A novel generating functions of binary products of Gaussian
\((p, q)\)-numbers with trivariate Fibonacci polynomials

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Abstract In this paper, we give some new generating functions of the products of Gaussian \((p, q)\)-Fibonacci numbers, Gaussian \((p, q)\)-Lucas numbers, Gaussian \((p, q)\)-Pell numbers, Gaussian \((p, q)\)-Pell Lucas numbers and \((p, q)\)-modified Pell numbers with 2-orthogonal Chebyshev polynomials of the first kind and trivariate Fibonacci polynomials.

Keywords symmetric functions · generating functions · Gaussian \((p, q)\)-Fibonacci numbers · Gaussian \((p, q)\)-Pell numbers · 2-orthogonal Chebyshev polynomials of the first kind · trivariate Fibonacci polynomials

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1 Introduction

Generating function were first introduced by Abraham de Moivre in 1730, in order to solve the general linear recurrence problem (see [14] Section 1.2.9, Generating Functions). One can generalize to formal power series in more than one indeterminate, to encode information about infinite multidimensional arrays of numbers.

This concept can be applied to solve many problems in mathematics. There is a huge chunk of mathematics concerning generating functions. It can be used to solve various kinds of counting problems easily, solve recurrence relations by translating the relation in terms of sequence to a problem about functions, prove combinatorial identities.

In simple words, generating functions can be used to translate problems about sequences to problems about functions which are comparatively easy to solve using maneuvers. (For more details, we can see [10,11,17,18,22,30]).

Given a sequence \((a_n)_{n\geq 0}\) of numbers (which can be integers, real numbers or even complex numbers) we try to describe the sequence in as simple a
form as possible. Where possible, the best way is usually to express \( a_n \) as a function of \( n \). Unfortunately, not all sequences can be described directly by such a formula, and in cases where they can, it is not always easy to find the formula. Therefore, in many cases we describe our sequence by a recurrence. Another way we could describe the sequence is to view the \( a_n \) as the coefficients of a formal power series \( F(x) := \sum_{n=0}^{\infty} a_n x^n \), \( F(x) \) is called the generating function of the sequence \( a_n \).

Note that, we can define the exponential (or Hurwitz) generating function of \( a_n \) by

\[
E(x) := \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.
\]

More generally, let \( \Omega = (\omega_0, \omega_1, \ldots) \) be a sequence of nonzero real numbers. Then, following Comtet (see [8] p. 137), we define the \( \omega \)-generating function of the sequence \( a_n \) by

\[
\Omega(x) := \sum_{n=0}^{\infty} a_n x^n \omega_n.
\]

Thus \( F(x) \) and \( E(x) \) are the special cases where \( \omega_n = 1 \) and \( \omega_n = 1/n! \) respectively.

The literature on these topics is extremely vast. See further examples in [2–4, 6, 7, 20, 21, 24, 29, 30].

In this paper, we give the generating functions for the products of each following numbers sequences [12, 13, 25]:

- The Gaussian \((p, q)\)-Fibonacci numbers \( \{GF_{p,q,n}\}_{n \geq 0} \), defined recursively by,

\[
\begin{align*}
GF_{p,q,0} & = i, GF_{p,q,1} = 1, \\
GF_{p,q,n} & = pGF_{p,q,n-1} + qGF_{p,q,n-2}, \quad n \geq 2.
\end{align*}
\]

- The Gaussian \((p, q)\)-Lucas numbers \( \{GL_{p,q,n}\}_{n \geq 0} \), defined by the recurrence relation,

\[
\begin{align*}
GL_{p,q,0} & = 2 - ip, GL_{p,q,1} = p + 2iq, \\
GL_{p,q,n} & = pGL_{p,q,n-1} + qGL_{p,q,n-2}, \quad n \geq 2.
\end{align*}
\]

