2-b\eta)[2z_2z_3-z_1]}\left[\tfrac{(2-b\eta)t^{\gamma}-1}{z_5}+\tfrac{2z_2-2(2-b\eta)z_2t^{\beta+\gamma}}{z_4}+\tfrac{b\eta^{\gamma+1}}{z_6}-\tfrac{2bz_2\eta^{\beta+\gamma+1}}{z_7}\right]\\
&\times\tfrac{1}{\Gamma(\alpha+\beta+\gamma-2\mu)}\int_0^1(1-\tau)^{\alpha+\beta+\gamma-2\mu-1}\left[\tfrac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))}{K(u(\tau))}\right.\\
&\left.+\tfrac{a_2 g(\tau,D^{\gamma}u(\tau),D^{\rho}D^{\gamma}u(\tau))+a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau + \tfrac{z_2}{(2-b\eta)[2z_2z_3-z_1]\Gamma(\alpha+\beta)}\\
&\times\left[\tfrac{(2-b\eta)z_1t^{\beta+\gamma}-z_1}{z_4}+\tfrac{z_3-(2-b\eta)z_3t^{\gamma}}{z_5}+\tfrac{bz_1\eta^{\beta+\gamma+1}}{z_7}-\tfrac{bz_3\eta^{\gamma+1}}{z_6}\right]\\
&\times\int_0^1(1-\tau)^{\alpha+\beta-1}\left[\tfrac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))+a_2 g(\tau,D^{\gamma}u(\tau),D^{\rho}D^{\gamma}u(\tau))+a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau.
\end{aligned} \tag{2.4}$$

We are ready to study the first part of our paper.

3 Main results

3.1 Part 1: Unique solutions

We are concerned with studying the uniqueness of one solution for (1.7). So, we consider the following hypotheses that can be seen as sufficient conditions, they can be changed by some other conditions (of Caratheodory type, for instance). But for our case, all what is needed is only the establishment of "some" conditions that allow us to assure the uniqueness of solutions. Therefore, we put:

$(R1)$: The functions f,g defined on $I\times\mathbb{R}^2$ and h defined on $I\times\mathbb{R}$ are continuous.

$(R2)$: There are non-negative continuous functions $\phi_1(t),\phi_2(t),\psi_1(t),\psi_2(t)$, $\delta(t)$, such that for any $t\in I$, $u_i,v_i\in\mathbb{R}$,

$$|f(t,u_1,u_2)-f(t,v_1,v_2)|\leq\sum_{i=1}^{2}\phi_i(t)|u_i-v_i|,$$
$$|g(t,u_1,u_2)-g(t,v_1,v_2)|\leq\sum_{i=1}^{2}\psi_i(t)|u_i-v_i|,$$

and for any $t\in I$, $u,v\in\mathbb{R}$,

$$|h(t,u)-h(t,v)|\leq\delta(t)|u-v|.$$

It is to note that we take:

$$\|\phi\|=Max(\sup_{t\in I}|\phi_1(t)|,\sup_{t\in I}|\phi_2(t)|),$$
$$\|\psi\|=Max(\sup_{t\in I}|\psi_1(t)|,\sup_{t\in I}|\psi_2(t)|),$$
$$\|\delta\|=\sup_{t\in I}|\delta(t)|.$$

$(R3)$: There exist continuous functions $L_1(t),L_2(t)$ and $L_3(t)$ such that, for any $t\in I$, $u,v\in\mathbb{R}$, we have

$$|f(t,v,v)|\leq\frac{|L_1(t)|}{1+|u|+|v|},|g(t,u,v)|\leq\frac{|L_2(t)|}{2+|u|^2+|v|^3},|h(t,u)|\leq\frac{|L_3(t)|}{e^2+e^u}.$$

We have needed $(R3)$ which is a new hypothesis that we can't find in the literature of fractional differential equations.
$(R4)$: The function K satisfies: $m\leq K(u)\leq M$, with $m,M\in\mathbb{R}_+^*$, for any $u\in\mathbb{R}$.
Also we need to consider the quantities to facilitate the fastidious calculations:

$$\begin{aligned}\Delta_1 &= \left[\frac{2|a_1|\|\phi\|+2|a_2|\|\psi\|+|a_3|\|\delta\|}{m}\right]\left[\frac{|3-b\eta|}{|2-b\eta|\Gamma(\alpha+\beta+\gamma+1)}+\frac{|b|}{|2-b\eta|\Gamma(\alpha+\beta+\gamma+2)}\right.\\ &+\frac{z_1z_3}{|2-b\eta||2z_2z_3-z_1|\Gamma(\alpha+\beta+\gamma-2\mu+1)}\left(\frac{|2-b\eta|+1}{z_5}+\frac{2z_2+2|2-b\eta|z_2}{z_4}+\frac{|b|\eta^{\gamma+1}}{z_6}\right.\\ &\left.+\frac{2|b|z_2\eta^{\beta+\gamma+1}}{z_7}\right)+\frac{z_2}{|2-b\eta||2z_2z_3-z_1|\Gamma(\alpha+\beta+1)}\left(\frac{|2-b\eta|z_1+z_1}{z_4}+\frac{z_3+|2-b\eta|z_3}{z_5}\right.\\ &\left.\left.+\frac{|b|z_1\eta^{\beta+\gamma+1}}{z_7}+\frac{|b|z_3\eta^{\gamma+1}}{z_6}\right)\right],\end{aligned}$$

