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Unification of relative versions of some star-covering properties

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Abstract In this paper, we introduce relative star-Δ-Menger property in topological spaces following the approach of Bonanzinga and Pansera (Acta Math. Hungar. 117 (3) 231– 243, 2007), which unifies the relative versions of star-K-Menger property, star-C-Menger property and star-L-Menger property in the same. We then study the properties of such spaces and also its relationships with different types of existing Menger properties present in the literature. Finally, we study the preservation properties of such spaces and discuss about the associated star-Δ-Menger game.

Keywords Menger spaces · star-Δ-Menger property · relative star-Δ-Menger property

Mathematics Subject Classification (2010) 54D20 · 54B20

1 Introduction

The theory of Selection principles is a field of Mathematics having a rich background and its study initiated with Borel [3], Menger [15], Hurewicz [9], Rothberger [17] and many others. A systematic study of selection principles in topology and its relations to Game theory and Ramsey theory was started with Scheepers [18], which led to a great deal of research into selection principles and their applications. After that, Kočinac [11] introduced another direction of study of selection principles concerning star type selection principles. During the last two decades or so, this study of selection principles has gained importance to mathematicians to become one of the most challenging areas of set-theoretic topology. Nowadays, this theory has deep connections with different branches of mathematics, such as Set theory and General Topology, Game Theory, Function Spaces and Hyperspaces, Dimension Thoery, Topological Groups and relatives etc.

In [15], Menger introduced a property for metric spaces (called now the Menger Basis property) and in [9], Hurewicz has proved that this property is equivalent to the property $\mathbf{S}_{fin}(\mathcal{O},\mathcal{O})$ (where $\mathcal{O}$ denotes the family of open

covers of the space), and now called the Menger property. This property is stronger than Lindelöfness and weaker than compactness. After that, this field was further enriched by L. D. R. Kočinac, where the author in [11] first introduced the star-Menger and different star versions of Menger spaces and studied their properties. In this connection, mention should be made about the paper [13] of L. D. R. Kočinac, which is an excellent reveiw of selected results on star selection principles in topology. In 2007, Bonanzinga and Pansera in [2] first introduced and studied the relative versions of some star selection principles. Also Song in [19,22] studied two new types of spaces named star-C-Menger and star-K-Menger spaces that are only special cases of the general selection principle $SS^*_{\mathcal{K}}(\mathcal{A},\mathcal{B})$ introduced in [[11], Definition 1.2], where these K and C stands for compact and countably compact subsets of a space respectively. These two types of sets satisfy some basic properties: they are closed under forming singletons and finite unions. Also these properties are absolute in nature (i.e., a subset A of a subspace $(Y,\tau_{_Y})$ of (X,τ) is compact or countably compact in Y if and only if A is respectively so in the whole space (X,τ)) and are preserved under continuous mappings. Observing these, we here consider a subcollection Δ of $\mathcal{P}(X)$, having the above mentioned properties.

Our aim here is to study in some sort, a unification of the relative versions of different types of star-Menger properties. The paper is organized as follows. In Section 2, the definitions of the terms used in this paper are provided. In Section 3, the concept of relativization of star-Δ-Menger property is studied and characterized using selection principles. Section 4 deals with the preservation properties of relative star-Δ-Menger property and also the star-Δ-Menger game associated with spaces having such properties.

2 Preliminaries

Let X be a topological space. We denote by $|A|$, the cardinality of a set A ($\subseteq X$). Let $\mathbb{N}$ denote the set of all positive integers. For a space X, a collection of subsets $\mathcal{U}$ of X and $A \subseteq X$, let $St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, let us write $St(x,\mathcal{U})$ instead of $St(\{x\},\mathcal{U})$.

Let us now recall two known concepts both defined in 1996 by M. Scheepers [18]. Given an infinite set X, let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of X.

- $\mathbf{S}_1(\mathcal{A},\mathcal{B})$ denotes the principle: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.
- $\mathbf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the principle: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{B_n : n \in \mathbb{N}\}$ such that B_n is a finite subset of A_n for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

In [11,12], Kočinac first introduced the following star selection principles.

– $\mathbf{S}_1^*(\mathcal{A},\mathcal{B})$ [Definition 1.1, [11]] denotes the principle: For each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of $\mathcal{B}$.
– $\mathbf{SS}^*_{comp}(\mathcal{A},\mathcal{B})(\mathbf{SS}^*_{fin}(\mathcal{A},\mathcal{B}))$ [Definition 1.2, [11]] denotes the principle: For each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{K_n : n \in \mathbb{N}\}$ of compact (resp., finite) subsets of X such that $\{St(K_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

Throughout the paper, by $\mathcal{A}$ and $\mathcal{B}$, we will denote the collections of the following open covers of X.

– $\mathcal{O}$: the collection of open covers of X.
– $\mathcal{O}_Y$: the collection of open covers of Y by sets open in X, for $Y \subseteq X$.
– Ω : the collection of all ω-covers of X, where, an open cover $\mathcal{U}$ of X is called an ω-cover [8] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in an element of $\mathcal{U}$.
– Ω_Y : the collection of all ω-covers of Y by sets open in X, for $Y \subseteq X$ [2].
– $\mathcal{O}^{wgp}$: the collection of all weakly groupable open covers of X, where, an open cover $\mathcal{U}$ of X is called weakly groupable [1], if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, for $n \in \mathbb{N}$, such that for each finite subset F of X, $F \subset \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, for some n.
– $\mathcal{O}^{wgp}_Y$: the collection of all weakly groupable open covers of Y by sets open in X, for $Y \subseteq X$ [2].

Also, let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [7].

Let us now recall the following definitions.