- The Gaussian \((p, q)\)-Pell numbers \( \{GP_{p,q,n}\}_{n \geq 0} \), defined by,

\[
\begin{align*}
GP_{p,q,0} & = i, GP_{p,q,1} = 1, \\
GP_{p,q,n} & = 2pGP_{p,q,n-1} + qGP_{p,q,n-2}, \quad n \geq 2.
\end{align*}
\]

- The Gaussian \((p, q)\)-Pell Lucas numbers \( \{GQ_{p,q,n}\}_{n \geq 0} \) defined as follows,

\[
\begin{align*}
GQ_{p,q,0} & = 2 - 2ip, GQ_{p,q,1} = 2p + 2iq, \\
GQ_{p,q,n} & = 2pGQ_{p,q,n-1} + qGQ_{p,q,n-2}, \quad n \geq 2.
\end{align*}
\]

And
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- The \((p,q)\)-modified Pell numbers \(\{M_{p,q,n}\}_{n \geq 0}\), given by,

\[
\begin{cases}
M_{p,q,0} = 1, M_{p,q,1} = p, \\
M_{p,q,n} = 2pM_{p,q,n-1} + qM_{p,q,n-2}, n \geq 2.
\end{cases}
\]

With the following polynomials sequences:
- The 2-orthogonal monic Chebyshev polynomials of the first kind \((\hat{T}_n(x))\) studied in [9], and defined by the following relation where \(\alpha\) and \(\gamma\) are constants (see also [19]),

\[
\begin{cases}
\hat{T}_0(x) = 1, \\
\hat{T}_1(x) = x, \\
\hat{T}_2(x) = x^2 - \alpha, \\
\hat{T}_{n+3}(x) = x\hat{T}_{n+2}(x) - \alpha\hat{T}_{n+1}(x) - \gamma\hat{T}_n(x), \quad \gamma \neq 0, \quad n \geq 0.
\end{cases}
\]

And
- The trivariate Fibonacci polynomials, introduced by E.G. Kocer and H. Gedikce in [15], and defined by the next relation,

\[
\begin{cases}
H_0(x, y, t) = 0, \\
H_1(x, y, t) = 1, \\
H_2(x, y, t) = x, \\
H_n(x, y, t) = xH_{n-1}(x, y, t) + yH_{n-2}(x, y, t) + tH_{n-3}(x, y, t), \quad n \geq 3.
\end{cases}
\]

The technique applied here is based on the so-called symmetric functions.

The further contents of this paper are as follows. Section 2 gives some preliminaries that we will need in the sequel. More precisely, we present and prove our main result which relates the symmetric function with the symmetrizing operator \(\delta_{e_1}^{\delta_{e_2}}\). In Section 3, we give some new generating functions related to another Gaussian \((p,q)\)-numbers and 2-orthogonal Chebyshev polynomials. Section 4 is devoted to give some generating functions of the products of Gaussian \((p,q)\)-numbers with the trivariate Fibonacci polynomials.

2 Preliminaries and main results

In this section, we introduce the notion of the symmetric functions and we give some properties (for more details, we can see [5, 16, 23, 26, 27]). Let us now start at the following definitions.

**Definition 2.1** [1] Let \(A\) and \(E\) be any two alphabets, then we give \(S_n(A - E)\) by the following form:

\[
\sum_{n=0}^{\infty} S_n(A - E)z^n = \prod_{e \in E} \frac{1 - ez}{1 - az}, \quad (2.1)
\]

with the condition \(S_n(A - E) = 0\) for \(n < 0\).
In particular, if we take \( A = \{0\} \), the relation (2.1) gives
\[
\sum_{n=0}^{\infty} S_n(-E)z^n = \prod_{e \in E}(1 - e z).
\] (2.2)

Further, in the case \( A = \{0\} \) or \( E = \{0\} \), we have
\[
\sum_{n=0}^{\infty} S_n(A - E)z^n = \left( \sum_{n=0}^{\infty} S_n(A)z^n \right) \times \left( \sum_{n=0}^{\infty} S_n(-E)z^n \right).
\] (2.3)