$$\begin{aligned}\Delta_2 &= \left[\frac{2|a_1|\|\phi\|+2|a_2|\|\psi\|+|a_3|\|\delta\|}{m}\right]\left[\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{z_1z_3}{|2z_2z_3-z_1|\Gamma(\alpha+\beta+\gamma-2\mu+1)}\right.\\ &\left.+\frac{z_2}{|2z_2z_3-z_1|\Gamma(\alpha+\beta+1)}\left(\frac{z_1}{z_2}+z_3\right)\right],\end{aligned}$$

and

$$\begin{aligned}\Delta_3 &= \left[\frac{2|a_1|\|\phi\|+2|a_2|\|\psi\|+|a_3|\|\delta\|}{m}\right]\left[\frac{1}{\Gamma(\alpha+\beta-\rho+1)}+\frac{z_1z_3}{|2z_2z_3-z_1|\Gamma(\alpha+\beta+\gamma-2\mu+1)}\right.\\ &\left.\times\left(\frac{2z_2}{\Gamma(\beta-\rho+1)}\right)+\frac{z_2}{|2z_2z_3-z_1|\Gamma(\alpha+\beta+1)}\left(\frac{z_1}{\Gamma(\beta-\rho+1)}\right)\right].\end{aligned}$$

Now, we pass to prove the main result.

Theorem 3.1 *Assume that the conditions* (R_2) *and* (R_4) *are valid and suppose that the constant satisfies* $\Delta : \Delta = \Delta_1+\Delta_2+\Delta_3 < 1$. *Then, problem* (1.7) *has a unique solution over* I.

Proof. We can use Banach principle to prove this result. For $(u_1,u_2)\in X^2$, we can write

$$\begin{aligned}\|Hu_1-Hu_2\|_\infty \leq & \left[\frac{2|a_1|\|\phi\|+2|a_2|\|\psi\|+|a_3|\|\delta\|}{m}\right]\left[\frac{|3-b\eta|}{|2-b\eta|\Gamma(\alpha+\beta+\gamma+1)}\right.\\ &+\frac{|b|}{|2-b\eta|\Gamma(\alpha+\beta+\gamma+2)}+\frac{z_1z_3}{|2-b\eta||2z_2z_3-z_1|\Gamma(\alpha+\beta+\gamma-2\mu+1)}\\ &\times\left(\frac{|2-b\eta|+1}{z_5}+\frac{2z_2+2|2-b\eta|z_2}{z_4}+\frac{|b|\eta^{\gamma+1}}{z_6}+\frac{2|b|z_2\eta^{\beta+\gamma+1}}{z_7}\right) \qquad (2.5)\\ &+\frac{z_2}{|2-b\eta||2z_2z_3-z_1|\Gamma(\alpha+\beta+1)}\left(\frac{|2-b\eta|z_1+z_1}{z_4}+\frac{z_3+|2-b\eta|z_3}{z_5}\right.\\ &\left.\left.+\frac{|b|z_1\eta^{\beta+\gamma+1}}{z_7}+\frac{|b|z_3\eta^{\gamma+1}}{z_6}\right)\right]\|u_1-u_2\|_X.\end{aligned}$$

We have also

$$\begin{aligned}D^\gamma Hu(t) = & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-\tau)^{\alpha+\beta-1}\left[\frac{a_1f(\tau,u(\tau),D^\gamma u(\tau))+a_2g(\tau,D^\gamma u(\tau),D^\rho D^\gamma u(\tau))}{K(u(\tau))}\right.\\ &\left.+\frac{a_3h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau+\frac{z_1z_3}{2z_2z_3-z_1}\left(1-2t^\beta\right)\frac{1}{\Gamma(\alpha+\beta+\gamma-2\mu)}\\ &\times\int_0^1(1-\tau)^{\alpha+\beta+\gamma-2\mu-1}\left[\frac{a_1f(\tau,u(\tau),D^\gamma u(\tau))+a_2g(\tau,D^\gamma u(\tau),D^\rho D^\gamma u(\tau))}{K(u(\tau))}\right. \qquad (2.6)\\ &\left.+\frac{a_3h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau+\frac{z_2}{2z_2z_3-z_1}\left(\frac{z_1}{z_2}t^\beta-z_3\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(t-\tau)^{\alpha+\beta-1}\\ &\times\left[\frac{a_1f(\tau,u(\tau),D^\gamma u(\tau))+a_2g(\tau,D^\gamma u(\tau),D^\rho D^\gamma u(\tau))+a_3h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau;\end{aligned}$$