Definition 2.1 *[15, 9, 10] A space X is said to have the Menger property if the selection hypothesis $S_{fin}(\mathcal{O},\mathcal{O})$ is true for X.*

Definition 2.2 *[11, 12, 19, 22] A space X is said to be star Menger (strongly star Menger, star-K-Menger, star-C-Menger, star-L-Menger) if it satisfies the selection hypothesis $S^*_{fin}(\mathcal{O},\mathcal{O})$ (resp., $SS^*_{fin}(\mathcal{O},\mathcal{O})$, $SS^*_{comp}(\mathcal{O},\mathcal{O})$, $SS^*_{countcomp}(\mathcal{O},\mathcal{O})$, $SS^*_{Lind}(\mathcal{O},\mathcal{O})$).*

Definition 2.3 *[6, 14, 20, 21] A space X is said to be $\mathcal{C}$-starcompact ($\mathcal{K}$-starcompact) if for each open cover $\mathcal{U}$ of X, there exists a countably compact (resp., compact) subset F of X such that $St(F,\mathcal{U}) = X$.*

Definition 2.4 *[6, 24] A space X is said to be starcompact if for every open cover $\mathcal{U}$ of X, there exists a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $St(\cup\mathcal{V},\mathcal{U}) = X$.*

In [19], the relation between the above defined spaces are given.

Recall that a nonempty subcollection $\mathcal{I}$ of $\mathcal{P}(X)$ is said to be an ideal on X if it has the following two properties: (i) $\mathcal{I}$ is closed under subsets, i.e. whenever $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$, (ii) $\mathcal{I}$ is closed under finite unions, i.e. whenever $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal $\mathcal{I}$ on a space X is said to be admissible or free if $\{x\} \in \mathcal{I}$ for each $x \in X$, and $\mathcal{I}$ is said to be a σ-ideal if it is closed under countable unions. An ideal $\mathcal{I}$ of subsets of a space X is said to be proper if $X \notin \mathcal{I}$. Throughout the paper, $\mathcal{I}$ denotes a proper ideal in X.

3 Relative star-Δ-Menger property

In this article we first consider a subcollection Δ of the power set $\mathcal{P}(X)$ of X, which is closed under finite unions and contains singletons. Also, a property satisfied by each member of Δ is absolute in nature, i.e. a subset $A \in \Delta$ with $A \subseteq Y \subseteq X$ has a topological property P with respect to Y if and only if it has P with respect to X. The following definition is a version of Definition 1.2 in [11] for Δ with special properties. In view of Definition 2.2, we can now have a general definition as follows.

Definition 3.1 *A space X is said to be star-Δ-Menger if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there is a sequence $\{A_n : n \in \mathbb{N}\}$ of members of Δ such that $\{St(A_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X, i.e. $X = \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{U}_n)$.*

Remark 3.1 If we put $\Delta = \mathcal{F}(X)$ (resp., $\mathcal{K}(X)$, the collection of all countably compact subsets of X, the collection of all Lindelöf subsets of X) in Definition 3.1, we get the notions of strongly star Menger (resp., star-K-Menger, star-C-Menger, star-L-Menger) spaces X.

We next define the relative version of the above as follows.

Definition 3.2 *A subspace A of a topological space (X, τ) is said to have the relative star-Δ-Menger property in X or A is a relatively star-Δ-Menger subspace of X if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there is a sequence $\{A_n : n \in \mathbb{N}\}$ of members of Δ such that $\{St(A_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of A, i.e. $A \subseteq \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{U}_n)$.*

Definition 3.3 *For an infinite set X, let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of X. Then*

– $\mathbf{SS}^*_\Delta(\mathcal{A}, \mathcal{B})$ *denotes the principle: For each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{A_n : n \in \mathbb{N}\}$ of members of Δ such that $\{St(A_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.*

Remark 3.2 In view of the above definition, we can now say that a subspace A of a topological space (X, τ) is said to have the relative star-Δ-Menger property in X or A is a relatively star-Δ-Menger subspace of X if $SS^*_\Delta(\mathcal{O}, \mathcal{O}_A)$ holds.

Remark 3.3 If we put $\Delta = \mathcal{F}(X)$ (resp., $\mathcal{K}(X)$, the collection of all countably compact subsets of X) in Definition 3.2, we get the notions of relatively strong star Menger (resp., relatively star-K-Menger, relatively star-C-Menger) subspaces of X.

Remark 3.4 A space X is strongly star Menger (resp., star-K-Menger, star-C-Menger) if and only if X is relatively strong star Menger (resp., relatively star-K-Menger, relatively star-C-Menger) subspace of itself.

Theorem 3.4 *If A is a subspace of a space X having the star-Δ-Menger property, then A has the relative star-Δ-Menger property in X.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then for each $n \in \mathbb{N}$, consider $\mathcal{V}_n = \{U \cap A : U \in \mathcal{U}_n\}$. Thus $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of open covers of A. As A has the star-Δ-Menger property, there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of members of Δ such that $\{St(A_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is an open cover of A. Then the sequence $\{A_n : n \in \mathbb{N}\}$ witnessess the relative star-Δ-Menger property of A in X.

□

The converse of the above theorem is not true. At first we prove the following theorems.

Theorem 3.5 *If $Y = \bigcup_{n\in\mathbb{N}} X_n$, where each X_n is relatively star-Δ-Menger in X, then Y is also relatively star-Δ-Menger in X.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of Y and let $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup ...$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. Since each X_n is relatively star-Δ-Menger in X, for the sequence $\{\mathcal{U}_m : m \in \mathbb{N}_n\}$ of open covers of X_n, there exists a sequence $\{A_m : m \in \mathbb{N}_n\}$ of Δ-subsets of X_n such that $X_n \subseteq \bigcup_{m\in\mathbb{N}_n} St(A_m, \mathcal{U}_m)$. Thus the sequence $\{A_n : n \in \mathbb{N}\}$ of members of Δ witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that Y is relatively star-Δ-Menger in X.