Thus,
\[
S_n(A - E) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-E) \quad (\text{see [1]}).
\]

**Definition 2.2** [28] Let \( n \) be positive integer and \( E = \{e_1, e_2\} \) are set of given variables. Then, the \( n^{th} \) symmetric function \( S_n(e_1 + e_2) \) is defined by
\[
S_n(E) = S_n(e_1 + e_2) = \frac{e_1^{n+1} - e_2^{n+1}}{e_1 - e_2},
\]
with
\[
S_0(E) = S_0(e_1 + e_2) = 1,
S_1(E) = S_1(e_1 + e_2) = e_1 + e_2,
S_2(E) = S_2(e_1 + e_2) = e_1^2 + e_1e_2 + e_2^2,
\]
\[
\vdots
\]

**Definition 2.3** [6] Given an alphabet \( E = \{e_1, e_2\} \), the symmetrizing operator \( \delta_{e_1e_2}^k \) is defined by
\[
\delta_{e_1e_2}^k f(e_1) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2}, \quad (k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots \}).
\]

In the next theorem, we will combine all these results in a unified way such that all these obtained results can be treated as special case of the following theorem.

**Theorem 2.4** Given two alphabets \( A = \{a_1, a_2, \ldots\} \) and \( E = \{e_1, e_2\} \), we have: for all \( n \in \mathbb{N}_0 \) and \( r \in \{0, 1, 2\} \),
\[
\sum_{n=0}^{\infty} S_n(A) S_{n-r+2}(E) z^n =
\frac{S_{2-r}(E) + e_1e_2S_1(-A)S_{1-r}(E)z + e_1^2 e_2^2 S_2(-A)S_{-r}(E)z^2}{\left( \sum_{n=0}^{\infty} S_n(-A) e_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-A) e_2^n z^n \right)}
\]
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\[ e_1^{3-r} e_2^{3-r} z^{4-r} \sum_{n=0}^{\infty} S_{n-r+4} (-A) S_n (E) z^n \]

\[ - \left( \sum_{n=0}^{\infty} S_n (-A) e_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n (-A) e_2^n z^n \right). \]  

(2.4)

Proof. By applying the operator \( \delta_1^{3-r} \) to the series

\[ f (e_1 z) = \sum_{n=0}^{\infty} S_n (A) e_1^n z^n, \]

we obtain

\[ \delta_1^{3-r} f (e_1 z) = \frac{e_1^{3-r} \sum_{n=0}^{\infty} S_n (A) e_1^n z^n - e_2^{3-r} \sum_{n=0}^{\infty} S_n (A) e_2^n z^n}{e_1 - e_2} \]

\[ = \sum_{n=0}^{\infty} S_n (A) \left( \frac{e_1^{n-r+3} - e_2^{n-r+3}}{e_1 - e_2} \right) z^n \]

\[ = \sum_{n=0}^{\infty} S_n (A) S_{n-r+2} (E) z^n. \]  

(2.5)

On the other hand, by applying the operator \( \delta_1^{3-r} \) to the series

\[ f (e_1 z) = \frac{1}{\sum_{n=0}^{\infty} S_n (-A) e_1^n z^n}, \]

we obtain

\[ \delta_1^{3-r} f (e_1 z) = \frac{e_1^{3-r} \sum_{n=0}^{\infty} S_n (-A) e_1^n z^n - e_2^{3-r} \sum_{n=0}^{\infty} S_n (-A) e_2^n z^n}{e_1 - e_2} \]

\[ = \frac{e_1^{3-r} \sum_{n=0}^{\infty} S_n (-A) e_1^n z^n - e_2^{3-r} \sum_{n=0}^{\infty} S_n (-A) e_2^n z^n}{(e_1 - e_2) \left( \sum_{n=0}^{\infty} S_n (-A) e_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n (-A) e_2^n z^n \right)} \]

\[ = \frac{\sum_{n=0}^{\infty} S_n (-A) e_1^n e_2^{3-r-n} - e_2^{3-r-n} e_1^n}{(e_1 - e_2) \sum_{n=0}^{\infty} S_n (-A) e_1^n z^n \sum_{n=0}^{\infty} S_n (-A) e_2^n z^n} \]

\[ = \left( \sum_{n=0}^{\infty} S_n (-A) e_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n (-A) e_2^n z^n \right). \]