As before, we have

$$\begin{aligned}\|D^\gamma Hu_1-D^\gamma Hu_2\|_\infty \leq & \left[\frac{2|a_1|\|\phi\|+2|a_2|\|\psi\|+|a_3|\|\delta\|}{m}\right]\left[\frac{1}{\Gamma(\alpha+\beta+1)}\right.\\ &+\frac{z_1z_3}{|2z_2z_3-z_1|\Gamma(\alpha+\beta+\gamma-2\mu+1)}+\frac{z_2}{|2z_2z_3-z_1|\Gamma(\alpha+\beta+1)} \qquad (2.7)\\ &\left.\times\left(\frac{z_1}{z_2}+z_3\right)\right]\|u_1-u_2\|_X.\end{aligned}$$

Also, taking into account that

$$
\begin{aligned}
D^{\rho}D^{\gamma}Hu(t) \;&= \tfrac{1}{\Gamma(\alpha+\beta-\rho)}\int_0^t (t-\tau)^{\alpha+\beta-\rho-1}\left[\tfrac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))}{K(u(\tau))}\right.\\
&\left.+\tfrac{a_2 g(\tau,D^{\gamma}u(\tau),D^{\rho}D^{\gamma}u(\tau))+a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau+\tfrac{z_1 z_3}{2z_2z_3-z_1}\left(\tfrac{-2z_2}{\Gamma(\beta-\rho+1)}t^{\beta-\rho}\right)\\
&\times\tfrac{1}{\Gamma(\alpha+\beta+\gamma-2\mu)}\int_0^1(1-\tau)^{\alpha+\beta+\gamma-2\mu-1}\left[\tfrac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))}{K(u(\tau))}\right.\\
&\left.+\tfrac{a_2 g(\tau,D^{\gamma}u(\tau),D^{\rho}D^{\gamma}u(\tau))+a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau+\tfrac{z_2}{(2z_2z_3-z_1)\Gamma(\alpha+\beta)}\\
&\times\left(\tfrac{z_1}{\Gamma(\beta-\rho+1)}t^{\beta-\rho}\right)\int_0^1(1-\tau)^{\alpha+\beta-1}\left[\tfrac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))}{K(u(\tau))}\right.\\
&\left.+\tfrac{a_2 g(\tau,D^{\gamma}u(\tau),D^{\rho}D^{\gamma}u(\tau))+a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau,
\end{aligned}
\tag{2.8}
$$

so, the reader can observe that

$$
\begin{aligned}
\|D^{\rho}D^{\gamma}Hu_1-D^{\rho}D^{\gamma}Hu_2\|_{\infty} \;&\le \left[\tfrac{2|a_1|\|\phi\|+2|a_2|\|\psi\|+|a_3|\|\delta\|}{m}\right]\left[\tfrac{1}{\Gamma(\alpha+\beta-\rho+1)}\right.\\
&+\tfrac{z_1z_3}{|2z_2z_3-z_1|\Gamma(\alpha+\beta+\gamma-2\mu+1)}\left(\tfrac{2z_2}{\Gamma(\beta-\rho+1)}\right)\\
&\left.+\tfrac{z_2}{|2z_2z_3-z_1|\Gamma(\alpha+\beta+1)}\left(\tfrac{z_1}{\Gamma(\beta-\rho+1)}\right)\right]\|u_1-u_2\|_X.
\end{aligned}
\tag{2.9}
$$

Now, thanks to (2.5), (2.7) and (2.9), we find that

$$
\begin{gathered}
\|Hu_1-Hu_2\|_{\infty}\le\Delta_1\|u_1-u_2\|_X,\\
\|D^{\gamma}Hu_1-D^{\gamma}Hu_2\|_{\infty}\le\Delta_2\|u_1-u_2\|_X,\\
\|D^{\rho}D^{\gamma}Hu_1-D^{\rho}D^{\gamma}Hu_2\|_{\infty}\le\Delta_3\|u_1-u_2\|_X.
\end{gathered}
$$

Finally, we get

$$
\|Hu_1-Hu_2\|_X\le(\Delta_1+\Delta_2+\Delta_3)\|u_1-u_2\|_X.
$$

We deduce that H is contractive. We conclude that H has a unique fixed point which is the solution of (1.7).

□

Example
As illustrative example for the first part of our results, we consider the following problem:

$$\begin{cases} D^{\frac{17}{20}}D^{\frac{9}{10}}D^{\frac{13}{20}}u(t) = \dfrac{2f(t,u(t),D^{\frac{13}{20}}u(t)) + \frac{1}{3}g(t,u(t),D^{\frac{6}{10}}D^{\frac{13}{20}}u(t)) + \frac{1}{2}h(t,u(t))}{K(u)}, \\ \\ u(0)+u(1) = 5\int_0^{0.6} u(s)ds, \\ D^{\frac{13}{20}}u(0)+D^{\frac{13}{20}}u(1) = 0, \\ D^{\frac{11}{20}}D^{\frac{11}{20}}u(0)+D^{\frac{11}{20}}D^{\frac{11}{20}}u(1) = 0, \end{cases} \tag{2.10}$$

with the conditions:

$$f(t,u,v) = \frac{u^2}{2(t^3+115)(u^2+1)} + \frac{|v|}{(200+t)(1+|v|)},$$

$$g(t,u,v) = \frac{4|u|}{(e^t+25)^2-25} + \frac{cosv}{320+t^2},$$

$$h(t,u) = \frac{3|u|}{15(t^3+70)(1+|u|)},$$

$$k(u) = \frac{1+2u^2}{10(1+u^2)},$$

and
$\alpha = \frac{17}{20},\ \beta = \frac{9}{10},\ \gamma = \frac{13}{20},\ \rho = \frac{6}{10},\ b = \frac{3}{10},\ \mu = \frac{11}{20},\ \eta = 0.6,\ m = 0.1,$