□

Theorem 3.6 *The set of all relatively star-Δ-Menger subsets of a space X forms an admissible σ-ideal in X.*

Proof. First let A, B be subsets of X with $A \subseteq B$, where B is relatively star-Δ-Menger in X and let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be any sequence of open covers of X. Since B is relatively star-Δ-Menger in X, there is a sequence $\{B_n : n \in \mathbb{N}\}$ of members of Δ such that $\{St(B_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ forms a cover of B. Clearly then $\{St(B_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ forms a cover of A also. Next, let C, D be two relatively star-Δ-Menger subsets of X and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be any sequence of

open covers of X. Then there exist sequences $\{C_n : n \in \mathbb{N}\}$ and $\{D_n : n \in \mathbb{N}\}$ of members of Δ such that $C \subseteq \bigcup_{n\in\mathbb{N}} St(C_n, \mathcal{U}_n)$and $D \subseteq \bigcup_{n\in\mathbb{N}} St(D_n, \mathcal{U}_n)$. For each $n \in \mathbb{N}$, define $P_n = C_n \cup D_n$. Then the sequence $\{P_n : n \in \mathbb{N}\}$ witnesses the relative star-Δ-Menger property of $C \cup D$ in X. Hence the set of all relatively star-Δ-Menger subsets of a space X forms an ideal in X. Finally as for each $x \in X$, $\{x\}$ is relatively star-Δ-Menger in X and the relative star-Δ-Menger property is preserved under coutable unions (as proved in Theorem 3.5), hence the set of all relatively star-Δ-Menger subsets of a space X forms an admissible σ-ideal in X.

□

We now give an example to show that the converse of Theorem 3.4 is not true in general.

Example 3.1 Consider the space $\mathbb{R}$ of real numbers with the usual topology and let $\mathbb{I}$ be the set of all irrationals. Then $\mathbb{I}$ is not star-C-Menger, but it is relatively star-C-Menger.

Remark 3.5 It follows from Theorem 3.6 that the relative star-Δ-Menger property is a hereditary property.

Definition 3.7 *A space X is said to be Δ-starcompact if for every open cover $\mathcal{U}$ of X, there exists a member $F \in \Delta$ in X such that $St(F, \mathcal{U}) = X$.*

X is said to be σ-Δ-starcompact if it is the union of countably many Δ-starcompact spaces.

Remark 3.6 If $\Delta = \mathcal{F}(X)$ (resp., $\mathcal{K}(X)$, the collection of all countably compact subsets), then we get the notions of starcompact (resp., $\mathcal{K}$-starcomp act, $\mathcal{C}$-starcompact) spaces.

Theorem 3.8 *A σ-Δ-starcompact subspace A of a space X is a relatively star-Δ-Menger subspace of X.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. As A is σ-Δ-starcompact, $A = \bigcup_{n\in\mathbb{N}} A_n$, where each A_n is Δ-starcompact, for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists a member $F_n \in \Delta$, where $F_n \subseteq A_n$, such that $A_n \subseteq St(F_n, \mathcal{U}_n)$. Clearly then $\{F_n : n \in \mathbb{N}\}$ witnesses the relative star-Δ-Menger property of A in X, for the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$.

□

Corollary 3.9 *A σ-Δ-starcompact space X is star-Δ-Menger.*

That the converse does not hold follows from the next example.

Example 3.2 Consider the set $\mathbb{I}$ of irrational numbers. Then $\mathbb{I}$ is relatively star-C-Menger in $(\mathbb{R}, \tau_u)$, but it is not a σ-$\mathcal{C}$-starcompact subspace, as every σ-$\mathcal{C}$-starcompact subspace is star-C-Menger and it is not a star-C-Menger subspace.

Next recall that given an ideal $\mathcal{I}$ in X, the filter generated by $\mathcal{I}$ (known as the dual filter of $\mathcal{I}$) is $\mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$.

Theorem 3.10 *Let $\mathcal{I}_X$ denote the set of all relatively star-Δ-Menger subspaces of a space X. If $\mathcal{I}_X \cap \mathcal{F}(\mathcal{I}_X) \neq \emptyset$, then X is star-Δ-Menger.*

Proof. Let $A \in \mathcal{I}_X \cap \mathcal{F}(\mathcal{I}_X)$. Then $X \setminus A$ and A both are members of $\mathcal{I}_X$. As $\mathcal{I}_X$ is an ideal in X (by Theorem 3.6), $X = A \cup (X \setminus A) \in \mathcal{I}_X$. Hence X is a star-Δ-Menger space.

□

Theorem 3.11 *For a topological space X, at least one of the following statements holds:*
(i) $\mathcal{F}(\mathcal{I}_X)$ is an ultrafilter.
(ii) $|\mathcal{P}(X) \setminus \mathcal{I}_X| \geq 2^{\omega_1}$.

Proof. First let, $\mathcal{F}(\mathcal{I}_X)$ be not an ultrafilter on X. Then there exists a subset A of X such that $A \notin \mathcal{I}_X$ and $X \setminus A \notin \mathcal{I}_X$. Then A and $X \setminus A$ both are uncountable or, $|A| \geq \omega_1$ and $|X \setminus A| \geq \omega_1$. This implies that $|X| = |A \cup (X \setminus A)| \geq \omega_1$ and hence $|\mathcal{P}(X)| \geq 2^{\omega_1}$. Also, $A, X \setminus A \in \mathcal{P}(X) \setminus \mathcal{I}_X$. Thus $A \cup B \in \mathcal{P}(X) \setminus \mathcal{I}_X$, for each $B \in \mathcal{P}(X \setminus A)$, so that

$$|\mathcal{P}(X) \setminus \mathcal{I}_X| \geq |\{A \cup B : B \in \mathcal{P}(X \setminus A)\}| \geq |\mathcal{P}(X \setminus A)| \geq 2^{|X \setminus A|} = 2^{\omega_1}.$$

Next suppose that (ii) does not hold. Then $|\mathcal{P}(X) \setminus \mathcal{I}_X| \leq 2^{\omega_1}$. Let A be a subset of X such that $A, X \setminus A \notin \mathcal{F}(\mathcal{I}_X)$. Then $|\mathcal{P}(X) \setminus \mathcal{I}_X| \geq 2^{\omega_1}$, a contradiction. Hence the proof.

□

Definition 3.12 *A subset A of a topological space X is said to have the relative Menger property in X or A is a relatively Menger subspace of X if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$, where for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ such that $\{\mathcal{V}_n : n \in \mathbb{N}\}$ forms an open cover of A.*

Next we present conditions under which the notions of relative Menger and relative star-Δ-Menger properties of a space are equivalent.