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Equivalently
\[
\delta^{3-r}_{e_1e_2} f (e_1 z) = \sum_{n=0}^{\infty} S_n (-A) e_1^n e_2^n S_{2-n-r} (E) z^n
\]
\[
= \sum_{n=0}^{2-r} S_n (-A) e_1^n e_2^n S_{2-n-r} (E) z^n + \sum_{n=2-r}^{\infty} S_n (-A) e_1^n e_2^n S_{2-n-r} (E) z^n
\]
\[
= \sum_{n=0}^{2-r} S_n (-A) e_1^n e_2^n S_{2-n-r} (E) z^n + \sum_{n=2-r}^{\infty} S_n (-A) e_1^n e_2^n S_{2-n-r} (E) z^n
\]
\[
= \sum_{n=0}^{2-r} S_n (-A) e_1^n e_2^n S_{2-n-r} (E) z^n + \sum_{n=2-r}^{\infty} S_n (-A) e_1^n e_2^n S_{2-n-r} (E) z^n
\]
which also gives
\[
\delta^{3-r}_{e_1e_2} f (e_1 z) = \frac{S_{2-r} (E) + e_1 e_2 S_1 (-A) S_{1-r} (E) z + e_1^2 e_2^2 S_2 (-A) S_{-r} (E) z^2}{\left( \sum_{n=0}^{\infty} S_n (-A) e_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n (-A) e_2^n z^n \right)}
\]
\[
= \frac{S_{2-r} (E) + e_1 e_2 S_1 (-A) S_{1-r} (E) z + e_1^2 e_2^2 S_2 (-A) S_{-r} (E) z^2}{\left( \sum_{n=0}^{\infty} S_n (-A) e_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n (-A) e_2^n z^n \right)}
\]
Hence, from (2.5), we obtain the desired result.

For \( A = \{a_1, a_2, a_3\}, E = \{e_1, e_2\} \) and \( r = 1, 2, 3 \) in the Theorem 2.4 we deduce the following lemmas.

Lemma 2.5 Given two alphabets \( A = \{a_1, a_2, a_3\} \) and \( E = \{e_1, e_2\} \), we have
\[
\sum_{n=0}^{\infty} S_n (a_1 + a_2 + a_3) S_{n+1} (e_1 + e_2) z^n = \frac{e_1 + e_2 - e_1 e_2 (a_1 + a_2 + a_3) z + e_1^2 e_2 a_1 a_2 a_3 z^3}{\prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 - a_i e_2 z)}.
\]
(2.6)
Note that, based on the relationship (2.6), we get
\[
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_n(e_1 + e_2)z^n = \frac{(e_1 + e_2)z - e_1e_2(a_1 + a_2 + a_3)z^2 + e_1^2e_2a_1a_2a_3z^4}{\prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 - a_i e_2 z)} \tag{2.7}
\]

**Lemma 2.6** Given two alphabets \(A = \{a_1, a_2, a_3\}\) and \(E = \{e_1, e_2\}\), we have
\[
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3)S_n(e_1 + e_2)z^n = 1 - e_1e_2(a_1a_2 + a_1a_3 + a_2a_3)z^2 + e_1e_2(e_1 + e_2)a_1a_2a_3z^3 \prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 - a_i e_2 z) \tag{2.8}
\]