$\Delta_1 = 0.5439,\ \ \Delta_2 = 0.0039,\ \ \Delta_3 = 0.0102,$

$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = 0.5580.$

The conditions of the first main result hold. Therefore, problem (2.10) has a unique solution on I.

3.2 Part 2: Traveling wave solutions by EFM method

The aim of this subsection is use the EFM method to study the following problem:

$$\left(T_x^{\beta}u(x,t)\right)\left(T_x^{\beta}\left(T_t^{\alpha}u(x,t)\right)\right) - \left(u(x,t)\right)\left(T_x^{2\beta}\left(T_t^{\alpha}u(x,t)\right)\right) = f\left(u, T_t^{\alpha}u\right),$$

where $T_x^{\beta}, T_t^{\alpha}$ are the conformable fractional derivative, with $0 < \alpha, \beta \leq 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function.

Note, when $\alpha = \beta = 1$ and $f(u,v) = u^2 v$, so problem (1.8) is transformed into the Ostrovsky equation [28]:

$$(u_x)(u_{xt}) - (u)(u_{xxt}) = \left(u^2\right)(u_t). \tag{3.1}$$

Now, we need to introduce the following preliminaries, see [1,9,18]:

Definition 3.2 *Let $f : (0,\infty) \to \mathbb{R}$. The conformable fractional derivative of order α is defined by*

$$(T^\alpha f)(t) = \frac{\partial^\alpha f(t,x)}{\partial t^\alpha} = \lim_{\varepsilon \to 0} \left(\frac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \right), \quad t > 0, \ \ 0 < \alpha \leq 1.$$

It is to note that when $\alpha = 1$, the above formula is reduced to the standard derivative of order one.

Definition 3.3 *The conformable fractional integral of a function $f : (0,\infty) \to \mathbb{R}$ of order α is defined as*

$$(J^\alpha f)(t) = \int_0^t \tau^{\alpha-1} f(\tau)\, d\tau, \qquad 0 < \alpha \leq 1.$$

The following properties are needed.

$$J^\alpha T^\alpha f(t) = f(t) - f(0)$$

and

$$(T^\alpha f)(t) = t^{1-\alpha} \frac{df(t)}{dt}.$$

The EFM method consists on considering the following problem

$$F\left(u, T_t^\alpha u, T_x^\beta u, T_t^\alpha (T_t^\alpha u), T_t^\alpha (T_x^\beta u), T_x^\beta (T_x^\beta u), \ldots\right) = 0, \tag{3.2}$$

where $T_t^\alpha u$ is the conformable fractional derivative of u of order $\alpha, 0 < \alpha \leq 1$.
It consists then on introducing a function $U(\xi)$ as follows:

$$U(\xi) = u(t,x), \tag{3.3}$$

with:

$$\xi = \frac{k}{\alpha} t^\alpha + \frac{\omega}{\beta} x^\beta, \tag{3.4}$$

where k and ω are constants.
So, the above FPDE will be converted, after that, to:

$$G\left(U, U', U'', U''', \ldots\right) = 0. \tag{3.5}$$

The EFM method is based on the fact that travelling wave solutions (as in [15]) can be expressed as follows:

$$U(\zeta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\zeta)}{\sum_{m=-p}^{q} b_m \exp(m\zeta)}. \tag{3.6}$$

We suppose that the solution of $Eq(3.5)$ can be expressed as:

$$U(\zeta) = \frac{a_d \exp(d\zeta)+...+a_{-c}\exp(-c\zeta)}{b_q \exp(q\zeta)+...+b_{-p}\exp(-p\zeta)}, \tag{3.7}$$

we can write $Eq(3.7)$ in the following form

$$U(\zeta) = \frac{a_c \exp(c\zeta)+...+a_{-d}\exp(-d\zeta)}{b_p \exp(p\zeta)+...+b_{-q}\exp(-q\zeta)}. \tag{3.8}$$

To determine the value of c and p, we balance the linear term of highest order of $Eq(3.5)$ with the highest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of $Eq(3.5)$ with lowest order nonlinear term.
After this separated algebraic equation, we can find a_n and b_m constants.

3.2.1 Two examples for finding traveling wave solutions

We first consider the problem under the form:

$$\left(T_x^{\beta} u(x,t)\right)\left(T_x^{\beta}\left(T_t^{\alpha} u(x,t)\right)\right) - \left(u(x,t)\right)\left(T_x^{2\beta}\left(T_t^{\alpha} u(x,t)\right)\right) = f\left(u, T_t^{\alpha} u\right),$$

where $T_x^{\beta}, T_t^{\alpha}$ are the conformable fractional derivatives, with $0 < \alpha, \beta \leq 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function.

