An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIX, 2023, f. 1
https://doi.org/10.47743/anstim.2023.00007

A study of Jordan involution on prime rings involving derivations

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Abstract In this manuscript, we discuss the concept of Jordan involution θ on rings. Moreover, we describe the structure of rings which satisfies some θ-differential identities. Few other related results have also been discussed.

Keywords prime ring · Jordan involution · derivation

Mathematics Subject Classification (2010) 16N60 · 16U80 · 16W10 · 16W25.

1 Preliminaries

The success of ring theory dates back to theorems addressing the commutativity of rings whose constituents were predicated on specific types of algebraic conditions. Herstein started these works and published numerous publications on it (specifically [10], [12]) and any references within. Many authors continued to study in this field after then, and the majority of their works made connections between the structure of rings and various maps, such as derivations, involutions, and automorphisms etc. (see [8], [9], [16], [18] and references therein).

We treat $\mathscr{R}$ as an associative ring with $\mathscr{Z}(\mathscr{R})$ as its centre throughout. For $t_1, t_2 \in \mathscr{R}$, $[t_1, t_2] = t_1t_2 - t_2t_1$ referred as Lie product and $t_1 \circ t_2 = t_1t_2 + t_2t_1$ referred as Jordan product, respectively. Assume that $\mathscr{M}$ and $\mathscr{N}$ are two subsets of $\mathscr{R}$. Then, all of the elements of the type $[t_1, t_2]$ represent the additive subgroup of $\mathscr{R}$ represented by $[\mathscr{M}, \mathscr{N}]$, where $t_1 \in \mathscr{M}$ and $t_2 \in \mathscr{N}$. In a same manner, we define $\mathscr{M} \circ \mathscr{N}$. A ring $\mathscr{R}$ is referred to as a prime ring if $t_1\mathscr{R}t_2 = (0)$ (where $t_1, t_2 \in \mathscr{R}$) results in either $t_1 = 0$ or $t_2 = 0$, and a semi prime ring in case $t_1\mathscr{R}t_1 = (0)$ implies $t_1 = 0$. "A mapping $d : \mathscr{R} \to \mathscr{R}$ is said to be a derivation if $d(t_1 + t_2) = d(t_1) + d(t_2)$ and $d(t_1t_2) = d(t_1)t_2 + t_1d(t_2)$ for all $t_1, t_2 \in \mathscr{R}$ and a derivation is said to be inner if $d(t_1) = at_1 - t_1a$ for all $t_1 \in \mathscr{R}$ and a fixed element a of $\mathscr{R}$". Numerous algebraists

have produced numerous conclusions over the past few decades that relate to establishing a link between some additive and multiplicative mappings and the ring structure. Divinski [9] is the one who got started on this task first. His result stated as: "*a simple Artinian ring is commutative if it has a commuting non-trivial automorphism*". A noteworthy finding in this field was provided by Posner [18] in 1957, which read: "*the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative*". Following that, numerous authors extrapolated these findings in numerous directions (see [1], [2], [3] and references therein). If (i) $(s_1s_2)^* = s_2^*s_1^*$ and (ii) $(s_1^*)^* = s_1$ holds for all $s_1, s_2 \in \mathscr{R}$, then an additive map $s_1 \mapsto s_1^*$ of $\mathscr{R}$ onto itself is said to be an involution. According to Yood [19], Jordan involution is an additive map $\theta : \mathscr{R} \to \mathscr{R}$, if for any $t_1, t_2 \in \mathscr{R}$, $t_1^{\theta\theta} = t_1$ and $(t_1t_2+t_2t_1)^\theta = t_1^\theta t_2^\theta + t_2^\theta t_1^\theta$. Obviously, every involution is a Jordan involution but the converse is not true. For example:

Example 1.1 Let $\mathscr{R} = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \;\middle|\; p, q, r, s \in \mathbb{C} \right\}$. Then it is easy to prove that $\mathscr{R}$ is a ring under usual matrix addition and matrix multiplication. Define $\theta : \mathscr{R} \longrightarrow \mathscr{R}$ such that

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}^\theta = \begin{pmatrix} \bar{p} & \bar{q} \\ \bar{r} & \bar{s} \end{pmatrix},$$

where $\bar{p}$ denotes the complex conjugate of p. Obviously, θ is a Jordan involution but not an involution.

