Approximation by means of Fourier trigonometric series in Morrey spaces

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Abstract In the present work we investigate the approximation of the functions by Cesaro, Zygmund and Abel-Poisson means of Fourier series in Morrey spaces $L^{p,\lambda}(T)$, $0 < \lambda \leq 2$, $1 < p < \infty$ in the terms of the modulus of continuity. These results applied to the estimates of approximation of Cesaro, Zygmund and Abel sums of Faber series in the Morrey-Smirnov classes $E^{p,\lambda}(G)$, $0 < \lambda \leq 2$ and $1 < p < \infty$ defined in the domains with Dini-smooth boundary of the complex plane.

Keywords Morrey spaces · Morrey-Smirnov classes · trigonometric polynomials · modulus of continuity · Cesaro mean · Zygmund mean · Abel-Poisson mean

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1 Introduction, some auxiliary results and main results

Let $T$ denote the interval $[0,2\pi]$. Let $L^p(T)$, $1 \leq p < \infty$ be the Lebesgue space of all measurable $2\pi$-periodic functions defined on $T$ such that

$$
\|f\|_p := \left(\int_T |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.
$$

The Morrey spaces $L^{p,\lambda}_0(T)$ for a given $0 \leq \lambda \leq 2$ and $p \geq 1$, we define as the set of functions $f \in L^p_{\text{loc}}(T)$ such that

$$
\|f\|_{L^{p,\lambda}_0(T)} := \left\{ \sup_I \frac{1}{|I|^{1-\frac{\lambda}{p}}} \int_I |f(t)|^p \, dt \right\}^{\frac{1}{p}} < \infty,
$$

where the supremum is taken over all intervals $I \subset [0,2\pi]$. Note that $L^{p,\lambda}_0(T)$ becomes a Banach spaces, $\lambda = 2$ coincides with $L^p(T)$ and for
\[ \lambda = 0 \text{ with } L^\infty (\mathbb{T}). \] If \( 0 \leq \lambda_1 \leq \lambda_2 \leq 2 \), then \( L^{p,\lambda_1}_0 (\mathbb{T}) \subset L^{p,\lambda_2}_0 (\mathbb{T}) \). Also, if \( f \in L^{p,\lambda}_0 (\mathbb{T}) \), then \( f \in L^p (\mathbb{T}) \) and hence \( f \in L^1 (\mathbb{T}) \). The Morrey spaces were introduced by C. B. Morrey in 1938. The properties of these spaces have been investigated intensively by several authors and together with weighted Lebesgue spaces \( L^p_\omega \) play an important role in the theory of partial equations, in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces \( L^p \). The detailed information about properties of the Morrey spaces can be found in [6]-[10], [14], [17]-[20], [22], [31], [39], [41], [43]-[45], [47] and [49].

Denote by \( C^\infty (\mathbb{T}) \) the set of all functions that are realized as the restriction to \( \mathbb{T} \) of elements in \( C^\infty (\mathbb{T}) \). We define \( L^{p,\lambda}_0 (\mathbb{T}) \) to be closure of \( C^\infty (\mathbb{T}) \) in \( L^{p,\lambda}_0 (\mathbb{T}) \). \( L^{p,\lambda}_0 (\mathbb{T}) \) is modified Morrey spaces which contains the set of trigonometric polynomials as a dense subset.

We define Steklov means \( f_h \) by

\[
f_h (x) := \frac{1}{2h} \int_{-h}^{h} f (x + t) \, dt, \quad 0 < h < \pi, \ x \in \mathbb{T}.
\]

According to [12] the inequality

\[
\| f_h \|_{L^{p,\lambda}(\mathbb{T})} \leq c \| f \|_{L^{p,\lambda}(\mathbb{T})}
\]

holds. Hence the operator \( f_h \) is bounded in the space \( L^{p,\lambda}(\mathbb{T}) \).

The function

\[
\Omega_{p,\lambda}(\delta, f) := \sup_{|h| \leq \delta} \| f - f_h \|_{L^{p,\lambda}(\mathbb{T})}, \quad \delta > 0
\]

is called the modulus of continuity of \( f \in L^{p,\lambda}(\mathbb{T}), \) \( 0 \leq \lambda \leq 2 \) and \( p \geq 1 \).

The modulus of continuity \( \Omega_{p,\lambda}(\delta, f) \) is a nondecreasing, nonnegative, continuous function and

\[
\Omega_{p,\lambda}(\delta, f + g) \leq \Omega_{p,\lambda}(\delta, f) + \Omega_{p,\lambda}(\delta, g)
\]

for \( f, g \in L^{p,\lambda}(\mathbb{T}), \) \( 0 \leq \lambda \leq 2 \) and \( p \geq 1 \).