So, we introduce the following transformations

$$u(t,x) = U(\zeta), \tag{3.9}$$

$$\zeta = \frac{k}{\alpha} t^{\alpha} + \frac{\omega}{\beta} x^{\beta}, \tag{3.10}$$

where k and ω are constants.
Substituting (3.10) into (1.8) can reduce (1.8) into the following ODE:

$$k\omega^2 \left(U_{\zeta}\right)\left(U_{\zeta\zeta}\right) - k\omega^2 U U_{\zeta\zeta\zeta} = f(U, kU_{\zeta}). \tag{3.11}$$

According to [15], we suppose that the solutions of $Eq(1.8)$ are given in the following form:

$$U(\zeta) = \frac{a_d \exp(d\zeta)+...+a_{-c}\exp(-c\zeta)}{b_q \exp(q\zeta)+...+b_{-p}\exp(-p\zeta)}, \tag{3.12}$$

Example 1:

We consider the following problem:

$$\left(T_x^{\beta}u\right)\left(T_x^{\beta}\left(T_t^{\alpha}u\right)\right)-\left(u\right)\left(T_x^{2\beta}\left(T_t^{\alpha}u\right)\right)=u^2T_t^{\alpha}u. \tag{3.13}$$

Substituting (3.10) into (3.13), so (3.13) is given by:

$$\left(U_{\zeta}\right)\left(U_{\zeta\zeta}\right)-UU_{\zeta\zeta\zeta}=\tfrac{1}{\omega^2}U^2U_{\zeta}. \tag{3.14}$$

Therefore,

$$\left(U_{\zeta}\right)^2-UU_{\zeta\zeta}-\tfrac{1}{3\omega^2}U^3=0. \tag{3.15}$$

Now, we can set $p=c=q=d=1$, then (3.12) is transformed to

$$U\left(\zeta\right)=\tfrac{a_1\exp(\zeta)+a_0+a_{-1}\exp(-\zeta)}{b_1\exp(\zeta)+b_0+b_{-1}\exp(-\zeta)}. \tag{3.16}$$

Putting $Eq(3.16)$ into $Eq(3.15)$ so this yields a set of algebraic equations for a_0,a_1, a_{-1},b_0,b_1,b_{-1},k and ω. We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of $Eq(3.13)$ as follows:

Family 1

$$a_1=0,a_0=\tfrac{1}{a_{-1}}\sqrt{-\tfrac{1}{a_{-1}}},a_{-1}=a_{-1},$$

$$b_1=0,b_0=\tfrac{1}{a_{-1}}\sqrt{-\tfrac{1}{a_{-1}}},b_{-1}=0,$$

$$U\left(\zeta\right)=1+\tfrac{a_{-1}^2}{\sqrt{-\frac{1}{a_{-1}}}}\exp\left(-\zeta\right).$$

Family 2

$$a_1=a_1,a_0=0,a_{-1}=0,$$

$$b_1=b_1,b_0=0,b_{-1}=0,$$

$$U\left(\zeta\right)=\tfrac{a_1}{b_1}.$$

Family 3

$$a_1=\tfrac{b_1a_{-1}}{b_{-1}},a_0=0,a_{-1}=a_{-1},$$

$$b_1=b_1,b_0=0,b_{-1}=b_{-1},$$

$$U\left(\zeta\right)=\tfrac{\frac{b_1a_{-1}}{b_{-1}}\exp(\zeta)+a_{-1}\exp(-\zeta)}{b_1\exp(\zeta)+b_{-1}\exp(-\zeta)}.$$

Fig.1 presents the graph of solution (Family 1) for Eq(3.13) with $a_{-1}=-3$, $k=2,\omega=1,\alpha=\frac{3}{4}$ and $\beta=\frac{3}{5}$.

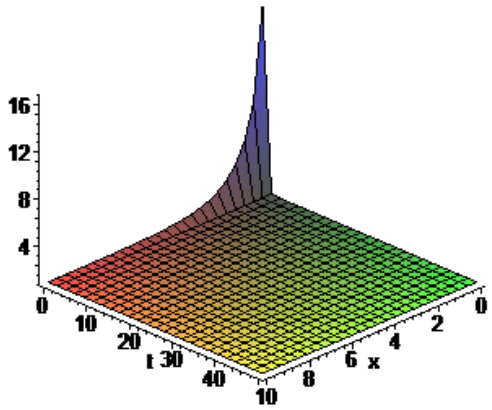

Fig. 1 *3D plot of traveling wave solution (Family 1) of (3.13) sketched within the intervals* $0 \leq x \leq 10$ *and* $0 \leq t \leq 50$.