Theorem 3.13 *In a Hausdorff, paracompact space X, a subspace A has the relative star-Δ-Menger property in X if and only if A has the relative Menger property in X.*

Proof. We need to prove just one part of the theorem as any relatively Menger subspace of X is always relatively star-Δ-Menger subspace of X. So let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then there exists a locally finite open refinement $\mathcal{V}_n$ of $\mathcal{U}_n$, for each $n \in \mathbb{N}$. As A is relatively star-Δ-Menger in X, there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of members of Δ, where each A_n is a subset of X, such that $\{St(A_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is an open cover of A. For each $n \in \mathbb{N}$, let $\mathcal{W}_n = \{V \in \mathcal{V}_n : V \cap A_n \neq \emptyset\}$. Then $\mathcal{W}_n$

is a finite subset of $\mathcal{V}_n$, as $\mathcal{V}_n$ is locally finite and also $\bigcup \mathcal{W}_n = St(A_n, \mathcal{V}_n)$. Hence $A = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n$. Also for each $V \in \mathcal{W}_n$, choose $U_V \in \mathcal{U}_n$ such that $V \subset U_V$. For each $n \in \mathbb{N}$, consider $\mathcal{P}_n = \{U_V : V \in \mathcal{W}_n\}$. Then the sequence $\{\mathcal{P}_n : n \in \mathbb{N}\}$ witnesses the relative Menger property of A for the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$.

□

Corollary 3.14 *A Hausdorff, paracompact space X has the Menger property if and only if it has the star-Δ-Menger property.*

Definition 3.15 *A collection $\mathcal{U}$ of subsets of a space X is said to be a Δ-finite collection if every member of Δ intersects only finitely many members of $\mathcal{U}$.*

Theorem 3.16 *In a Hausdorff space X, if every open cover of X has a Δ-finite open refinement, then a subspace A of X has the relative star-Δ-Menger property if and only if it has the relative Menger property in X.*

Proof. Let A be a relatively star-Δ-Menger subspace of the Hausdorff space X and let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. By hypothesis, there exists a Δ-finite open refinement $\mathcal{V}_n$ of $\mathcal{U}_n$, for each $n \in \mathbb{N}$. So there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of members of Δ such that $\{St(A_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ forms a cover of A. By the given condition, there exists a finite subset $\mathcal{V}_{n,1} = \{V_{n,1}, ..., V_{n,k_n}\}$ of $\mathcal{V}_n$ such that $A_n \subseteq \cup\{V_{n,i} : 1 \leq i \leq k_n\}$ and $St(A_n, \mathcal{V}_n) = \cup\mathcal{V}_{n,1}$. Thus for each $V \in \mathcal{V}_{n,1}$, there exists $U_V \in \mathcal{U}_n$ such that $V \subseteq U_V$ and consider $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_{n,1}\}$. Then $\{\mathcal{W}_n : n \in \mathbb{N}\}$ witnessess the relative Menger property of A for the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$.

□

Recall here that a subspace Y of a space X is said to be relatively star Menger [2] in X if and only if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an open cover of Y.

We next investigate the equivalence of all the relative versions of the above defined Menger spaces.

Theorem 3.17 *For a subspace A of a Hausdorff, paracompact space X, the following are equivalent:*

(i) A is relatively Menger.
(ii) A is relatively strong star Menger.
(iii) A is relatively star-K-Menger.
(iv) A is relatively star-C-Menger.
(v) A is relatively star Menger.

Proof. We only prove here (v) ⇒ (i). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then by the Stone characterization of paracompactness

[7], for each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be an open star refinement of $\mathcal{U}_n$. As A is relatively star Menger, there exists a sequence $\{\mathcal{W}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and $A = \bigcup_{n\in\mathbb{N}} St(\cup\mathcal{W}_n, \mathcal{V}_n)$. For every $W \in \mathcal{W}_n$, let U_W be the member of $\mathcal{U}_n$ such that $St(W, \mathcal{V}_n) \subset U_W$. Then $\mathcal{H}_n = \{U_W : W \in \mathcal{W}_n\}$ is a finite subset of $\mathcal{U}_n$, for each $n \in \mathbb{N}$ and $A = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{H}_n$. Hence (i) is proved.

□

Recall that a space X is said to be para-Lindelöf if every open cover $\mathcal{U}$ of X has a locally countable open refinement and a subspace A of X is said to be relatively Lindelöf if every open cover $\mathcal{U}$ of X has a countable subcover for A.

Theorem 3.18 *Every relatively star-Δ-Menger subspace A of a Hausdorff, para-Lindelöf space X is relatively Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. Then there exists a locally countable open refinement $\mathcal{V}$ of $\mathcal{U}$. As A is a relatively star-Δ-Menger subspace of X, there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of Δ-subsets of X such that $\{St(A_n, \mathcal{V}) : n \in \mathbb{N}\}$ is a cover of A. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{V \in \mathcal{V} : V \cap A_n \neq \emptyset\}$. Then $\mathcal{V}_n$ is a countable subset of $\mathcal{V}$, as $\mathcal{V}$ is locally countable. For each $n \in \mathbb{N}$, let $\mathcal{W}_n = \{V \in \mathcal{V} :$ there exists some $V' \in \mathcal{V}_n$ such that $V \cap V' \neq \emptyset\}$. Then $\mathcal{W}_n$ is countable, as $\mathcal{V}$ is locally countable. Let $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$. Then $\mathcal{W}$ is a countable open cover of A. For each $V \in \mathcal{W}$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then $\{U_V : V \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$, so that A is relatively Lindelöf.

□

As every paracompact space is para-Lindelöf, we have

Corollary 3.19 *Every relatively star-Δ-Menger subspace A of a Hausdorff, paracompact space is relatively Lindelöf.*

Theorem 3.20 *A subset A of X satisfies the relative star-Δ-Menger property if and only if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of Δ-subsets of X such that for every $x \in A$, $St(x,\mathcal{U}_n) \cap F_n \neq \emptyset$, for some $n \in \mathbb{N}$.*

Proof. First let, A be a subset of X satisfying the relative star-Δ-Menger property and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of Δ-subsets of X such that $A \subseteq \bigcup_{n\in\mathbb{N}} St(F_n,\mathcal{U}_n)$. Now, for each $x \in A$, there exists $m \in \mathbb{N}$ such that $x \in St(F_m,\mathcal{U}_m)$, i.e. $x \in \bigcup\{P \in \mathcal{U}_m : P \cap F_m \neq \emptyset\}$, i.e. $St(x,\mathcal{U}_m) \cap F_m \neq \emptyset$, for some $m \in \mathbb{N}$.