From (2.8), we get
\[
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_{n-1}(e_1 + e_2)z^n = \frac{z - e_1e_2(a_1a_2 + a_1a_3 + a_2a_3)z^3 + e_1e_2(e_1 + e_2)a_1a_2a_3z^4}{\prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 - a_i e_2 z)} \tag{2.9}
\]

**Lemma 2.7** Given two alphabets \(A = \{a_1, a_2, a_3\}\) and \(E = \{e_1, e_2\}\), we have
\[
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3)S_{n-1}(e_1 + e_2)z^n = \frac{(a_1 + a_2 + a_3)z - (e_1 + e_2)(a_1a_2 + a_1a_3 + a_2a_3)z^2 + a_1a_2a_3((e_1 + e_2)^2 - e_1e_2)z^3}{\prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 - a_i e_2 z)} \tag{2.10}
\]

### 3 Generating functions of the products of Gaussian \((p, q)\)-numbers with 2-orthogonal Chebyshev polynomials

In this part, we now derive the new generating functions of the products of 2-orthogonal Chebyshev polynomials with Gaussian \((p, q)\)-Fibonacci numbers, Gaussian \((p, q)\)-Lucas numbers, Gaussian \((p, q)\)-Pell numbers, Gaussian \((p, q)\)-Pell Lucas numbers and \((p, q)\)-modified Pell numbers.

Replacing \(e_2\) by \((-e_2)\) in (2.8) and (2.10), we obtain
\[
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3)S_n(e_1 + [-e_2])z^n =
\]
\[
1 + e_1 e_2 (a_1 a_2 + a_1 a_3 + a_2 a_3) z^2 - e_1 e_2 (e_1 - e_2) a_1 a_2 a_3 z^3 \prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 + a_i e_2 z) \]  
(3.1)

\[
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n =

(a_1 + a_2 + a_3) z - (e_1 - e_2) (a_1 a_2 + a_1 a_3 + a_2 a_3) z^2 + a_1 a_2 a_3 ((e_1 - e_2)^2 + e_1 e_2) z^3 \prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 + a_i e_2 z)
\]  
(3.2)

This case consists of two related parts. **Firstly**, the substitutions

\[
\begin{align*}
& a_1 + a_2 + a_3 = x \\
& a_1 a_2 + a_1 a_3 + a_2 a_3 = \alpha \\
& a_1 a_2 a_3 = -\gamma
\end{align*}
\]

and we have the following results.

in (3.1) and (3.2), we obtain

\[
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n = \frac{1 + \alpha q z^2 + \gamma pq z^3}{f_{\alpha, \gamma}(z)},
\]  
(3.3)

\[
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n = \frac{x z - \alpha pz^2 - \gamma (p^2 + q) z^3}{f_{\alpha, \gamma}(z)},
\]  
(3.4)

with

\[
f_{\alpha, \gamma}(z) = 1 - pxz + (\alpha (p^2 + 2q) - qx^2) z^2 + p (\gamma (p^2 + 3q) + \alpha qx) z^3 + q (\gamma x (p^2 + 2q) + \alpha^2 q) z^4 + \alpha \gamma pq^2 z^5 - \gamma^2 q^3 z^6,
\]

and we have the following results.

**Theorem 3.1** For \( n \in \mathbb{N} \), the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with Gaussian \((p, q)\)-Fibonacci numbers is given by

\[
\sum_{n=0}^{\infty} \hat{T}_n(x) GF_{p,q,n} z^n =

\frac{i + x (1 - pi) z + \alpha (i (p^2 + q) - p) z^2 + \gamma (ip(p^2 + 2q) - (p^2 + q)) z^3}{f_{\alpha, \gamma}(z)}
\]  
(3.5)
Proof. We have \[ GF_{p,q,n} = i S_n(e_1 + [-e_2]) + (1 - ip) S_{n-1}(e_1 + [-e_2]). \]