Throughout our discussion, we consider $(\mathscr{R}, \theta)$ as a ring with Jordan involution θ. In light of the above definition, we starts with the following:

Definition 1.1 *Let$(\mathscr{R}, \theta)$ be a ring. An element of $\mathscr{R}$ is called θ-symmetric and θ-skew symmetric respectively, if $t_1^\theta = t_1$ and $t_1^\theta = -t_1$, for all $t_1 \in \mathscr{R}$ and the collection of such type elements are denoted by J_H and J_S respectively.*

Now, we have the following remarks:

Remark 1.1 Let $(\mathscr{R}, \theta)$ is 2-torsion free ring and if θ-commuting i.e., $t_1t_1^\theta = t_1^\theta t_1$ for any $t_1 \in \mathscr{R}$. Then $t_1t_1^\theta$ is θ-symmetric.

Remark 1.2 Let $(\mathscr{R}, \theta)$ be a ring. Then $t_1 + t_1^\theta$ and $t_1 - t_1^\theta$ are θ-symmetric and θ-skew symmetric elements of $\mathscr{R}$.

Remark 1.3 Let$(\mathscr{R}, \theta)$ be a 2-torsion free ring. Then every element of $\mathscr{R}$ can be written as sum of θ-symmetric and θ-skew symmetric elements of $\mathscr{R}$, uniquely.

Remark 1.4 Let $(\mathscr{R}, \theta)$ be a 2-torsion free ring. Then for any $t_1 \in \mathscr{R}$, $(t_1\alpha)^\theta = -t_1^\theta\alpha$, where α is a θ-skew symmetric central element of $\mathscr{R}$.

Proof. We have

$$2(t_1\alpha)^\theta = (t_1\alpha + \alpha t_1)^\theta = -\alpha t_1^\theta - t_1^\theta\alpha = -2t_1^\theta\alpha.$$

Since $\mathscr{R}$ is 2-torsion free, so $(t_1\alpha)^\theta = -t_1^\theta\alpha$.

□

One can easily deduce that

Remark 1.5 Let $(\mathscr{R}, \theta)$ be a 2-torsion free ring. Then for any $t_1 \in \mathscr{R}$, $(t_1\beta)^\theta = t_1^\theta\beta$, where is β is a θ-symmetric central element of $\mathscr{R}$.

Remark 1.6 Let $(\mathscr{R}, \theta)$ be a 2-torsion free prime ring and $J_S \cap \mathscr{Z}(\mathscr{R}) \neq (0)$. If $d(\beta) = 0$, where β is a nonzero central θ-symmetric element of $\mathscr{R}$, then $d(z) = 0, \quad \text{for all} \quad z \in \mathscr{Z}(\mathscr{R})$.

Remark 1.7 $[d(t_1^2), d(t_1)] \in \mathscr{Z}(\mathscr{R}) \iff [d(t_1)^2, t_1] \in \mathscr{Z}(\mathscr{R})$.

Let $\mathscr{C}$ be its extended centroid and $\mathscr{Q}$ its Martindale symmetric ring of quotients of $\mathscr{R}$ (see [17] for details). The center of $\mathscr{Q}$ is $\mathscr{C}$, a field, and $\mathscr{RC}$ (the central closure of $\mathscr{R}$) is a prime ring. In fact, $\mathscr{C}$ is the centralizer of $\mathscr{R}$ in $\mathscr{Q}$. The involution θ is said to be first kind if $\lambda^\theta = \lambda, \quad \text{for all} \quad \lambda \in \mathscr{C}$. If not, then θ is said to be second kind (see [5], [7], [10]). Now, we are enough equipped to begin following lemma:

Lemma 1.2 *Let $(\mathscr{R}, \theta)$ be a prime ring and $J_S \cap \mathscr{Z}(\mathscr{R}) \neq (0)$ satisfying $char(\mathscr{R}) \neq 2$. Then $\mathscr{R}$ is commutative $\iff \quad [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}), \quad$ for all $t_1 \in \mathscr{R}$.*

Proof. Since θ is additive, so by [4, Proposition 3.1] we conclude that $[t_1,t_1^\theta] = 0, \quad \text{for all} \quad t_1 \in \mathscr{R}$. Again additivity of θ implies from [4, Theorem 3.2] that

$$t_1^\theta = \lambda t_1 + \mu(t_1) \tag{1.1}$$

for all $t_1 \in \mathscr{R}$, where $\lambda \in \mathscr{C}$, the extended centroid of $\mathscr{R}$ and μ is an additive map from $\mathscr{R}$ to $\mathscr{C}$. Taking t_1 as t_1^θ in last expression gives

$$t_1 = \lambda t_1^\theta + \mu(t_1^\theta) \tag{1.2}$$

for all $t_1 \in \mathscr{R}$. In view of (1.1), we have

$$t_1 = \lambda(\lambda t_1 + \mu(t_1)) + \mu(t_1^\theta) = \lambda^2 t_1 + \lambda\mu(t_1) + \mu(t_1^\theta) \tag{1.3}$$