Conversely, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then by the given condition, there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of Δ-subsets of X such that for each $x \in A$, $St(x,\mathcal{U}_n) \cap F_n \neq \emptyset$, for some $n \in \mathbb{N}$, i.e.

$A \subseteq \bigcup_{n\in\mathbb{N}} St(F_n, \mathcal{U}_n)$. Hence A satisfies the relative star-Δ-Menger property in X.

□

We next give the characterization of the relative star-Δ-Menger property by selection principles.

Theorem 3.21 *Let A be a subset of X such that for every $n \in \mathbb{N}$, A^n satisfies the relative star-Δ-Menger property in X^n. Then A satisfies $SS^*_\Delta(\mathcal{O}, \Omega_A)$.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X and let $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup ...$ be a partition of $\mathbb{N}$ into infinite, pairwise disjoint sets. For each $k \in \mathbb{N}$ and every $m \in \mathbb{N}_k$, let $\mathcal{W}_k = \mathcal{U}^k_m$. Then $\{\mathcal{W}_m : m \in \mathbb{N}_k\}$ is a sequence of open covers of X^k. Hence there exists a sequence $\{K_m : m \in \mathbb{N}_k\}$ of Δ-subsets of X^k such that $\{St(K_m, \mathcal{W}_m) : m \in \mathbb{N}_k\}$ is a cover of A^k. For each $m \in \mathbb{N}_k$, take a Δ-subset S_m of A such that $K_m \subset S^k_m$. Then $\{St(S_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an ω-cover of A : infact, let $K = \{x_1, ..., x_p\}$ be a finite subset of A. Then $(x_1, ..., x_p) \in A^p$. Hence there exists $n \in \mathbb{N}_p$ such that $(x_1, ..., x_p) \in St(K_n, \mathcal{W}_n) \subset St(S^p_n, \mathcal{W}_n)$ and so $K \subset St(S_n, \mathcal{U}_n)$.

□

Corollary 3.22 *If for every $n \in \mathbb{N}$, X^n satisfies the relative star-Δ-Menger property, then X satisfies $SS^*_\Delta(\mathcal{O}, \Omega)$.*

We next give another version of the previous theorem as follows.

Theorem 3.23 *For a subspace A of X, the following are equivalent:*
(i) X satisfies $SS^*_\Delta(\mathcal{O}, \Omega_A)$.
(ii) X satisfies $SS^*_\Delta(\mathcal{O}, \Omega^{wgp}_A)$.

Proof. (i) $\Rightarrow$ (ii): Follows from the fact that each countable ω-cover is weakly groupable.
(ii) $\Rightarrow$ (i): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \bigwedge_{i\leq n} \mathcal{U}_i$ (where, the symbol $\mathcal{U} \wedge \mathcal{V}$ denotes the set $\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$, where $\mathcal{U}$ and $\mathcal{V}$ are families of subsets of a topological space X). Then for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is an open cover of X. By (ii) to the sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$, there exists a sequence $\{W_n : n \in \mathbb{N}\}$ of Δ-subsets of X such that $\{St(W_n, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a weakly groupable cover of A. Hence there exists a partition of $\mathbb{N}$ into finite sets M_n, $n \in \mathbb{N}$, such that for each finite subset F of A, there exists $n \in \mathbb{N}$ such that $F \subset \cup\{St(W_i, \mathcal{G}_i) : i \in M_n\}$. Denote by i_n, the first element of $M_n, n \in \mathbb{N}$. Then $\mathcal{G}_i \preceq \mathcal{G}_{i_n}$, for each $i \in M_n$ and $n \in \mathbb{N}$. Hence $St(W_i, \mathcal{G}_i) \subset St(W_i, \mathcal{G}_{i_n})$. Put $A_{i_n} = \bigcup_{i\in M_n} W_i$ and for i different from all $i_n, n \in \mathbb{N}$, $A_i = \emptyset$. Then $\{A_n : n \in \mathbb{N}\}$ is a sequence of Δ-subsets of X. For every finite subset F of A, $F \subset \bigcup\{St(W_i, \mathcal{G}_i) : i \in M_n\} \subset St(A_{i_n}, \mathcal{G}_{i_n}) \subset St(A_{i_n}, \mathcal{U}_{i_n}), n \in \mathbb{N}$. Hence X satisfies $SS^*_\Delta(\mathcal{O}, \Omega_A)$.

□

Corollary 3.24 *Let A be a subset of X such that for every $n \in \mathbb{N}$, A^n satisfies the relative star-Δ-Menger property in X^n. Then A satisfies $SS^*_\Delta(\mathcal{O}, \mathcal{O}^{wgp}_A)$.*

Corollary 3.25 *If for every $n \in \mathbb{N}$, X^n satisfies the relative star-Δ-Menger property, then X satisfies $SS^*_\Delta(\mathcal{O}, \mathcal{O}^{wgp})$.*

4 Preservation properties of relatively star-Δ-Menger spaces

In this article we investigate the preservation properties of a relative star-Δ-Menger space. It is to be noted here that under any surjection $f : X \to Y$, whenever we consider $\Delta \subseteq \mathcal{P}(X)$ with the properties that (i) it is closed under finite unions, (ii) contains singletons and (iii) absolute in nature; $f(\Delta)$ also enjoys the same properties. Also same is true for $f^{-1}(\Delta)$, whenever $\Delta \subseteq \mathcal{P}(Y)$ enjoys the above mentioned properties and $f : X \to Y$ is a surjection. So without loss of generality, irrespective of the spaces where Δ has been taken, we continue to call such a collection as Δ, instead of $f(\Delta)$ or $f^{-1}(\Delta)$.