Let

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x, f)
\]

be the Fourier series of the function \( f \in L^1 (\mathbb{T}), \) where \( A_k(x, f) := (a_k(f) \cos kx + b_k(f) \sin kx) \), \( a_k(f) \) and \( b_k(f) \) are Fourier coefficients of the function \( f \in L^1 (\mathbb{T}). \)
The $n$–th partial sums, Cesaro means, Zygmund means of order 2 and Abel- Poisson means of the series (1.1) are defined, respectively as

$$S_n(x, f) = \frac{a_0}{2} + \sum_{\nu=1}^{n} A_{\nu}(x, f),$$

$$\sigma_n(x, f) = \frac{1}{n+1} \sum_{\nu=1}^{n} S_{\nu}(x, f),$$

$$Z_{n,2}(x, f) = \frac{a_0}{2} + \sum_{\nu=1}^{n} \left(1 - \frac{\nu^2}{(n+1)^2}\right) A_{\nu}(x, f),$$

$$U_r(x, f) = \frac{1}{2\pi} \int_{\mathbb{T}} P_r(x-t)f(t)\,dt,$$

where

$$P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}, 0 \leq r < 1$$

is the Poisson kernel.

It is clear that

$$S_0(x, f) = Z_{0,2}(x, f) = \frac{a_0}{2}.$$

The best approximation to $f \in L^{p,\lambda} (\mathbb{T})$, $0 < \lambda \leq 2$, $1 < p < \infty$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding $n$ is defined by

$$E_n(f)_{L^{p,\lambda} (\mathbb{T})} := \inf \left\{ \|f - T_n\|_{L^{p,\lambda} (\mathbb{T})} : T_n \in \prod_n \right\}$$

Let $G$ be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve $\Gamma$, and let $G^- := ext\, \Gamma$. Further let

$$\Gamma_r := \{ w \in \mathbb{C} : |w| = 1 \}, \quad D := int\, T \quad and \quad D^- := ext\, T.$$

Let $w = \varphi(z)$ be the conformal mapping of $G^-$ onto $D^-$ normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0,$$

and $\psi$ stands for the inverse of $\varphi$.

Let $w = \varphi_1(z)$ indicate a function that maps the domain $G$ conformally onto the disk $|w| < 1$. The inverse mapping of $\varphi_1$ will be shown by $\psi_1$. Let $\Gamma_r$ be the image of the circle $|\varphi_1(z)| = r$, $0 < r < 1$ under the mapping $z = \psi_1(w)$. 

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Let us denote by $E_p$, where $p > 0$, the class of all functions $f(z) \neq 0$ that are analytic in $G$ and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p |dz|$$

is uniformly bounded for $0 < r < 1$. We shall call the $E_p$-class the Smirnov class. If the function $f(z)$ belongs to $E_p$, then $f(z)$ has definite limiting values $f(z')$ almost everywhere on $\Gamma$, over all nontangential paths; $|f(z')|$ is summable on $\Gamma$; and

$$\lim_{r \to 1} \int_{\Gamma_r} |f(z)|^p |dz| = \int_{\Gamma} |f(z')|^p |dz'|.$$

It is known that $\phi' = E_1(G^-)$ and $\psi' \in E_1(D^-)$. Note that the general information about Smirnov classes can be found in [15] (pp. 168-185).

Let $\Gamma$ be a rectifiable Jordan curve in the complex plane. The Morrey spaces $L^{p,\lambda}(\Gamma)$ for a given $0 \leq \lambda \leq 2$ and $p \geq 1$, we define as the set of functions $f \in L^{p,\lambda}_{\text{loc}}(\Gamma)$ such that

$$\|f\|_{L^{p,\lambda}(\Gamma)} := \left\{ \sup_{F} \frac{1}{|F \cap \Gamma|^{1-\frac{\lambda}{2}}} \int_{F} |f(z)|^p |dz| \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all disks $F$ centered on $\Gamma$. Let $G := \text{int} \Gamma$ and $L^{p,\lambda}(\Gamma), 0 \leq \lambda \leq 2$ and $1 < p < \infty$, is a Morrey space defined on $\Gamma$.

We also define the Morrey-Smirnov classes $E^{p,\lambda}(G)$ as

$$E^{p,\lambda}(G) := \left\{ f \in E_1(G) : f \in L^{p,\lambda}(\Gamma) \right\}.$$

Hence for $f \in E^{p,\lambda}(G)$ we can define the $E^{p,\lambda}(G)$ norm as

$$\|f\|_{E^{p,\lambda}(G)} := \|f\|_{L^{p,\lambda}(\Gamma)}.$$

Note that if $G = D = \{ z : |z| < 1 \}$, then we have the space $H^{p,\lambda}(D) := E^{p,\lambda}(D)$. The space $H^{p,\lambda}(D)$ is called Morrey-Hardy space on the unit disk $D$. For $f \in L^{p,\lambda}(\Gamma)$ we define the function

$$f_0(t) := f(\psi(t)), \ t \in \mathbb{T}.$$

Let $h$ be a continuous function on $[0,2\pi]$. Its modulus of continuity is defined by

$$\omega(t,h) := \sup \{|h(t_1) - h(t_2)| : t_1, t_2 \in [0,2\pi], |t_1 - t_2| \leq t\}, \ t \geq 0.$$

The curve $\Gamma$ is called Dini-smooth if it has a parameterization

$$\Gamma : \varphi_0(s), 0 \leq s \leq 2\pi$$
such that $\varphi'_0(s)$ is Dini-continuous, i.e.

$$\int_0^\pi \frac{\omega(\varphi'_0, t)}{t} dt < \infty$$

and $\varphi'_0(s) \neq 0$ [46], p. 48.