for all $t_1 \in \mathscr{R}$. By commuting above relation with t_2, it follows

$$[t_1, t_2] = \lambda^2[t_1, t_2] \tag{1.4}$$

for all $t_1, t_2 \in \mathscr{R}$. Now commuting (1.1) with t_2, we obtain

$$[t_1^\theta, t_2] = \lambda[t_1, t_2] \tag{1.5}$$

for all $t_1, t_2 \in \mathscr{R}$. Left multiplication of (1.5) with $\lambda \in \mathscr{C}$ and application of (1.4) yields

$$\lambda[t_1^\theta, t_2] = \lambda^2[t_1, t_2] = [t_1, t_2] \tag{1.6}$$

for all $t_1, t_2 \in \mathscr{R}$. Now for $0 \neq \alpha \in J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{R})$, a central θ-skew symmetric element of $\mathscr{R}$, take t_1 as $t_1^\theta \alpha$ in (1.6). Application of Remark 1.4 yields

$$-\lambda[t_1^\theta, t_2]\alpha = [t_1, t_2]\alpha \tag{1.7}$$

for all $t_1, t_2 \in \mathscr{R}$. Multiply (1.6) by α and combine with (1.7), we infer that $[t_1, t_2]\alpha = 0$ for all $t_1, t_2 \in \mathscr{R}$. Since $0 \neq \alpha$ is central, so we have $t_1 t_2 = t_2 t_1$, for all $t_1, t_2 \in \mathscr{R}$. This gives the required result.

□

Following proposition is also one of the key result:

Proposition 1.3 *Let $(\mathscr{R}, d)$ be a prime ring. If $[d(t_1)^2, t_1] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$, then $dim_{\mathscr{C}} \mathscr{R}\mathscr{C} = 4$.*

Proof. We have

$$[d(t_1)^2, t_1] \in \mathscr{Z}(\mathscr{R}) \tag{1.8}$$

for all $t_1 \in \mathscr{R}$. This can be rewritten as

$$[[d(t_1)^2, t_1], t_2] = 0 \tag{1.9}$$

for all $t_1, t_2 \in \mathscr{R}$. Replace t_2 by $t_2 d(t_1)$ in above expression, we get

$$[[d(t_1)^2, t_1], t_2]d(t_1) + t_2[[d(t_1)^2, t_1], d(t_1)] = 0 \tag{1.10}$$

for all $t_1, t_2 \in \mathscr{R}$. From (1.9), we have

$$t_2[[d(t_1)^2, t_1], d(t_1)] = 0 \tag{1.11}$$

for all $t_1, t_2 \in \mathscr{R}$. Primeness of $\mathscr{R}$ forces that

$$[[t_1, d(t_1)^2], d(t_1)] = 0 \tag{1.12}$$

for all $t_1 \in \mathscr{R}$. In view of [15, Theorem 2.2], we have the required result.

□

Since $[d(t_1)^2, t_1] = [d(t_1^2), d(t_1)]$, for all $t_1 \in \mathscr{R}$, so we have the following:

Corollary 1.4 *Let $(\mathscr{R}, d)$ be a prime ring. If $[d(t_1^2), d(t_1)] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$, then $dim_{\mathscr{C}} \mathscr{R}\mathscr{C} = 4$.*

2 Main results

Many results in the literature assert that certain subsets of a ring $\mathscr{R}$, defined by some sort of commutativity condition, must coincide with its center $\mathscr{Z}(\mathscr{R})$. Herstein [10] proved that for a ring $\mathscr{R}$ with no nil ideals the hypercenter

$$\mathscr{S}(\mathscr{R}) = \{a \in \mathscr{R} |\ at_1^n = t_1^n a, \quad \text{for all} \quad t_1 \in \mathscr{R}, \quad \text{for some} \quad n = n(t_1, a) \geq 1\}$$

coincides with the center $\mathscr{Z}(\mathscr{R})$ of $\mathscr{R}$. Motivated by Herstein's hypercenter, Chacron [6] introduced a more general concept that he called the cohypercenter $T(\mathscr{R})$ of $\mathscr{R}$ defined by

$$T(\mathscr{R}) = \{a \in \mathscr{R} |\ [a, t_1 - t_1^2 p(t_1)] = 0, \quad \text{for all} \quad t_1 \in \mathscr{R}, \quad \text{and} \quad p(t_1) \in \mathscr{Z}(\mathscr{R})[t_1]\}$$

depends on (a, t_1). Shortly, he proved "the cohypercenter of a semiprime ring $\mathscr{R}$ is exactly the center of $\mathscr{R}$ (see [6, Theorem 1])". Inspired by his results, we define the following: let $\mathscr{R}$ be a ring with Jordan involution and $f_{i,j}$ (where $i \neq j$) be mappings from $\mathscr{R}$ to $\mathscr{R}$, then $f_{i,j}$ are said to be $\mathscr{C}_1^\theta$ if

$$f_i([t_1, t_1^\theta]) + [f_j(t_1), f_j(t_1^\theta)] \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1 \in \mathscr{R}, i \neq j \in \{1, 2\};$$

and said to be $\mathscr{C}_2^\theta$, if

$$[f_i(t_1), f_i(t_1^\theta)] + f_j(t_1) \circ f_j(t_1^\theta) \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1 \in \mathscr{R}, i \neq j \in \{1, 2\}.$$