Then we obtain
\[
\sum_{n=0}^{\infty} \hat{T}_n(x) GF_{p,q,n} z^n = \\
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) (i S_n(e_1 + [-e_2]) + (1 - ip) S_{n-1}(e_1 + [-e_2])) z^n \\
= i \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n \\
+ (1 - ip) \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n.
\]

According to the relations (3.3) and (3.4), we obtain
\[
\sum_{n=0}^{\infty} \hat{T}_n(x) GF_{p,q,n} z^n = \frac{i (1 + \alpha qz^2 + \gamma pqz^2) + (1 - ip) (xz - \alpha pz^2 - \gamma (p^2 + q)z^3)}{f_{\alpha,\gamma}(z)} \\
= \frac{i + x (1 - pi) z + \alpha (i (p^2 + q) - p) z^2 + \gamma (ip(p^2 + 2q) - (p^2 + 2q)) z^3}{f_{\alpha,\gamma}(z)}.
\]

Hence the result.

\[\square\]

**Theorem 3.2** For \( n \in \mathbb{N} \), the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with Gaussian \((p,q)\)-Lucas numbers is given by
\[
\sum_{n=0}^{\infty} \hat{T}_n(x) GL_{p,q,n} z^n = \frac{2 - ip + x (i (p^2 + 2q) - p) z + \alpha (p^2 + 2q - ip (p^2 + 3q)) z^2}{f_{\alpha,\gamma}(z)} + \frac{\gamma (p (p^2 + 3q) - i (p^4 + 4p^2q + 2q^2)) z^3}{f_{\alpha,\gamma}(z)}. \tag{3.6}
\]

**Proof.** We know that 
\[ GL_{p,q,n} = (2 - ip) S_n(e_1 + [-e_2]) + (i (p^2 + 2q) - p) S_{n-1}(e_1 + [-e_2]), \] (see [25]).

We can see that
\[
\sum_{n=0}^{\infty} \hat{T}_n(x) GL_{p,q,n} z^n = \\
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) ((2 - ip) S_n(e_1 + [-e_2]) + (i (p^2 + 2q) - p) S_{n-1}(e_1 + [-e_2])) z^n
\]

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\[ = (2 - ip) \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3)S_n(e_1 + [-e_2])z^n + (i(p^2 + 2q) - p) \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3)S_{n-1}(e_1 + [-e_2])z^n. \]

Then, according to the relationships (3.3) and (3.4), we obtain

\[ \sum_{n=0}^{\infty} \hat{T}_n(x) \text{GL}_{p,q,n}z^n = \frac{(2 - ip)(1 + \alpha qz^2 + \gamma pqz^3)}{f_{\alpha,\gamma}(z)} \]

\[ + \frac{(ip^2 + 2q - p)(xz - \alpha p^2 - \gamma(p^2 + q)z^3)}{f_{\alpha,\gamma}(z)} \]

\[ = 2 - ip + x(i(p^2 + 2q) - p)z + \alpha (p^2 + 2q - ip(p^2 + 3q))z^2 \]

\[ + \gamma \left( p(p^2 + 3q) - i(p^4 + 4p^2q + 2q^2) \right)z^3. \]

This completes the proof. \(\square\)

By the relationships (3.5) and (3.6), we have two cases.

**Case 1.** With \(\alpha = 0\), we obtain

\[ f_{0,\gamma}(z) = 1 - pxz - qx^2z^2 + \gamma p(p^2 + 3q)z^3 + \gamma qx(p^2 + 2q)z^4 - \gamma^2q^3z^6, \]

and we have the following corollaries.