If $\Gamma$ is a Dini-smooth curve, then there exist [56] the constants $c_1$ and $c_2$ such that

$$0 < c_1 \leq |\psi'(t)| \leq c_2 < \infty, \; |t| > 1.$$  \hspace{1cm} (1.2)

Note that if $\Gamma$ is a Dini-smooth curve, then by (1.2) we have $f_0 \in L^{p,\lambda} (T)$ if $f \in L^{p,\lambda} (\Gamma)$.

Let $\varphi_k(z)$, $k = 0, 1, 2, ...$ be the Faber polynomials for $G$. The Faber polynomials $\varphi_k(z)$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{\varphi_k(z)}{t^{k+1}}, \; z \in G, \; t \in D^-$$  \hspace{1cm} (1.3)

and the equalities

$$\varphi_k(z) = \varphi^k(z) + \frac{1}{2\pi i} \int_T \frac{\varphi^k(s)}{s - z} ds, \; z \in G^-, \; k = 0, 1, 2, ...$$

hold [48], p. 33-38.

Let $f \in E^{p,\lambda}(G)$. Since $f \in E_1(G)$, we have

$$f(z) = \frac{1}{2\pi i} \int_T \frac{f(s) ds}{s - z} = \frac{1}{2\pi i} \int_T \frac{f(\psi(w)) \psi'(w)}{\psi(w) - z} dw,$$  \hspace{1cm} (1.4)

for every $z \in G$. Considering formula (1.3) and expansion (1.4), we can associate with $f$ the formal series [30]

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f) \varphi_k(z), \; z \in G,$$  \hspace{1cm} (1.5)

where

$$a_k(f) := \frac{1}{2\pi i} \int_T \frac{f(\psi(w))}{w^{k+1}} dw, \; k = 0, 1, 2, ...$$

This series is called the Faber series expansion of $f$, and the coefficients $a_k(f)$, $k = 0, 1, 2, ...$ are said to be the Faber coefficients of $f$.  

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The $n$-th partial sums, Cesaro sums, Zygmund sums of order 2 and Abel sum of the series (1.5) are defined, respectively, as

\[ S_n(z, f) = \sum_{\nu=0}^{n} a_\nu(f) \varphi_\nu(z), \]
\[ \sigma_n(z, f) = \frac{1}{n+1} \sum_{\nu=0}^{n} S_\nu(z, f), \]
\[ Z_{n,2}(z, f) = \frac{a_0}{2} + \sum_{\nu=0}^{n} \left( 1 - \frac{\nu^2}{(n+1)^2} \right) a_k(f) \varphi_k(z), \]
\[ U_r(z, f) \sim \sum_{\nu=0}^{\infty} r^\nu a_\nu(f) \varphi_\nu(z), \quad 0 \leq r < 1. \]

Note that if $f \in E^{p,\lambda}(G)$, $0 < \lambda \leq 2$ and $1 < p < \infty$, then by [29] and [30] the function

\[ f_0^+(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f_0(t)}{t-w} dt, \quad w \in D \]

belongs to $H^{p,\lambda}(D)$.

Let $\Gamma$ be a Dini-smooth curve. We define the modulus of continuity of the function $f \in E^{p,\lambda}(G)$, $0 < \lambda \leq 2$ and $1 < p < \infty$ by

\[ \Omega_{\Gamma, p,\lambda}(f, \delta) := \Omega_{p,\lambda}(f_0^+, \delta), \quad \delta > 0. \]

Let $\mathcal{P} := \{ \text{all polynomials (with no restriction on the degree)} \}$, and let $\mathcal{P}(D)$ be the set of traces of members of $\mathcal{P}$ on $D$. We define the operator $T$ as follows:

\[ T := \mathcal{P}(D) \rightarrow E^{p,\lambda}(G), \]
\[ T(P)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{P(w)\psi(w)}{\psi(w) - z} dt, \quad z \in G. \]

Then taking into account (1.4) and (1.5) we have

\[ T \left( \sum_{k=0}^{n} b_k w^k \right) = \sum_{k=0}^{n} b_k \varphi_k(z), \quad z \in G. \]

We use the constants $c, c_1, c_2, \ldots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.
The approximation of the functions by trigonometric polynomials in non-weighted and weighted Morrey spaces have been investigated in [3]-[5], [12], [13], [21], [29], [30], [34], [35], and [40]. In this work, the order of approximation of Cesaro, Zygmund and Abel-Poisson means of Fourier trigonometric series were estimated by the modulus of continuity in Morrey spaces. Later, these results were applied to estimate the rate of approximation of Cesaro, Zygmund and Abel sums of Faber series in Morrey-Smirnov classes defined on simply connected domains of the complex plane. Similar results in different spaces were studied by several authors (see, for example, [1], [2], [16], [23]-[28], [32], [33], [37], [38], [50]-[55]).