The above definition prompt us to prove the following results:

Theorem 2.1 *Let $(\mathscr{R}, \theta)$ be a prime ring ($J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{R}) \neq (0)$) satisfying char$(\mathscr{R}) \neq 2$, and $\mathscr{R}$ admits derivations $d_{i,j}$'s where $i, j \in \{1, 2\}$ and $i \neq j$. If $d_{i,j}$'s are $\mathscr{C}_1^\theta$ mappings, then either $\mathscr{R} = \mathscr{Z}(\mathscr{R})$ or $\mathscr{R}$ satisfies the standard identity $s_4(x_1, \ldots, x_4)$.*

Proof. Assume that $\mathscr{R}$ does not satisfy $s_4(x_1, \ldots, x_4)$. We will show that $\mathscr{R} = \mathscr{Z}(\mathscr{R})$. Clearly, $\mathscr{Z}(\mathscr{R}) \subseteq \mathscr{R}$, thus we only need to show that $\mathscr{R} \subseteq \mathscr{Z}(\mathscr{R})$. By the given assumption, we have

$$d_i([t_1, t_1^\theta]) + [d_j(t_1), d_j(t_1^\theta)] \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1 \in \mathscr{R}, \tag{2.1}$$

where $i \neq j \in \{1, 2\}$. Now we divide in two cases:

Case I. If $d_i = 0$ for $i \neq j \in \{1, 2\}$, then we have

$$[d_j(t_1), d_j(t_1^\theta)] \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1 \in \mathscr{R}. \tag{2.2}$$

First we handle the case for $'+'$ and linearizing it, we get

$$[d_j(t_1), d_j(t_2^\theta)] + [d_j(t_2), d_j(t_1^\theta)] + [t_1, t_2^\theta] + [t_2, t_1^\theta] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}. \tag{2.3}$$

For any central $0 \neq \beta \in J_{\mathscr{H}}$, replace t_2 by βt_2 in (2.3) and making use of (1.2) gives

$$[([d_j(t_1), t_2^\theta] + [t_2, d_j(t_1^\theta)]), \mathscr{R}] d_j(\beta) = 0, \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}. \tag{2.4}$$

This implies either $[([d_j(t_1), t_2^\theta] + [t_2, d_j(t_1^\theta)]), \mathscr{R}] = 0$ or $d_j(\beta) = 0$, for all $t_1, t_2 \in \mathscr{R}$ and $\beta \in J_{\mathscr{H}} \cap \mathscr{Z}(\mathscr{R})$. Assume that $d_j(\beta) = 0$, for all $\beta \in J_{\mathscr{H}} \cap \mathscr{Z}(\mathscr{R})$. This gives $d(\beta) = 0$, for all $\beta \in J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{R})$. Keep in mind the last argument, put t_2 as $t_2\beta$ where $\beta \in J_{\mathscr{H}} \cap \mathscr{Z}(\mathscr{R})$ in (2.3), we obtain

$$-[d_j(t_1), d_j(t_2^\theta)]\beta + [d_j(t_2), d_j(t_1^\theta)]\beta - [t_1, t_2^\theta]\beta + [t_2, t_1^\theta]\beta \in \mathscr{Z}(\mathscr{R}) \tag{2.5}$$

for all $t_1, t_2 \in \mathscr{R}$. Combination of (2.2) and (2.5) yields

$$([d_j(t_2), d_j(t_1^\theta)] + [t_2, t_1^\theta])\beta \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}. \tag{2.6}$$

This can be further written as

$$([d_j(t_2), d_j(t_1)] + [t_2, t_1])\beta \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}. \tag{2.7}$$

Since β is non-zero central skew θ-symmetric element as α is non-zero, so we have

$$[d_j(t_2), d_j(t_1)] + [t_2, t_1] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}. \tag{2.8}$$

It follows from [14, Theorem 1]. Now consider $[([d_j(t_1), t_2^\theta] + [t_2, d_j(t_1^\theta)]), \mathscr{R}] = 0$. One can easily deduce from last relation that $[d_j(t_1), t_2] = 0$, for all $t_1, t_2 \in \mathscr{R}$. Thus $\mathscr{R}$ must be a commutative from Posner [18]. Similarly we can prove the case for $[d_j(t_1), d_j(t_2^\theta)] + [d_j(t_2), d_j(t_1^\theta)] - [t_1, t_2^\theta] + [t_2, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$ for all $t_1, t_2 \in \mathscr{R}$.