Example 2:

We consider the following problem:

$$\left(T_x^{\beta} u\right)\left(T_x^{\beta}\left(T_t^{\alpha} u\right)\right) - (u)\left(T_x^{2\beta}\left(T_t^{\alpha} u\right)\right) = 0. \tag{3.17}$$

Substituting (3.10) into (3.17), this can reduce (3.17) to the equation

$$\left(U_{\zeta}\right)\left(U_{\zeta\zeta}\right) - UU_{\zeta\zeta\zeta} = 0, \tag{3.18}$$

and by integrating of (3.18), we obtain

$$\left(U_{\zeta}\right)^2 - UU_{\zeta\zeta} = 0. \tag{3.19}$$

Hence, we get

$$U(\zeta) = \frac{a_1 \exp(\zeta) + a_0 + a_{-1}\exp(-\zeta)}{b_1 \exp(\zeta) + b_0 + b_{-1}\exp(-\zeta)}, \tag{3.20}$$

By substitution as before, we have:

$$\begin{cases} -4a_1a_{-1}b_0^2+4b_1b_{-1}a_0^2=0, \\ -a_{-1}a_0b_1^2+b_0b_{-1}a_1^2+b_1b_0a_0^2+6a_1a_0b_1b_{-1}-6a_1a_{-1}b_1b_0-a_1a_0b_0^2=0, \\ b_1b_0a_{-1}^2+b_1b_{-1}a_0^2+6a_{-1}a_0b_1b_{-1}-6a_{-1}a_1b_0b_{-1}-6a_{-1}b_0^2-a_1a_0b_{-1}^2=0, \\ -4a_1a_{-1}b_1^2+4b_1b_{-1}a_1^2=0, \\ 4b_1b_{-1}a_{-1}^2-4a_1a_{-1}b_{-1}^2=0, \\ -a_0a_1b_1^2+b_1b_0a_1^2=0, \\ b_0b_{-1}a_{-1}^2-a_{-1}a_0b_{-1}^2=0, \end{cases} \tag{3.21}$$

where $A=b_1\exp(\zeta)+b_0+b_{-1}\exp(-\zeta)$.
Solving the system, we find the sets:
Family 1

$$a_1=a_1,a_0=0,a_{-1}=0,$$

$$b_1=0,b_0=b_0,b_{-1}=0,$$

$$u(x,t)=\tfrac{a_1}{b_0}\exp\left(\tfrac{k}{\alpha}t^\alpha+\tfrac{\omega}{\beta}x^\beta\right).$$

Family 2

$$a_1=a_1,a_0=0,a_{-1}=0,$$

$$b_1=0,b_0=0,b_{-1}=b_{-1},$$

$$u(x,t)=\tfrac{a_1}{b_{-1}}\exp\left(\tfrac{2k}{\alpha}t^\alpha+\tfrac{2\omega}{\beta}x^\beta\right).$$

Family 3

$$a_1=\tfrac{a_0b_0}{b_{-1}},a_0=a_0,a_{-1}=0,$$

$$b_1=0,b_0=b_0,b_{-1}=b_{-1},$$

$$u(x,t)=\frac{a_0+\frac{a_0b_0}{b_{-1}}\exp\left(\frac{k}{\alpha}t^\alpha+\frac{\omega}{\beta}x^\beta\right)}{b_0+b_{-1}\exp\left(-\frac{k}{\alpha}t^\alpha-\frac{\omega}{\beta}x^\beta\right)}.$$

Family 4

$$a_1=0,a_0=0,a_{-1}=a_{-1},$$

$$b_1=b_1,b_0=0,b_{-1}=0,$$

$$u(x,t) = \frac{a_{-1}}{b_1} \exp\left(-\frac{2k}{\alpha}t^{\alpha} - \frac{2\omega}{\beta}x^{\beta}\right).$$

Family 5

$$a_1 = 0, a_0 = a_0, a_{-1} = 0,$$

$$b_1 = b_1, b_0 = 0, b_{-1} = 0,$$

$$u(x,t) = \frac{a_0}{b_1} \exp\left(-\frac{k}{\alpha}t^{\alpha} - \frac{\omega}{\beta}x^{\beta}\right)$$

Family 6

$$a_1 = 0, a_0 = \frac{a_{-1}b_1}{b_0}, a_{-1} = a_{-1},$$

$$b_1 = b_1, b_0 = b_0, b_{-1} = 0,$$

$$u(x,t) = \frac{\frac{a_{-1}b_1}{b_0} + a_{-1}\exp\left(-\frac{k}{\alpha}t^{\alpha} - \frac{\omega}{\beta}x^{\beta}\right)}{b_1\exp\left(\frac{k}{\alpha}t^{\alpha} + \frac{\omega}{\beta}x^{\beta}\right) + b_0}.$$

Family 7

$$a_1 = a_1, a_0 = 0, a_{-1} = 0,$$

$$b_1 = b_1, b_0 = 0, b_{-1} = 0,$$

$$u(x,t) = \frac{a_1}{b_1}.$$

Family 8

$$a_1 = \frac{a_0 b_1}{b_0}, a_0 = a_0, a_{-1} = 0,$$

$$b_1 = b_1, b_0 = b_0, b_{-1} = 0,$$

$$u(x,t) = \frac{\frac{a_0 b_1}{b_0}\exp\left(\frac{k}{\alpha}t^{\alpha} + \frac{\omega}{\beta}x^{\beta}\right) + a_0}{b_1\exp\left(\frac{k}{\alpha}t^{\alpha} + \frac{\omega}{\beta}x^{\beta}\right) + b_0}.$$