Our main results are the following.

**Theorem 1.1.** Let $f \in L^{p,\lambda}(T)$, $0 < \lambda \leq 2$ and $1 < p < \infty$. Then the inequality

$$\|f - \sigma_n(\cdot, f)\|_{L^{p,\lambda}(T)} \leq c_3 \Omega_{p,\lambda}(\frac{1}{n}, f)$$

holds with a constant $c_3 > 0$, not depend on $n$.

**Theorem 1.2.** Under the conditions of Theorem 1.3, there is a constant $c_4 > 0$, not depend on $n$, such that the inequality

$$\|f - Z_{n,2}(\cdot, f)\|_{L^{p,\lambda}(T)} \leq c_4 \Omega_{p,\lambda}(\frac{1}{n}, f)$$

holds.

**Theorem 1.3.** Under the conditions of Theorem 1.3, there is a constant $c_5 > 0$, not depend on $r$, such that the inequality

$$\|f - U_r(\cdot, f)\|_{L^{p,\lambda}(\Gamma)} \leq c_5 \frac{1}{1 - r} \Omega_{p,\lambda}(1 - r, f)$$

holds.

**Theorem 1.4.** Let $\Gamma$ be a Dini-smooth curve. Then for $f \in E^{p,\lambda}(G)$, $0 < \lambda \leq 2$, $p > 1$ the inequality

$$\|f - \sigma_n(\cdot, f)\|_{L^{p,\lambda}(G)} \leq c_6 n \Omega_{G, p,\lambda}(\frac{1}{n}, f)$$

holds with a constant $c_6 > 0$ independent of $n$.

**Theorem 1.5.** Let $\Gamma$ be a Dini-smooth curve. Then for $f \in E^{p,\lambda}(G)$, $0 < \lambda \leq 2$, $p > 1$ there exists a constant $c_7 > 0$, not depend on $r$, such that

$$\|f - Z_{n,2}(\cdot, f)\|_{L^{p,\lambda}(\Gamma)} \leq c_7 \Omega_{G, p,\lambda}(\frac{1}{n}, f)$$

holds with a constant $c_7 > 0$ independent of $n$.

**Theorem 1.6.** Let $\Gamma$ be a Dini-smooth curve. Then for $f \in E^{p,\lambda}(G)$, $0 < \lambda \leq 2$, $p > 1$ there exists a constant $c_8 > 0$, not depend on $r$, such that

$$\|f - U_r(\cdot, f)\|_{L^{p,\lambda}(\Gamma)} \leq c_8 \frac{1}{1 - r} \Omega_{G, p,\lambda}(1 - r, f), \quad 0 \leq r < 1.$$
Note that similar results in the Smirnov classes $E_p(G)$, $p > 1$ were investigated in \[36\]. When $\Gamma$ is regular curve the approximation of the functions by means $S_n(z,f)$ in the weighted Smirnov classes $E_p(G,\omega)$, $p > 1$ were investigated in \[25\]-\[27\]. If $\Gamma$ is Dini-smooth curve this problem in the weighted Smirnov-Orlicz classes was investigated in study \[27\].

We set

$$\Delta_\nu(x,f) := \sum_{k=2^{\nu-1}}^{2^\nu-1} A_k(x,f).$$

In the proof of the main results we use the following auxiliary results.

Lemma 1.7. Let $f \in L_p^\lambda(T)$, $0 < \lambda \leq 2$ and $1 < p < \infty$. Then the inequalities

$$c_9 \left\| \left( \sum_{\nu=k}^{\infty} \Delta^2_\nu(\cdot,f) \right)^{1/2} \right\|_{L_p^\lambda(T)} \leq \left\| \sum_{\nu=k}^{\infty} A_\nu(\cdot,f) \right\|_{L_p^\lambda(T)} \leq c_{10} \left\| \left( \sum_{\nu=k}^{\infty} \Delta^2_\nu(\cdot,f) \right)^{1/2} \right\|_{L_p^\lambda(T)},$$

hold, where the constants $c_9, c_{10} > 0$ depend only on $p$ and $\lambda$.