Theorem 4.1 *Let f be a continuous mapping from a space X onto Y. If A is relatively star-Δ-Menger in X, then so is $f(A)$ in Y.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of Y. Since f is continuous and onto, for each $n \in \mathbb{N}$, $f^{-1}(\mathcal{U}_n) = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ is an open cover of X. Hence $\{f^{-1}(\mathcal{U}_n) : n \in \mathbb{N}\}$ is a sequence of open covers of X. As A is relatively star-Δ-Menger in X, there exists a sequence $\{K_n : n \in \mathbb{N}\}$ of Δ-subsets of X such that each $x \in A$ belongs to $St(K_n, f^{-1}(\mathcal{U}_n))$, for some $n \in \mathbb{N}$. Then $f(K_n)$ is a Δ-subset of Y, for $n \in \mathbb{N}$. Hence the sequence $\{f(K_n) : n \in \mathbb{N}\}$ witnesses the relative star-Δ-Menger property of $f(A)$ as a subspace of Y. For each $y \in f(A)$, there is some $a \in A$ such that $y = f(a)$. Then $a \in St(K_n, f^{-1}(\mathcal{U}_n))$, for $n \in \mathbb{N}$, so that $a \in f^{-1}(U)$, for some $f^{-1}(U) \in f^{-1}(\mathcal{U}_n)$ and $f^{-1}(U) \cap K_n \neq \emptyset$. Then $f(a) \in U \in \mathcal{U}_n$ and $f(K_n) \cap U \neq \emptyset$ implies that $y = f(a) \in St(f(K_n), \mathcal{U}_n)$, for $n \in \mathbb{N}$. Hence $f(A)$ is relatively star-Δ-Menger in Y.

□

Next we turn our attention to consider preimages. We shall give an example to show that the preimage of a relatively star-Δ-Menger space under a closed 2-to-1 continuous mapping need not be relatively star-Δ-Menger. For the example, we need the following definitions.

Let X be any topological space. The Alexandroff duplicate $A(X)$ of X is $X \times \{0, 1\}$; where each point of $X \times \{1\}$ is isolated and a basic neighbourhood of a point $(x, 0) \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{(x, 1)\})$, where U is a neighbourhood of x in X.

Also recall that a family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is said to be almost disjoint if every element of $\mathcal{A}$ is infinite and the sets $A \cap B$ are finite, for all distinct elements

$A, B \in \mathcal{A}$. Let $\psi(\mathcal{A}) = \mathcal{A} \cup \mathbb{N}$, for an almost disjoint family $\mathcal{A}$. A topology on $\psi(\mathcal{A})$ is defined as follows. The natural numbers are isolated and for each element $A \in \mathcal{A}$ and each finite set $F \subseteq \mathbb{N}$, $\{A\} \cup (A \setminus F)$ is a basic open neighbourhood of A. The spaces constructed in this manner are called Isbell–Mrówka ψ-spaces [16].

Define ω^ω as the set of all functions from ω to itself. Recall that for $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n (and $f \leq g$ means that $f(n) \leq g(n)$, for all n). A subset D of ω^ω is said to be dominating if for each $g \in \omega^\omega$, there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of a dominating subset of ω^ω is denoted by $\mathfrak{d}$. The value of $\mathfrak{d}$ does not change if one considers the relation $\leq$ instead of $\leq^*$ ([5]). It is not difficult to show that $\omega_1 \leq \mathfrak{d} \leq \mathfrak{c}$.

We next investigate when a subspace of $\psi(\mathcal{A})$ has the relative strong star-Menger property. We thus have the following.

Theorem 4.2 *A subspace M of $X = \psi(\mathcal{A})$ has the relative strong star-Menger property if and only if $|M \cap \mathcal{A}| < \mathfrak{d}$.*

Proof. First, let M has the relative strong star-Menger property in $\psi(\mathcal{A})$ and if possible, let $|M \cap \mathcal{A}| \geq \mathfrak{d}$. Then there is $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$ with the property that for every $f \in \omega^\omega$, there exists f_α such that $f_\alpha(n) \geq f(n)$, for all $n \in \omega$. Choose distinct points $p_{\alpha,\beta} \in \mathcal{A}$, for all $\alpha, \beta < \mathfrak{d}$ and put $P = \{p_{\alpha,\beta} : \alpha, \beta < \mathfrak{d}\}$. For every $\alpha, \beta < \mathfrak{d}$ and $n \in \omega$, put $O_n(p_{\alpha,\beta}) = \{p_{\alpha,\beta}\} \cup \{m \in \omega : m > f_\alpha(n)\}$. Then for each $n \in \omega$, $\mathcal{U}_n = \{O_n(p_{\alpha,\beta}) : \alpha, \beta < \mathfrak{d}\} \cup \{X \setminus P\}$ is an open cover of X, so that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open covers of X. Since M has the relative strong star-Menger property in X, there exists a sequence $\{K_n : n \in \omega\}$ of finite subsets of X such that $M \cap \mathcal{A} = \bigcup\{St(K_n, \mathcal{U}_n) : n \in \omega\}$. For $n \in \omega$, put

$$f^*(n) = \begin{cases} max(K_n \cap \omega) + 1, \text{ if } K_n \cap \omega \neq \emptyset \\ 1, \text{ otherwise.} \end{cases}$$

Then there is $\alpha < \mathfrak{d}$ such that $f_\alpha(n) \geq f^*(n)$, for every $n \in \omega$. Also, there is $\beta < \mathfrak{d}$ such that $p_{\alpha,\beta} \notin \bigcup\{K_n : n \in \omega\}$, where $p_{\alpha,\beta} \in M \cap \mathcal{A}$. Then $O_n(p_{\alpha,\beta}) \cap K_n = \emptyset$, for all $n \in \omega$. Since $O_n(p_{\alpha,\beta})$ is the only element of $\mathcal{U}_n$ that contains $p_{\alpha,\beta}$, it follows that $p_{\alpha,\beta} \notin \bigcup\{St(K_n, \mathcal{U}_n) : n \in \omega\}$, a contradiction.

Conversely, let $|M \cap \mathcal{A}| < \mathfrak{d}$ and $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. For each $n \in \omega$ and $A \in \mathcal{A}$, choose $U_{n,A} \in \mathcal{U}_n$ containing A. For each $A \in \mathcal{A}$, define a function $g_A \in \omega^\omega$ by

$$g_A(n) = min\{k \in \omega : k \in U_{n,A}\}.$$

Since $|M \cap \mathcal{A}| < \mathfrak{d}$, there exists a function $f^* \in \omega^\omega$ such that $f^* \not\leq^* g_A$, for all $A \in \mathcal{A}$. Let $A_n = [0, max\{f^*(n), n\}]$. Then $A \in St(A_n, \mathcal{U}_n)$, for some $n \in \omega$. Now for $m \in \omega$, add m to A_n, for each $n \geq m + 1$, if $m \notin A_n$. Hence $M \cap \mathcal{A} = \cup\{St(A_n, \mathcal{U}_n) : n \in \omega\}$.