Family 9

$$a_1 = \frac{a_{-1}b_1}{b_{-1}}, a_0 = \frac{a_{-1}b_0}{b_{-1}}, a_{-1} = a_{-1},$$

$$b_1 = b_1, b_0 = b_0, b_{-1} = b_{-1},$$

$$u(x,t) = \frac{\frac{a_{-1}b_1}{b_{-1}}\exp\left(\frac{k}{\alpha}t^{\alpha} + \frac{\omega}{\beta}x^{\beta}\right) + \frac{a_{-1}b_0}{b_{-1}} + a_{-1}\exp\left(-\frac{k}{\alpha}t^{\alpha} - \frac{\omega}{\beta}x^{\beta}\right)}{b_1\exp\left(\frac{k}{\alpha}t^{\alpha} + \frac{\omega}{\beta}x^{\beta}\right) + b_0 + b_{-1}\exp\left(-\frac{k}{\alpha}t^{\alpha} - \frac{\omega}{\beta}x^{\beta}\right)}.$$

Fig. 2 presents the graph of solution (Family 9) for Eq(3.17) with $a_{-1} = 1$, $b_1 = -2, b_0 = 1, b_{-1} = -1, k = 1, \omega = 2, \alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$.

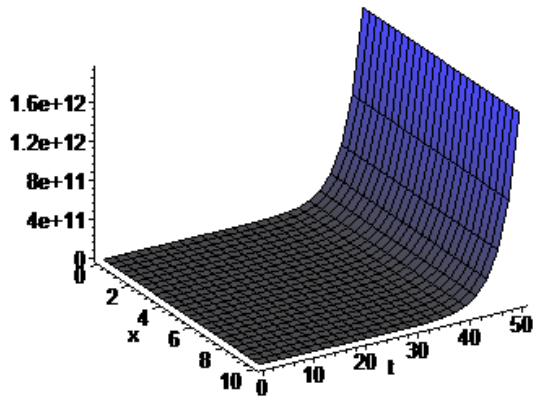

Fig. 2 *3D plot of traveling wave solution (Family 9) of (3.17) sketched within the intervals* $0 \leq x \leq 10$ *and* $0 \leq t \leq 50$.

4 Conclusion

We have proposed the new general fractional problem (1.7) by using Caputo approach. By application of Banach contraction principle, an existence and uniqueness result has been established. To show the applicability of the existence and uniqueness main result, an illustrative example has been discussed. The considered problem (1.7) has allowed us, under some particular case on its input data, to be concerned with the existence of some relationship with (1.7) and the second part of this work. In fact, in the second part, we have used Khalil approach and the EFM Method to find new travelling wave solutions for some interesting fractional differential equations that are similar to (1.8), we cite Ostrovsky and KdV type equations. Some graphics on the obtained traveling waves have been plotted. It has been found that the EFM method is a useful method for finding new travelling wave solutions for any nonlinear fractional differential equations using Khalil derivatives. In the future, we will be concerned with the stability analysis of the problem (1.7).

References

1. ABDELJAWAD, T. – *On conformable fractional calculus*, J. Comput. Appl. Math, **279** (2015), 57-66.
2. ASKARI, H.; ANSARI, A. – *Fractional Ostrovsky equation*, Comput. Methods Differ. Equ., **8** (2020), no. 4, 648-660.
3. BATIHA, I.M.; OUDETALLAH, J.; OUANNAS, A.; AL-NANA, A.A.; JEBRIL, I.H. – *Tuning the fractional-order PID-controller for blood glucose level of diabetic patients*, International Journal of Advances in Soft Computing and its Applications, **13** (2021), no. 2, 1-10.
4. BEZZIOU, M.; JEBRIL, I.; DAHMANI, Z. – *A new nonlinear Duffing system with sequential fractional derivatives*, Chaos Solitons Fractals, **151** (2021), 1-7.