Proof. First, let’s prove the second inequality. Let $I$, be a subinterval of $T$, $\chi_I$ his characteristic function and $M_{\chi_I}$ Hardy-Littlewood maximal function of $\chi_I$. According to \[11\] the function $M_{\chi_I}$ is the $A_1$ Muckenhoupt weight, that is, almost everywhere on $T$ is $M(M_{\chi_I}) \leq cM_{\chi_I}$. Hence, using the Littlewood-Paley inequality on weighted Lebesgue spaces \[36\], Theorem 1 we reach

$$\int_I \left| \sum_{\nu=2^{k-1}}^{\infty} A_\nu(x,f) \right|^p dx = \int_T \left| \sum_{\nu=2^{k-1}}^{\infty} A_\nu(x,f) \right|^p \chi_I(x) dx \leq \int_T \left| \sum_{\nu=2^{k-1}}^{\infty} A_\nu(x,f) \right|^p M_{\chi_I(x)} dx \leq \int_T \left| \sum_{\nu=k}^{\infty} \Delta^2_\nu(x,f) \right|^p M_{\chi_I(x)} dx. \quad (1.6)$$

On the other hand, the equivalence

$$M_{\chi_I(x)} \approx \chi_I(x) + \sum_{s=0}^{\infty} 2^{-2s} \chi(2^{s+1}T \setminus 2^sT) \quad (1.7)$$

holds. Taking into account the relations (1.6) and (1.7) we have
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Let \( L^{p,\lambda}(T) \) be a Morrey space with \( 0 \leq \lambda \leq 1 \) and \( 1 < p < \infty \). Let \( \{\lambda_k\}_0^\infty \) be a sequence of numbers such that

\[
\int \left| \sum_{i=2^{k-1}}^{\infty} A_{\nu_i} (x, f) \right|^p dx
\]

\[
\leq c_{11} \int \left| \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right|^{p/2} \left( \chi_{i} (x) + \sum_{s=0}^{\infty} 2^{-2s} \chi_{(2^{s+1} \lambda, 2^{s+1} \lambda)} (x) \right) dx
\]

\[
\leq c_{12} \int \left| \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right|^{p/2} \chi_{i} (x) + c \int \left| \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right|^{p/2} \sum_{s=0}^{\infty} 2^{-2s} \chi_{(2^{s+1} \lambda, 2^{s+1} \lambda)} (x)
\]

\[
= c_{13} \int \left| \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right|^{p/2} dx + c_{13} \sum_{s=0}^{\infty} 2^{-2s} \int_{(2^{s+1} \lambda, 2^{s+1} \lambda)} \left| \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right|^{p/2} dx.
\]

If the norm of the last inequality is taken and use of \( \sum_{s=0}^{\infty} 2^{-2s+(s+1)\lambda/2} < \infty \), gives us

\[
\sup_{i} \frac{1}{|I|^{1/2}} \int_{I} \sum_{i=2^{k-1}}^{\infty} |A_{\nu_i} (x, f)|^p dx
\]

\[
\leq c_{14} \left\{ \left( \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right)^{1/2} \right\}^p + \sup_{i} \frac{1}{|I|^{1/2}} \sum_{s=0}^{\infty} 2^{-2s} \int_{(2^{s+1} \lambda, 2^{s+1} \lambda)} \left| \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right|^{p/2} dx
\]

\[
\leq c_{15} \left\{ \left( \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right)^{1/2} \right\}^p + \sum_{s=0}^{\infty} 2^{-2s} \sup_{i} \frac{1}{|I|^{1/2}} \int_{(2^{s+1} \lambda, 2^{s+1} \lambda)} \left| \sum_{i=k}^{\infty} \Delta^2_{i} (f, x) \right|^{p/2} dx
\]

\[
\leq c_{16} \left\{ \left( \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right)^{1/2} \right\}^p + \sum_{s=0}^{\infty} 2^{-2s+(s+1)\lambda/2} \sup_{i} \frac{1}{|I|^{1/2}} \int_{2^{s+1} \lambda} \left| \sum_{i=k}^{\infty} \Delta^2_{i} (f, x) \right|^{p/2} dx
\]

\[
\leq c_{16} \left\{ \left( \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right)^{1/2} \right\}^p
\]

\[
+ c_{17} \sum_{s=0}^{\infty} 2^{-2s+(s+1)\lambda/2} \sup_{i} \frac{1}{|I|^{1/2}} \int_{I} \left| \sum_{i=k}^{\infty} \Delta^2_{i} (f, x) \right|^{p/2} dx
\]

\[
\leq c_{18} \left\{ \left( \sum_{i=k}^{\infty} \Delta^2_{i} (x, f) \right)^{1/2} \right\}^p
\]

First inequality of Lemma 1.7. is proved similarly.

Note that, Lemma 1.7 has been proved in the thesis entitled "N. P. Tozman, Some problems of approximation theory in Morrey spaces, PhD thesis, Balıkesir University, Graduate School of Natural and Applied Sciences, Balıkesir, Turkey, (2009), (in Turkish). We give the proof of Lemma 1.7 in the article, as it is difficult for readers to reach the thesis.