**Corollary 3.3** For \(n \in \mathbb{N}\), the new generating function of the product of 2-orthogonal Chebyshev MPS of the second kind with Gaussian \((p, q)\)-Fibonacci numbers is given by

\[ \sum_{n=0}^{\infty} \hat{U}_n(x) \text{GF}_{p,q,n}z^n = \frac{i + x(1 - pi)z + \gamma(ip(p^2 + 2q) - (p^2 + q))z^3}{f_{0,\gamma}(z)}. \]

**Corollary 3.4** For \(n \in \mathbb{N}\), the new generating function of the product of 2-orthogonal Chebyshev MPS of the second kind with Gaussian \((p, q)\)-Lucas numbers is given by

\[ \sum_{n=0}^{\infty} \hat{U}_n(x) \text{GL}_{p,q,n}z^n = \frac{2 - ip + x(i(p^2 + 2q) - p)z}{f_{0,\gamma}(z)} \]

\[ + \frac{\gamma \left( p(p^2 + 3q) - i(p^4 + 4p^2q + 2q^2) \right)z^3}{f_{0,\gamma}(z)}. \]
Case 2. With $\alpha = 3$ and $\gamma = -1$, we obtain
\[
f_{3,-1}(z) = 1 - pxz + (3p^2 + 6q - qx^2) z^2 + p (3qx - p^2 - 3q) z^3 + q (9q - x (p^2 + 2q)) z^4 - 3pq^2 z^5 - q^3 z^6,
\]
and we have the following results.

**Corollary 3.5** For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with Gaussian $(p, q)$-Fibonacci numbers is given by
\[
\sum_{n=0}^{\infty} \tilde{T}_n(x) GF_{p,q,n} z^n = \frac{i + x (1 - pi) z + 3 \left( i \left( p^2 + q \right) - p \right) z^2}{f_{3,-1}(z)} + \frac{- (ip^2 + 2q) - (p^2 + q)) z^3}{f_{3,-1}(z)}.
\]

**Corollary 3.6** For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with Gaussian $(p, q)$-Lucas numbers is given by
\[
\sum_{n=0}^{\infty} \tilde{T}_n(x) GL_{p,q,n} z^n = \frac{2 - ip + x \left( i \left( p^2 + 2q \right) - p \right) z + 3 \left( p^2 + 2q - ip \left( p^2 + 3q \right) \right) z^2}{f_{3,-1}(z)} - \frac{\left( p \left( p^2 + 3q \right) - i \left( p^3 + 4p^2 q + 2q^2 \right) \right) z^3}{f_{3,-1}(z)}.
\]

Secondly, the substitutions
\[
\begin{align*}
\begin{cases}
a_1 + a_2 + a_3 = x \\
a_1 a_2 + a_1 a_3 + a_2 a_3 = \alpha \\
a_1 a_2 a_3 = -\gamma
\end{cases}
\quad\text{and}\quad
\begin{cases}
e_1 - e_2 = 2p \\
e_1 e_2 = q
\end{cases},
\end{align*}
\]
in (3.1) and (3.2), we obtain
\[
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n = \frac{1 + \alpha q z^2 + 2 \gamma pq z^3}{g_{\alpha, \gamma}(z)}, \quad (3.7)
\]
\[
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n = \frac{xz - 2 \alpha p z^2 - \gamma \left( 4p^2 + q \right) z^3}{g_{\alpha, \gamma}(z)}, \quad (3.8)
\]
with
\[
g_{\alpha, \gamma}(z) = 1 - 2pxz + \left( 2\alpha \left( 2p^2 + q \right) - qx^2 \right) z^2 + 2p \left( \gamma \left( 4p^2 + 3q \right) + \alpha qx \right) z^3 + q \left( \alpha^2 q + 2\gamma x \left( 2p^2 + q \right) \right) z^4 + 2\alpha \gamma pq^2 z^5 - \gamma^2 q^3 z^6,
\]
and we have the following results.
Theorem 3.7 For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with Gaussian $(p,q)$-Pell numbers is given by

$$\sum_{n=0}^{\infty} \hat{T}_n(x) GP_{p,q,n} z^n =$$

$$i + x (1 - 2ip) z + \alpha (i (4p^2 + q) - 2p) z^2 + \gamma (4ip (2p^2 + q) - (4p^2 + q)) z^3 \frac{g_{\alpha,\gamma}(z)}{g_{\alpha,\gamma}(z)}.$$