Case II. If $d_j = 0$ for $j \neq i \in \{1, 2\}$, then we have

$$d_i([t_1, t_1^\theta]) \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1 \in \mathscr{R}. \tag{2.9}$$

Follow the same steps as we used in previous case, we arrive at

$$d_i([t_1, t_2]) \pm [t_1, t_2] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}. \tag{2.10}$$

Thus in view of [13, Theorem 1], we get the required result.

Case III. If $d_i = d_j = 0$ for $i = j \in \{1, 2\}$, then in view of Lemma 1.2, we get the desired conclusion.

Case IV. If both d_i and d_j are non-zero, then we have

$$d_i([t_1, t_1^\theta]) + [d_j(t_1), d_j(t_1^\theta)] \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}) \tag{2.11}$$

for all $t_1 \in \mathscr{R}$. Linearization of the above relation gives

$$d_i([t_1, t_2^\theta]) + d_i([t_2, t_1^\theta]) + [d_j(t_1), d_j(t_2^\theta)] + [d_j(t_2), d_j(t_1^\theta)] \quad (2.12)$$
$$\pm[t_1, t_2^\theta] \pm [t_2, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$$

for all $t_1, t_2 \in \mathscr{R}$. Replacing t_2 by $t_2\beta$ in (2.12), where β is a non-zero central θ-symmetric. element of $\mathscr{R}$, and making use of Remark 1.5, we obtain

$$(d_i([t_1, t_2^\theta]) + d_i([t_2, t_1^\theta]) + [d_j(t_1), d_j(t_2^\theta)] + [d_j(t_2), d_j(t_1^\theta)] \pm [t_1, t_2^\theta] \quad (2.13)$$
$$\pm[t_2, t_1^\theta])\beta + ([t_1, t_2^\theta] + [t_2, t_1^\theta])d_i(\beta) + ([d_j(t_1), t_2^\theta] + [t_2, d_j(t_1^\theta)])d_j(\beta) \in \mathscr{Z}(\mathscr{R})$$

for all $t_1, t_2 \in \mathscr{R}$. Applying of (2.12), we get

$$([t_1, t_2^\theta] + [t_2, t_1^\theta])d_i(\beta) + ([d_j(t_1), t_2^\theta] + [t_2, d_j(t_1^\theta)])d_j(\beta) \in \mathscr{Z}(\mathscr{R}) \quad (2.14)$$

for all $t_1, t_2 \in \mathscr{R}$. Taking t_2 by αt_2 in (2.14), where α is a non-zero central θ-skew-symmetric. Thus in light of Remark 1.4, we obtain

$$[t_1, t_2^\theta]\alpha d_i(\beta) + [t_2, t_1^\theta]\alpha d_i(\beta) + [d_j(t_1), t_2^\theta]\alpha d_j(\beta) \quad (2.15)$$
$$+[t_2, d_j(t_1^\theta)]\alpha d_j(\beta) \in \mathscr{Z}(\mathscr{R})$$

for all $t_1, t_2 \in \mathscr{R}$. Combination of (2.14) and (2.15) yields

$$2([t_2, t_1^\theta]d_i(\beta) + [t_2, d_j(t_1^\theta)]d_j(\beta))\alpha \in \mathscr{Z}(\mathscr{R}) \quad (2.16)$$

for all $t_1, t_2 \in \mathscr{R}$. Since $\mathrm{char}(\mathscr{R}) \neq 2$ and $\alpha \neq 0$, so we have

$$[t_2, t_1^\theta]d_i(\beta) + [t_2, d_j(t_1^\theta)]d_j(\beta) \in \mathscr{Z}(\mathscr{R}) \quad (2.17)$$

for all $t_1, t_2 \in \mathscr{R}$. Taking t_2 by t_1^θ, we obtain $[t_1^\theta, d_j(t_1^\theta)]d_j(\beta) \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$. Since $\mathscr{R}$ is prime, yields $[t_1^\theta, d_j(t_1^\theta)] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$ or $d_j(\beta) = 0$. Considering case $[t_1^\theta, d_j(t_1^\theta)] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$, by Posner's result, we are done. Now suppose $d_j(\beta) = 0$, using this in (2.17), we get $[t_2, t_1^\theta]d_i(\beta) \in \mathscr{Z}(\mathscr{R})$, for all $t_1, t_2 \in \mathscr{R}$. As $\mathscr{R}$ is prime, $[t_2, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$, for all $t_1, t_2 \in \mathscr{R}$ or $d_i(\beta) = 0$. If $[t_2, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$, for all $t_1, t_2 \in \mathscr{R}$, putting t_1 for t_2, we obtain $[t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$. Hence by Lemma 1.2, we are done. If $d_i(\beta) = 0$, then $d_i(z) = 0$, for all $z \in \mathscr{Z}(\mathscr{R})$ by Remark 1.6. Now taking t_2 by β, a non-zero θ-symmetric. element of $\mathscr{R}$, in (2.12), we get