5. BOYD, J.P.; CHEN, G.Y. – *Five regimes of the quasi-cnoidal, steadily translating waves of the ratation-modified Korteweg-de Vries ("Ostrovsky") equation*, Wave Motion, **35** (2002), no. 2, 141-155.
6. COFFEY, W.T.; KALMYKOV, Y.P.; WALDRON, J.T. – *The Langevin Equation. With applications to stochastic problems in physics, chemistry and electrical engineering, Second edition*, World Scientific Series in Contemporary Chemical Physics, 14, World Scientific Publishing Co., Inc., River Edge, NJ (2004).
7. DAHMANI, Z.; ABDELLAOUI, M.A.; HOUAS, M. – *Coupled systems of fractional integro-differential equations involving several functions*, Theory Appl. Math. Comput. Sci., **5** (2015), no. 1, 53-61.
8. DAHMANI, Z.; BAHOUS, Y.; BEKKOUCHE, Z. – *A two parameter singular fractional differential equation of Lane Emden type*, Turkish J. Ineq, **3** (2019), no. 1, 35-53.
9. DAHMANI, Z.; ANBER, A.; GOUARI, Y.; KAID, M.; JEBRIL, I. – *Extension of a Method for Solving Nonlinear Evolution Equations Via Conformable Fractional Approach*, International Conference on Information Technology, (2021), 38-42.
10. EBAID, A. – *An improvement on the Exp-function method when balancing the highest order linear and nonlinear terms*, J. Math. Anal. Appl, **392** (2012), 1-5.
11. EL BAHI, M.I.; HILAL, K. – *Lie symmetry analysis, exact solutions, and conservation laws for the generalized time-fractional KdV-like equation*, J. Funct. Spaces, (2021).
12. GIL'MAN, O.A.; GRIMSHAW, R.; STEPAN'YANTS, Y.A. – *Approximate analytical and numerical solutions of the stationary Ostrovsky equation*, Stud. Appl. Math, **95** (1995), no. 1, 115-126.
13. GOUARI, Y.; DAHMANI, Z.; FAROOQ, S.E.; AHMAD, F. – *Fractional Singular Differential Systems of Lane-Emden Type: Existence and Uniqueness of Solutions*, Axioms, **9** (2020), no. 3, 1-18.
14. GOUARI, Y.; DAHMANI, Z.; SARIKAYA, M.Z. – *A nonlocal multi-point singular fractional integro-differential problem of Lane-Emden type*, Math. Methods Appl. Sci., **43** (2020), no. 11, 6938-6949.
15. HE, J.H.; WU, X.H. – *Exp-function method for nonlinear wave equations*, Chaos Solitons Fractals, **30** (2006), no. 3, 700-708.
16. HE, J.H.; ABDOU, M.A. – *New periodic solutions for nonlinear evolution equations using Exp-function method*, Chaos Solitons Fractals, **34** (2007), no. 5, 1421-1429.
17. HE, J.H. – *Exp-function method for fractional differential equations*, Int. J. Nonlinear Sci. Numer. Simul, **14** (2013), no. 6, 363-366.
18. KHALIL, R.; AL HORANI, M.; YOUSEF, A.; SABABHEH, M. – *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65-70.
19. KILBAS, A.A.; SRIVASTAVA, H.M.; TRUJILLO, J.J. – *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam (2006).
20. KÖROĞLU, C.; ÖZIŞ, T. – *A novel traveling wave solution for Ostrovsky equation using Exp-function method*, Comput. Math. Appl., **58** (2009), no. 11-12, 2142-2146.
21. LI, C.; SARWAR, S. – *Existence and continuation of solutions for Caputo type fractional differential equations*, Electron. J. Differential Equations, (2016), 1-14.
22. B. LU – *The first integral method for some time fractional differential equations*, J. Math. Anal. Appl., **395** (2012), no. 2, 684-693.
23. MILLER, K.S.; ROSS, B. – *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, Inc., New York (1993).
24. NAVICKAS, Z.; RAGULSKIS, M.; BIKULCIENE, L. – *Be careful with the Exp-function method additional remarks*, Commun. Nonlinear Sci. Numer. Simul, **15** (2010), no. 12, 3874-3886.
25. NOOR, M.A.; MOHYUD-DIN, S.T.; WAHEED, A. – *Exp-function method for generalized traveling solutions of master partial differential equation*, Acta Appl. Math, **104** (2008), no. 2, 131-137.
26. OSTROVSKY, L.A. – *Nonlinear internal waves in a rotating ocean*, Oceanology, **18** (1978), no. 2, 119-125.
27. STEPANYANTS, Y.A. – *On stationary solutions of the reduced Ostrovsky equation: periodic waves, compactons and compound solitons*, Chaos Solitons Fractals, **28** (2006), no. 1, 193-204.
28. TALARPOSHTI, R.A.; GHASEMI, S.E.; RAHMANI, Y.; GANJI, D.D. – *Application of Exp-function method to wave solutions of the sine-Gordon and Ostrovsky equations*, Acta Math. Appl. Sin. Engl. Ser., **32** (2016), 571-578.
29. VAKHNENKO, V.O.; PARKES, E.J. – *The two loop soliton solution of the Vakhnenko equation*, Nonlinearity, **11** (1998), no. 6, 1457-1464.

30. TANG, B.; HE, Y.; WEI, L.; ZHANG, X. – *A generalized fractional sub-equation method for fractional differential equations with variable coefficients*, Phys. Lett. A, **376** (2012), no. 38-39, 2588-2590.
31. ZHANG, S.; ZONG, Q.-A.; LIU, D.; GAO, Q. – *A generalized exp function method for fractional riccati differential equations*, Communications in Fractional Calculus, **1** (2010), 48-51.
32. ZHANG,Y. – *Solution of the KdV equation with fractional time derivative via variational method*, Electron. J. Differential Equations, **66** (2014), 1-12.
33. ZHENG, B. – *(G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics*, Commun. Theor. Phys. (Beijing), **58** (2012), no. 5, 623-630.

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