Theorem 1.8. Let \( L^{p,\lambda}(\mathbb{T}) \) be a Morrey space with \( 0 \leq \lambda \leq 1 \) and \( 1 < p < \infty \). Let \( \{\lambda_k\}_0^\infty \) be a sequence of numbers such that
\[ |\lambda_k| \leq c_{19} \quad \text{and} \quad \sum_{k = 2^{m-1}}^{2^m - 1} |\lambda_k - \lambda_{k+1}| \leq c_{20}, \]  

(1.8)

where \( c_{20} > 0 \) does not depend on \( k \) and \( m \). If \( f \in L^{p, \lambda}(\mathbb{T}) \) has the Fourier series

\[
a_0 + \sum_{k=1}^{\infty} a_k(x, f),
\]

then there exists a function \( F \in L^{p, \lambda}(\mathbb{T}) \) with the Fourier series

\[
\lambda_0 a_0 + \sum_{k=1}^{\infty} \lambda_k A_k(x, f),
\]

and

\[
\| F \|_{L^{p, \lambda}(\mathbb{T})} \leq c_{11} \| f \|_{L^{p, \lambda}(\mathbb{T})}.
\]

Proof. We suppose that \( l \geq 2^{\nu - 1}, \nu = 1, 2, \ldots \) Also, we set

\[
\Delta_{\nu, l}(x, f) := \sum_{k=2^{\nu-1}}^{l} A_k(x, f), \quad A_k(x, f) := a_k \cos kx + b_k \sin kx,
\]

\[
\Delta'_\nu(x, f) := \sum_{k=2^{\nu-1}}^{2^\nu - 1} \lambda_k A_k(x, f)
\]

Note that according to [57], p.347 the estimation

\[
|\Delta'_\nu(x, f)|^2 \leq 2M \left( \sum_{k=2^{\nu-1}}^{2^\nu - 1} |\Delta_{\nu, l}(x, f)|^2 |\lambda_l - \lambda_{l+1}| + |\nu|^2 |\lambda_{2^\nu}| \right)
\]

holds. Use of (1.9) we have

\[
\int_T \left( \sum_{\nu=1}^{\infty} |\Delta'_\nu(x, f)|^2 \right)^{p/2} dx
\]

\[
\leq \int_T \left( \sum_{\nu=1}^{\infty} 2M \left( \sum_{k=2^{\nu-1}}^{2^\nu - 1} |\Delta_{\nu, l}(x, f)|^2 |\lambda_l - \lambda_{l+1}| + |\nu|^2 |\lambda_{2^\nu}| \right) \right)^{p/2} dx
\]

\[
\leq \left( 2M \right)^{p/2} \left( \sum_{\nu=1}^{\infty} \left( \sum_{k=2^{\nu-1}}^{2^\nu - 1} |\Delta_{\nu, l}(x, f)|^2 |\lambda_l - \lambda_{l+1}| + |\nu|^2 |\lambda_{2^\nu}| \right) \right)^{p/2} dx
\]

\[
\leq c_{21} \left( 2M \right)^{p/2} \left( \sum_{\nu=1}^{\infty} |\Delta'_\nu(x, f)|^2 \left( \sum_{k=2^{\nu-1}}^{2^\nu - 1} |\Delta_{\nu, l}(x, f)|^2 |\lambda_l - \lambda_{l+1}| + |\nu|^2 |\lambda_{2^\nu}| \right) \right)^{p/2} dx
\]

\[
\leq c_{22} \left( 2M \right)^{p/2} \left( \sum_{\nu=1}^{\infty} |\Delta'_\nu(x, f)|^2 \right)^{p/2} dx \leq c_{23} \int_T \left( \sum_{\nu=1}^{\infty} |\Delta'_\nu(x, f)|^2 \right)^{p/2} dx.
\]

The last inequality and Lemma 1.7 imply that
Approximation by means of Fourier trigonometric series in Morrey spaces

\[ \| F \|_{L^{p,\lambda}(\mathbb{T})} \leq c_{24} \left( \sum_{\nu=1}^{\infty} \left| \Delta_\nu (\cdot, f) \right|^2 \right)^{1/2} \]
\[ \leq c_{25} \left( \sum_{\nu=1}^{\infty} \left| \Delta_\nu (\cdot, f) \right|^2 \right)^{1/2} \| f \|_{L^{p,\lambda}(\mathbb{T})} \leq c_{26} \| f \|_{L^{p,\lambda}(\mathbb{T})}. \]

The proof of Theorem 1.8 is completed.

Lemma 1.9. [29], [30]. Let \( \Gamma \) be a Dini-smooth curve. Then the linear operator \( T := \mathcal{P}(\mathbb{D}) \rightarrow E^{p,\lambda}(G) \) with \( 1 < \lambda \leq 2 \) and \( 1 < p < \infty \), is bounded.