$$d_i([t_1, \beta]) + d_i([\beta, t_1^\theta]) + [d_j(t_1), d_j(\beta)] \quad (2.18)$$
$$+[d_j(\beta), d_j(t_1^\theta)] \pm [t_1, \beta] \pm [\beta, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$$

for all $t_1 \in \mathscr{R}$ and $\beta \in J_{\mathscr{H}}(\mathscr{R})$. Substituting $t_1\alpha_0$ for t_1 in (2.18), where α_0 is a θ-skew symmetric non-zero central element of $\mathscr{R}$ and using $d_i(z) = 0$ and $d_j(z) = 0$, for all $z \in \mathscr{Z}(\mathscr{R})$, we get

$$(d_i([t_1, \beta]) - d_i([\beta, t_1^\theta]) + [d_j(t_1), d_j(\beta)] \quad (2.19)$$
$$-[d_j(\beta), d_j(t_1^\theta)] \pm [t_1, \beta] \mp [\beta, t_1^\theta])\alpha_0 \in \mathscr{Z}(\mathscr{R})$$

for all $t_1 \in \mathscr{R}$. Combining (2.19) and (2.18), we get

$$d_i([t_1, \beta]) + [d_j(t_1), d_j(\beta)] \pm [t_1, \beta] \in \mathscr{Z}(\mathscr{R}) \tag{2.20}$$

for all $t_1 \in \mathscr{R}$. Changing β by $\alpha\alpha_0$ in (2.20), where α and α_0 are non-zero θ-symmetric. and central θ-symmetric. elements, respectively. Thus

$$d_i([t_1, \alpha]) + [d_j(t_1), d_j(\alpha)] \pm [t_1, \alpha] \in \mathscr{Z}(\mathscr{R}) \tag{2.21}$$

for all $t_1 \in \mathscr{R}$. Now consider

$$\begin{aligned} 2d_i([t_1, t_2]) + 2[d_j(t_1), d_j(t_2)] \pm 2[t_1, t_2] &= d_i([t_1, 2t_2]) + [d_j(t_1), d_j(2t_2)] \pm [t_1, 2t_2] \\ &= d_i[t_1, \alpha + \beta] + [d_j(t_1), d_j(\alpha) \\ &\quad + d_j(\beta)] \pm [t_1, \alpha + \beta] \\ &= d_i([t_1, \beta]) + d_i([t_1, \alpha]) + [d_j(t_1), d_j(\beta)] \\ &\quad + [d_j(t_1), d_j(\alpha)] \pm [t_1, \beta] \pm [t_1, \alpha]. \end{aligned}$$

Then by (2.20) and (2.21), and Remark 1.3, we get

$$2(d_i([t_1, t_2]) + [d_j(t_1), d_j(t_2)] \pm [t_1, t_2]) \in \mathscr{Z}(\mathscr{R}) \tag{2.22}$$

for all $t_1, t_2 \in \mathscr{R}$. As $\mathrm{char}(\mathscr{R}) \neq 2$, so

$$d_i([t_1, t_2]) + [d_j(t_1), d_j(t_2)] \pm [t_1, t_2] \in \mathscr{Z}(\mathscr{R}) \tag{2.23}$$

for all $t_1, t_2 \in \mathscr{R}$. Changing t_2 by t_1^2 in (2.23), we get $[d_j(t_1), d_j(t_1^2)] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$. Then we get a conclusion from Corollary 1.4. This completes the proof.

□

Corollary 2.2 *Let $(\mathscr{R}, \theta)$ be a prime ring of characteristic different from 2. If $\mathscr{R}$ admits a derivation d such that $[d(t_1), d(t_1^\theta)] \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$, then $\mathscr{R}$ is commutative.*

Corollary 2.3 *Let $(\mathscr{R}, \theta)$ be a prime ring of characteristic different from 2. If $\mathscr{R}$ admits a derivation d such that $d([t_1, t_1^\theta]) \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$, then $\mathscr{R}$ is commutative.*

Theorem 2.4 *Let $(\mathscr{R}, \theta)$ be a prime ring $(J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{R}))$ of characteristic different from 2, and $\mathscr{R}$ admits derivations $d_{i,j}$'s where $i, j \in \{1, 2\}$ and $i \neq j$. If $d_{i,j}$'s are C_2^θ mappings, the either $\mathscr{R} = \mathscr{Z}(\mathscr{R})$ or $\mathscr{R}$ satisfies $s_4(x_1, \ldots, x_4)$.*