□

Theorem 4.3 *Assume $\mathfrak{d} = \mathfrak{c}$ and $\neg CH$. Then there exists a closed 2-to-1 continuous mapping $f : X \to Y$ such that $A \subseteq Y$ is a relatively star-Δ-Menger subspace of Y, but $f^{-1}(A)$ is not a relatively star-Δ-Menger subspace of X.*

Proof. Let $Y = \psi(\mathcal{A}) = \omega \cup \mathcal{A}$, with $|\mathcal{A}| = \omega_1$ and $M = \mathcal{A}$ be a subspace of Y. Consider $\Delta =$ the collection of all countably compact subsets of Y. Then by Theorem 4.2, M is relatively star-Δ-Menger in Y. Let $X = A(Y)$. We claim that $A(M)$ is not relatively star-Δ-Menger in X. Since $\mathcal{A}$ is a discrete closed subset of Y with $|\mathcal{A}| = \omega_1$, $\mathcal{A} \times \{1\}$ is an open and closed subset of $A(Y)$ with $|\mathcal{A} \times \{1\}| = \omega_1$ and each point $(a, 1)$ of $\mathcal{A} \times \{1\}$ is isolated. Since M is relatively star-Δ-Menger, so is $\mathcal{A} \times \{1\}$, a contradiction.

Let $f : X \to Y$ be the projection. Then f is the required closed 2-to-1 continuous mapping. Hence the proof.

□

Remark 4.1 From the above example it follows that the Alexandroff duplicate $A(M)$ of a relatively star-Δ-Menger subspace M of a space X need not be relatively star-Δ-Menger.

To get an affirmative answer of the above, we need the following.

Lemma 4.4 *Let A be an open and closed subset of a space X. Then A is a relative star-Δ-Menger subspace of X if and only if A has the star-Δ-Menger property.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of A. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n = \mathcal{U}_n \cup \{X \setminus A\}$ is open cover of X, as A is open in X. By hypothesis, there exists a sequence $\{K_n : n \in \mathbb{N}\}$ of Δ-subsets X such that $A = \cup\{St(K_n, \mathcal{V}_n) : n \in \mathbb{N}\}$. Then $K'_n = K_n \cap A$ is a Δ-subset of A. We claim that the sequence $\{K'_n : n \in \mathbb{N}\}$ witnesses the star-Δ-Menger property of A, for the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$. For $x \in St(K_n, \mathcal{V}_n)$, there exists $U \in \mathcal{V}_n$ such that $x \in U$ and $U \cap K_n \neq \emptyset$. Then $U \cap K_n \subseteq U \subseteq A$ and $U \cap (K_n \cap A) \neq \emptyset$. This implies that $x \in St(K'_n, \mathcal{U}_n)$. Hence A has the star-Δ-Menger property. The converse follows from Theorem 3.4.

□

Next we recall the definition of extent of a space as follows.

Definition 4.5 *For a space X, the extent of X is the supremum of the cardinalities of the closed discrete subsets of X. We denote the extent of X by $e(X)$.*

Lemma 4.6 *[4] For a subspace M of a T_1-space X, $e(M) = e(A(M))$.*

Theorem 4.7 *If X is a T_1-space and $A(M)$ is a relatively star-Δ-Menger subspace of $A(X)$, for some closed subset M of X, then $e(M) < \omega_1$.*

Proof. Suppose that $e(M) \geq \omega_1$. Then there exists a discrete closed subset B of M such that $|B| \geq \omega_1$. Then B is a closed discrete subset of X also. Hence, $B \times \{1\}$ is an open and closed subset of $A(X)$ and each point $(b, 1)$ is isolated for every $b \in B$. Since $B \times \{1\} \subseteq A(M)$, $B \times \{1\}$ is also relatively star-Δ-Menger in $A(X)$. Then by Lemma 4.4, $B \times \{1\}$ has the star-Δ-Menger property, a contradiction.

□

We are now to give a positive answer to our above raised question.

Theorem 4.8 *Let f be an open bijective mapping from a space X to Y. If A is a relative star-Δ-Menger subspace of Y, then $f^{-1}(A)$ is also a relative star-Δ-Menger subspace of X.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{f(U) : U \in \mathcal{U}_n\}$ is an open cover of Y. As A is relatively star-Δ-Menger in Y, there is a sequence $\{K_n : n \in \mathbb{N}\}$ of Δ-subsets of Y such that $A = \cup\{St(K_n, \mathcal{V}_n) : n \in \mathbb{N}\}$. Take $H_n = f^{-1}(K_n)$, for each $n \in \mathbb{N}$. Then the sequence $\{H_n : n \in \mathbb{N}\}$ witnesses the relative star-Δ-Menger property of $f^{-1}(A)$ in X, for the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ in X. Choose $x \in f^{-1}(A)$. Then $f(x) \in A$ and thus $f(x) \in St(K_n, \mathcal{V}_n)$. This implies that $f(x) \in f(U) \in V_n$ and $K_n \cap f(U) \neq \emptyset$. Since f is one-to-one, $K_n \cap U \neq \emptyset$ and $x \in U$. So $x \in St(K_n, \mathcal{U}_n)$, so that $f^{-1}(A) = \cup\{St(K_n, \mathcal{U}_n) : n \in \mathbb{N}\}$.