2 Proofs of the results

Proof of Theorem 1.1. Let \( f \in L^{p,\lambda}(\mathbb{T}) \), \( 0 < \lambda \leq 2 \) and \( 1 < p < \infty \). Note that the Fourier series of a function \( f \) converges to \( f \) in the norm \( L^{p,\lambda}(\mathbb{T}) \), \( 0 < \lambda \leq 2 \) and \( 1 < p < \infty \). According to [12], Theorem 1.1 and relation (2.2) the estimate
\[ \| f - S(\cdot, f) \|_{L^{p,\lambda}(\mathbb{T})} \leq c_{27} \Omega_{p,\lambda} \left( \frac{1}{n}, f \right), \quad n = 1, 2, \ldots \] (2.1)
holds. It is clear that [30] if \( f \in L^{p,\lambda}(\mathbb{T}) \), \( 0 < \lambda \leq 2 \) and \( 1 < p < \infty \) has the Fourier series
\[ a_0 + \sum_{m=1}^{\infty} A_m (x, f) \] (2.2)
then according to [38] for the Steklov function \( f_h \) the following estimate holds:
\[ f_h (x) \sim a_0 + \sum_{m=1}^{\infty} \frac{\sin mh}{mh} A_m (x, f). \] (2.3)
Using (2.2) and (2.3) we have
\[ f (x) - f_h (x) \sim \sum_{m=1}^{\infty} \left( 1 - \frac{\sin mh}{mh} \right) A_m (x, f) \]
for \( h > 0 \).

Note that for the Cesaro means the equality
\[ \sigma_n (x, f) = \sum_{\nu=0}^{n} \left( 1 - \frac{\nu}{n+1} \right) A_\nu (x, f) \]
holds. Then it is clear that
\[ S_n (x, f) - \sigma_n (x, f) = \sum_{m=1}^{n} \frac{m}{n+1} A_m (x, f). \] (2.4)
Now we consider the sequence
\[ \lambda_m^{(n)} = \begin{cases} \frac{m/(n+1)}{m/n}, & m \leq n \\ 0, & m > n \end{cases} \]
Note that the sequence \( \lambda_m^{(n)} \) satisfies the condition (1.8). Using (2.4), Theorem 1.8 and the definition of the modulus of continuity we have

\[
\| S_n(\cdot, f) - \sigma_n(\cdot, f) \|_{L^p, \lambda(T)} = \left\| \sum_{m=1}^{n} \frac{m}{n+1} A_m(\cdot, f) \right\|_{L^p, \lambda(T)}
\]

\[
= \left\| \sum_{m=1}^{n} \lambda_m^{(n)} \left( 1 - \frac{\sin (m/n)}{m/n} \right) A_m(\cdot, f) \right\|_{L^p, \lambda(T)}
\]

\[
= n \left\| \sum_{m=1}^{n} \lambda_m^{(n)} \left( 1 - \frac{\sin (m/n)}{m/n} \right) A_m(\cdot, f) \right\|_{L^p, \lambda(T)}
\]

\[
\leq c_{28} n \left\| \sum_{m=1}^{n} \left( 1 - \frac{\sin (m/n)}{m/n} \right) A_m(\cdot, f) \right\|_{L^p, \lambda(T)}
\]

\[
= c_{29} \| f - f_{1/n} \|_{L^p, \lambda(T)} \leq c_{30} \Omega_{p, \lambda}(\frac{1}{n}, f). \tag{2.5}
\]

Use of (2.1) and (2.5) gives us

\[
\| f - \sigma_n(\cdot, f) \|_{L^p, \lambda(T)} \leq \| f - S(\cdot, f) \|_{L^p, \lambda(T)} + \| S_n(\cdot, f) - \sigma_n(\cdot, f) \|_{L^p, \lambda(T)}
\]

\[
\leq c_{30} n \Omega_{p, \lambda}(\frac{1}{n}, f).
\]

The proof of Theorem 1.1. is completed.

Proof of Theorem 1.2. It is clear that

\[
f \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} A_m(x, f).
\]

Then the equality

\[
S_n(x, f) - Z_{n,2}(x, f) = \sum_{m=1}^{\infty} \frac{m^2}{(n+1)^2} A_m(x, f) \tag{2.6}
\]

holds.

We consider the sequence

\[
\mu_m^{(n)} = \begin{cases} 
\frac{m^2}{(n+1)^2}, & m \leq n \\
\frac{m^2/n}{m/n}, & m > n
\end{cases}
\]

According to [38] the sequence \( \mu_m^{(n)} \) satisfies the condition (1.8). If we apply Theorem 1.8 and using (2.6) we have
By combining the relations (2.1) and (2.7), we have

\[\| f - Z_n(\cdot, f) \|_{L^p, \lambda(T)} \leq \| f - S(\cdot, f) \|_{L^p, \lambda(T)} + \| S_n(\cdot, f) - Z_n, 2(\cdot, f) \|_{L^p, \lambda(T)} \leq c_{32} \Omega_{p, \lambda}(\frac{1}{n}, f). \] (2.7)

and the proof is completed.