Proof. Assume that $\mathscr{R}$ does not satisfy the standard identity $s_4(x_1, \ldots, x_4)$. Then, cleary, $\mathscr{Z}(\mathscr{R}) \subseteq \mathscr{R}$. Only we have to show $\mathscr{R} \subseteq \mathscr{Z}(\mathscr{R})$. We have

$$[d_i(t_1), d_i(t_1^\theta)] + d_j(t_1) \circ d_j(t_1^\theta) \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}) \tag{2.24}$$

for all $t_1 \in \mathscr{R}$. We divide the proof in following cases:

Case I. If $d_i = 0$ for $i \neq j \in \{1, 2\}$, then we have

$$d_j(t_1) \circ d_j(t_1^\theta) \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}) \tag{2.25}$$

taking $\beta \in J_{\mathscr{H}}$, a non-zero θ-symmetric. element, for t_1 in (2.25), we get

$$(d_j(\beta))^2 \in \mathscr{Z}(\mathscr{R}) \tag{2.26}$$

for all $t_1 \in \mathscr{R}$. Substituting $\beta + \beta_1$ in (2.26) for β, where β_1 is a non-zero central θ-symmetric. element, this yields

$$d_j(\beta)d_j(\beta_1) \in \mathscr{Z}(\mathscr{R}). \tag{2.27}$$

This is same as $[d_j(\beta), r]d_j(\beta_1) = 0, \quad \text{for all} \quad r \in \mathscr{R}$. As $\mathscr{R}$ is prime, implies $[d_j(\beta), r] = 0, \quad \text{for all} l \quad r \in \mathscr{R}$ or $d_j(\beta_1) = 0$. First suppose $[d_j(\beta), r] = 0, \quad \text{for all} \quad \beta \in J_{\mathscr{H}}$ and $r \in \mathscr{R}$. Taking β as $\alpha\beta_1$, where β_1 is a non-zero central θ-symmetric. and α is a θ- skew symmetric. This implies $[d_j(\alpha), r] = 0, \quad \text{for all} \quad \alpha \in J_{\mathscr{S}}$ and $r \in \mathscr{R}$. In view of Remark 1.3 and application of Posner's theorem, we get $\mathscr{R}$ is a commutative. Now, consider $d_j(\beta_1) = 0$, which implies $d_j(z) = 0, \quad \text{for all} \quad z \in \mathscr{Z}(\mathscr{R})$ by Remark 1.6. Next, linearizing (2.24), which gives

$$\begin{aligned} &[d_i(t_1), d_i(t_2^\theta)] + [d_i(t_2), d_i(t_1^\theta)] + d_j(t_1) \circ d_j(t_2^\theta) \\ &+d_j(t_2) \circ d_j(t_1^\theta) \pm [t_1, t_2^\theta] \pm [t_2, t_1^\theta] \in \mathscr{Z}(\mathscr{R}) \end{aligned} \tag{2.28}$$

for all $t_1, t_2 \in \mathscr{R}$. Replacing t_2 by $t_2\alpha$ in (2.28), where α is a non-zero central θ-skew symmetric., and using $d_j(z) = 0, \quad \text{for all} \quad z \in \mathscr{Z}(\mathscr{R})$, implies

$$\begin{aligned} &-[d_i(t_1), d_i(t_2^\theta)]\alpha - (d_j(t_1) \circ d_j(t_2^\theta))\alpha \mp [t_1, t_2^\theta]\alpha \\ &+[d_i(t_2), d_i(t_1^\theta)]\alpha + (d_j(t_2) \circ d_j(t_1^\theta)\alpha \\ &\pm [t_2, t_1^\theta]\alpha - [d_i(t_1), t_2^\theta]d_i(\alpha) + [t_2, d_i(t_1^\theta)]d_1(\alpha) \in \mathscr{Z}(\mathscr{R}) \end{aligned} \tag{2.29}$$

for all $t_1, t_2 \in \mathscr{R}$. Combining (2.28) with (2.29), we obtain

$$\begin{aligned} &-[d_i(t_1), d_i(t_2^\theta)]d_1(\alpha) + 2[d_i(t_2), d_i(t_1^\theta)]\alpha^2 + [t_2, d_i(t_1^\theta)]\alpha d_i(\alpha) \\ &+2(d_j(t_2) \circ d_j(t_1^\theta))\alpha^2 \pm 2[t_2, t_1^\theta]\alpha^2 \in \mathscr{Z}(\mathscr{R}) \end{aligned} \tag{2.30}$$

for all $t_1, t_2 \in \mathscr{R}$. Again using similar arguments for (2.30), we obtain

$$-2[d_i(t_1), t_2^\theta]\alpha d_1(\alpha) - 4[t_2, d_i(t_1^\theta)]\alpha d_i(\alpha) \in \mathscr{Z}(\mathscr{R}). \tag{2.31}$$