□

Corollary 4.9 *The star-Δ-Menger property is an inverse invariant under an open bijective mapping.*

Theorem 4.10 *Let f be an open perfect mapping from a space X onto Y. If A is a relative star-Δ-Menger subspace of Y, then $f^{-1}(A)$ is also a relative star-Δ-Menger subspace of X.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then for each $y \in Y$ and each $n \in \mathbb{N}$, there is a finite subfamily $\mathcal{U}_{n_y}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_{n_y}$. Let $U_{n_y} = \bigcup \mathcal{U}_{n_y}$. Then $V_{n_y} = Y \setminus f(X \setminus U_{n_y}) \cap \{f(U) : U \in \mathcal{U}_{n_y}\}$ is an open neighbourhood of y, as f is closed as well as open. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is an open cover of Y. Thus $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of open covers of Y. As A is a relatively star-Δ-Menger subspace of Y, there is a sequence $\{A_n : n \in \mathbb{N}\}$ of Δ-subsets of Y such that $A = \cup\{St(A_n, \mathcal{V}_n) : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $B_n = f^{-1}(A_n)$. Since f is a perfect mapping and hence continuous, B_n is a Δ-subset of X. Hence the sequence $\{B_n : n \in \mathbb{N}\}$ witnesses the relative star-Δ-Menger property of $f^{-1}(A)$ as a subspace of X for $\{\mathcal{U}_n : n \in \mathbb{N}\}$. Let $x \in f^{-1}(A)$ so that $f(x) \in A$. Then $f(x)$ belongs to $St(A_n, \mathcal{V}_n)$. This implies that $f(x) \in V_{n_y} \in \mathcal{V}_n$ and $V_{n_y} \cap A_n \neq \emptyset$. As $V_{n_y} \subseteq f(U)$, for each $U \in \mathcal{U}_{n_y}$, $A_n \cap f(U) \neq \emptyset$. This implies that $U \cap f^{-1}(A_n) \neq \emptyset$. Since $f^{-1}(V_{n_{y_i}}) \subseteq U_{n_{y_i}}$, $U \cap U_{n_{y_i}} \neq \emptyset$. Then there is $U \in \mathcal{U}_n$ containing x and $U \cap B_n \neq \emptyset$.

□

Corollary 4.11 *The star-Δ-Menger property is an inverse invariant under open perfect mappings from a space X onto Y.*

We now discuss about Menger games and star-Δ-Menger games in paracompact spaces.

Hurewicz discovered a very useful description of spaces which have the Menger propery. This description is given most economically in the language of game theory. The Menger game [18] is defined as follows.

Let X be a topological space. Players ONE and TWO play the following infinitely long game: they play an inning for each positive integer. In the n^{th} inning, ONE chooses an open cover $\mathcal{U}_n$ of X; TWO responds by selecting a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. TWO wins the play $\mathcal{U}_1, \mathcal{V}_1, ..., \mathcal{U}_n, \mathcal{V}_n, ...$ of this game if $\bigcup_{n=1}^{\infty} \mathcal{V}_n$ is a cover of X.

We now define the star-Δ-Menger game on X as follows.

Definition 4.12 *The star-Δ-Menger game on a topological space X is defined as follows: for each positive integer n, the player ONE chooses an open cover $\mathcal{O}_n$ of X and TWO responds by choosing a Δ-subset K_n of X. A play $(\mathcal{O}_1, K_1, ..., \mathcal{O}_n, K_n, ...)$ is won by TWO if $\bigcup_{n=1}^{\infty} St(K_n, \mathcal{O}_n)$ is a cover of X.*

Theorem 4.13 *For a paracompact Hausdorff space X, if TWO has a winning strategy in the star-Δ-Menger game on X, then TWO has a winning strategy in the Menger game on X.*

Proof. First, let ONE choose an open cover $\mathcal{U}_1$ of X in the Menger game on X. Since X is paracompact, $\mathcal{U}_1$ has a locally finite open refinement $\mathcal{V}_1$. Let in the first inning of the star-Δ-Menger game, ONE chooses the open cover $\mathcal{V}_1$. Suppose TWO responds by choosing a Δ-subset K_1 of X.

For each $n \in \mathbb{N}$, let $\mathcal{V}_1' = \{V \in \mathcal{V}_1 : V \cap K_1 \neq \emptyset\}$. Then $\mathcal{V}_1'$ is a finite subset of $\mathcal{V}_1$, since $\mathcal{V}_1$ is locally finite. For each $V \in \mathcal{V}_1'$, choose $U_V \in \mathcal{U}_1$ such that $V \subseteq U_V$. Let $\mathcal{W}_1 = \{U_V \in \mathcal{U}_1 : V \in \mathcal{V}_1'\}$. Suppose TWO responds by choosing $\mathcal{W}_1$ in the Menger game.

Next, in the 2^{nd} inning of the Menger game, let ONE chooses an open cover $\mathcal{U}_2$ of X. In the 2^{nd} inning of the star-Δ-Menger game, ONE chooses open cover $\mathcal{V}_2'$ (as $\mathcal{V}_1'$ constructed above) and TWO responds by choosing a Δ-subset K_2 of X. Similarly construct $\mathcal{W}_2 = \{U_V \in \mathcal{U}_2 : V \in \mathcal{V}_2'\}$ as above. Suppose TWO responds by choosing $\mathcal{W}_2$ in the Menger game.

In this way in the n-th inning, ONE choose an open cover $\mathcal{U}_n$ of X. In the 2^{nd} inning of the star-Δ-Menger game, ONE chooses open cover $\mathcal{V}_n'$ (as $\mathcal{V}_1'$ constructed above) and TWO responds by choosing a Δ-subset K_n of X. Construct similarly $\mathcal{W}_n = \{U_V \in \mathcal{U}_n : V \in \mathcal{V}_n'\}$ as in the 1^{st} inning and the game has been played in this way.

Now for each $n \in \mathbb{N}$, we have $St(K_n, \mathcal{V}_n') \subseteq \cup \mathcal{W}_n$. As in the star-$\Delta$-Menger game $(\mathcal{V}_1', K_1, ..., \mathcal{V}_n', K_n, ...)$ TWO has a winning strategy, so for each $x \in X$, $x \in St(K_n, \mathcal{V}_n')$. Therefore for each $x \in X$, $x \in \bigcup \mathcal{W}_n$ and so TWO has a winning strategy in the Menger game on X.

□

In [23] R. Telgársky proved that for a metrizable space X, player TWO has a winning strategy in the Menger game if and only if X is σ-compact. Thus we have the following.

Theorem 4.14 *In a metrizable space X, if TWO has a winning strategy in the star-Δ-Menger game, then X is σ-compact.*

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