Proof of Theorem 1.3. It is clear that \([30]\) if \(f \in L^{p, \lambda}(T), 0 < \lambda \leq 2 \) and \(1 < p < \infty\), then the function \(f\) has the Fourier series

\[\hat{f}_0^2 + \sum_{m=1}^{\infty} A_m(x, f). \] (2.8)

Then the following estimation holds:

\[U_r(x, f) \sim \hat{f}_0^2 + \sum_{m=1}^{\infty} (1 - r^m) A_m(x, f). \] (2.9)

Now we consider the sequence

\[\alpha_{m}^{(r)} = \frac{(1 - r^m) \alpha_0^2}{1 - \sin \frac{m(1 - r)}{m(1 - r)}}, m = 1, 2, \ldots\]

Note that the sequence \(\alpha_{m}^{(r)}\) satisfies the condition (1.8) (see, for example [23] and [38]). If we apply Theorem 1.8 using (2.8) and (2.9) we have
\[ \|f - U_r(\cdot, f)\|_{L^p, \lambda(T)} = \left\| \sum_{m=1}^{\infty} (1 - r^m) A_m(\cdot, f) \right\|_{L^p, \lambda(T)} \]

\[ = \left\| \sum_{m=1}^{\infty} a_m^{(r)} \left( 1 - \frac{\sin m(1-r)}{m(1-r)} \right) A_m(\cdot, f) \right\|_{L^p, \lambda(T)} \]

\[ = \frac{1}{1 - r} \left\| \sum_{m=1}^{\infty} a_m^{(r)} \left( 1 - \frac{\sin m(1-r)}{m(1-r)} \right) A_m(\cdot, f) \right\|_{L^p, \lambda(T)} \]

\[ \leq c_{34} \frac{1}{1 - r} \left\| \sum_{m=1}^{\infty} \left( 1 - \frac{\sin m(1-r)}{m(1-r)} \right) A_m(\cdot, f) \right\|_{L^p, \lambda(T)} \]

\[ = c_{35} \frac{1}{1 - r} \|f - f_1\|_{L^p, \lambda(T)} \leq c_{21} \frac{1}{1 - r} \Omega_{p, \lambda}(1 - r, f). \]

The proof of Theorem 1.3 is completed.

**Proof of Theorem 1.4.** Let \( f \in E^{p, \lambda}(G) \), \( 0 < \lambda \leq 2 \) and \( 1 < p < \infty \). According to Lemma 1.9 the operator \( T := \mathcal{P}(\mathbb{D}) \rightarrow E^{p, \lambda}(G) \) is bounded. Note that the subspace \( \mathcal{P}(\mathbb{D}) \) is dense in \( H^{p, \lambda}(\mathbb{D}) \). Extending the operator \( T := \mathcal{P}(\mathbb{D}) \rightarrow E^{p, \lambda}(G) \) from \( \mathcal{P}(\mathbb{D}) \) to the space \( H^{p, \lambda}(\mathbb{D}) \) as a linear and bounded operator, for the extension \( T : H^{p, \lambda}(\mathbb{D}) \rightarrow E^{p, \lambda}(G) \), \( 0 < \lambda \leq 2 \) and \( 1 < p < \infty \), the representation

\[ T(h)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\bar{h}(t)\psi'(t)}{\psi(t) - z} dt, \quad z \in G, \quad h \in H^{p, \lambda}(\mathbb{D}) \]

holds. Since \( \Gamma \) Dini smooth curve, by virtue of [29] (see also, [30]) the operator \( T : H^{p, \lambda}(\mathbb{D}) \rightarrow E^{p, \lambda}(G) \), \( 0 < \lambda \leq 2 \) and \( 1 < p < \infty \), is bounded, one-to-one, onto and \( T(f_0^+) = f \). For the function \( f \in E^{p, \lambda}(G) \) the following Faber series holds [29]:

\[ f(z) = \sum_{k=0} a_k(f) \varphi_k(z), \quad z \in G, \]

where,

\[ a_k(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw, \quad k \in \mathbb{N} \]

According to reference [30] we get \( f_0^+ \in H^{p, \lambda}(\mathbb{D}) \). Then for the function \( f_0^+ \) we can write the following Taylor expansion

\[ f_0^+(w) = \sum_{k=0}^{\infty} a_k(f)w^k. \]

Note that \( f_0^+ \in E_1(\mathbb{D}) \) and boundary function \( f_0^+ \in L^{p, \lambda}(\mathbb{T}) \). Then according to [42], Theorem 3.4, pp.38 the function \( f_0^+(w) \) has the following Fourier expansion:

\[ f_0^+(t) = \sum_{k=0}^{\infty} a_k(f)e^{ikt}. \]
The operator $T$ is bounded. Taking into account the boundedness of the operator $T$ and Theorems 1.3 we have

\[
\| f - \sigma(\cdot, f) \|_{L_p^r, \lambda(T)} = \| T(f^+) - T(\sigma_n(\cdot, f^+)) \|_{L_p^r, \lambda(T)} \\
\leq c_{36} \| f^+ - \sigma_n(\cdot, f^+) \|_{L_p^r, \lambda(T)} \leq c_{37} \pi \lambda(f^+ (1/n) = c_{38} \pi \lambda(f^+ (1/n))
\]

which proves Theorem 1.4.

For the proofs of Theorems 1.5 and 1.6 we use Theorem 1.2 and 1.3 and the method in the proof of Theorem 1.4.

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