Changing t_2 by $t_2\alpha$ in (2.31) and using (2.31), we obtain $8[t_2, d_i(t_1^\theta)]\alpha^2$ $d_i(\alpha) \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}$. Since $\text{char}(\mathscr{R}) \neq 2$ and $\alpha \neq 0$, so $[t_2, d_i(t_1^\theta)]d_i(\alpha) \in \mathscr{Z}(\mathscr{R},) \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}$. As $\mathscr{R}$ is prime, $[t_2, d_i(t_1^\theta)] \in \mathscr{Z}(\mathscr{R}), \quad \text{for all} \quad t_1, t_2 \in \mathscr{R}$ or $d_i(\alpha) = 0$. If $[t_2, d_i(t_1^\theta)] \in \mathscr{Z}(\mathscr{R})$, then $\mathscr{R}$ is

commutative by Posner's theorem. Now consider $d_i(\alpha) = 0$. Taking $t_2\alpha$ for t_2 in (2.28), using $d_i(\alpha) = 0$ and $d_j(\alpha) = 0$, we obtain

$$\begin{aligned} &-[d_i(t_1), d_i(t_2^\theta)]\alpha - (d_j(t_1) \circ d_j(t_2^\theta))\alpha \mp [t_1, t_2^\theta]\alpha \\ &+[d_i(t_2), d_i(t_1^\theta)]\alpha + (d_j(t_2) \circ d_j(t_1^\theta)) \pm [t_2, t_1^\theta]\alpha \in \mathscr{Z}(\mathscr{R}) \end{aligned} \tag{2.32}$$

for all $t_1, t_2 \in \mathscr{R}$. Combine (2.32) and (2.28), we get

$$[d_i(t_2), d_i(t_1^\theta)] + d_j(t_2) \circ d_j(t_1^\theta) \pm [t_2, t_1^\theta] \in \mathscr{Z}(\mathscr{R}) \tag{2.33}$$

for all $t_1, t_2 \in \mathscr{R}$. Taking t_1^θ for t_2 in (2.33) gives

$$(d_2(t_1))^2 \in \mathscr{Z}(\mathscr{R}) \tag{2.34}$$

for all $t_1 \in \mathscr{R}$. Replacing t_1 by $t_1 + t_2$, we get

$$d_j(t_1) \circ d_j(t_2) \in \mathscr{Z}(\mathscr{R}) \tag{2.35}$$

for all $t_1, t_2 \in \mathscr{R}$. Taking t_2 for t_1, it gives $d_j(t_1)^2 \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$. This leads to a contradiction by Herstein's theorem (see [11]). Hence the proof is completed.

Case II. If $d_j = 0$ for $j \neq i \in \{1, 2\}$, then we have

$$[d_i(t_1), d_i(t_1^\theta)] \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R}) \tag{2.36}$$

for all $t_1 \in \mathscr{R}$. Now using case (i) of Theorem 2.1, reached to the conclusion.

Case III. Taking both d_i and d_j are zero in (2.24), gives $[t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$. Hence, Lemma 1.2 implies $\mathscr{R} = \mathscr{Z}(\mathscr{R})$.

Case IV. Both d_i and d_j be nonzero, taking β, a non-zero θ-symmetric element of $\mathscr{R}$, for t_1 in (2.24), we get

$$(d_j(\beta))^2 \in \mathscr{Z}(\mathscr{R}). \tag{2.37}$$

for all $\beta \in J_{\mathscr{H}}$. The last expression is same as (2.26). Proceeding same as above, we get the required result.

□

Corollary 2.5 *Let $(\mathscr{R}, \theta)$ be a prime ring satisfying char$(\mathscr{R}) \neq 2$. If $d(t_1) \circ d(t_1^\theta) \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$, then $\mathscr{R}$ is commutative.*

Corollary 2.6 *Let $(\mathscr{R}, \theta)$ be a prime ring satisfying char$(\mathscr{R}) \neq 2$. If $[d(t_1), d(t_1^\theta)] \pm [t_1, t_1^\theta] \in \mathscr{Z}(\mathscr{R})$, for all $t_1 \in \mathscr{R}$, then $\mathscr{R}$ is commutative.*

Acknowledgements The authors are greatly indebted to the referee for his/her valuable suggestions, which have immensely improved the paper.

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Received: 25.VII.2022 / Revised: 29.X.2022 / Accepted: 01.XI.2022

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