(3.9)

Proof. Recall that, we have

$$GP_{p,q,n} = i S_n(e_1 + [-e_2]) + (1 - 2ip) S_{n-1}(e_1 + [-e_2])$$

(see [25]). We see that

$$\sum_{n=0}^{\infty} \hat{T}_n(x) GP_{p,q,n} z^n =$$

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) (i S_n(e_1 + [-e_2]) + (1 - 2ip) S_{n-1}(e_1 + [-e_2])) z^n$$

$$= i \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n$$

$$+ (1 - 2ip) \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n.$$

Using the relationships (3.7) and (3.8), we obtain

$$\sum_{n=0}^{\infty} \hat{T}_n(x) GP_{p,q,n} z^n =$$

$$\frac{i (1 + \alpha q z^2 + 2\gamma p q z^3)}{g_{\alpha,\gamma}(z)} +$$

$$\frac{(1 - 2ip) (x z - 2 \alpha p z^2 - \gamma (4p^2 + q) z^3)}{g_{\alpha,\gamma}(z)}$$

$$= i + x (1 - 2ip) z + \alpha (i (4p^2 + q) - 2p) z^2 +$$

$$\frac{\gamma (4ip (2p^2 + q) - (4p^2 + q)) z^3}{g_{\alpha,\gamma}(z)}.$$

This completes the proof. \qed
A novel generating functions of binary products of Gaussian \((p,q)\)-numbers

**Theorem 3.8** For \(n \in \mathbb{N}\), the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with Gaussian \((p,q)\)-Pell Lucas numbers is given by

\[
\sum_{n=0}^{\infty} \hat{T}_n(x) GQ_{p,q,n} z^n = \frac{2 - 2ip + 2x \left( i \left( 2p^2 + q \right) - p \right) z}{g_{\alpha,\gamma}(z)} + \\
\frac{2\alpha \left( 2p^2 + q - ip \left( 4p^2 + 3q \right) \right) z^2}{g_{\alpha,\gamma}(z)} + \\
\frac{2\gamma \left( p(4p^2 + 3q) - i \left( 8p^4 + 8p^2q + q^2 \right) \right) z^3}{g_{\alpha,\gamma}(z)}.
\]  

(3.10)

**Proof.** Recall that, we have

\[GQ_{p,q,n} = (2 - 2ip) S_n(e_1 + [-e_2]) + (i \left( 4p^2 + 2q \right) - 2p) S_{n-1}(e_1 + [-e_2]) \text{ (see [25])}.
\]

We see that

\[
\sum_{n=0}^{\infty} \hat{T}_n(x) GQ_{p,q,n} z^n = \\
\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) (2 - 2ip) S_n(e_1 + [-e_2]) + (i \left( 4p^2 + 2q \right) - 2p) S_{n-1}(e_1 + [-e_2]) z^n
\]

\[= (2 - 2ip) \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n + (i \left( 4p^2 + 2q \right) - 2p) \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n.
\]

By using (3.7) and (3.8), we obtain

\[
\sum_{n=0}^{\infty} \hat{T}_n(x) GQ_{p,q,n} z^n = \frac{2 - 2ip \left( 1 + \alpha qz^2 + 2\gamma pqz^3 \right)}{g_{\alpha,\gamma}(z)} + \\
\frac{ \left( i \left( 4p^2 + 2q \right) - 2p \right) \left( xz - 2\alpha pz^2 - \gamma \left( 4p^2 + q \right) z^3 \right) }{g_{\alpha,\gamma}(z)}
\]

\[= \frac{2 - 2ip + 2x \left( i \left( 2p^2 + q \right) - p \right) z + 2\alpha \left( 2p^2 + q - ip \left( 4p^2 + 3q \right) \right) z^2}{g_{\alpha,\gamma}(z)} + \\
\frac{2\gamma \left( p(4p^2 + 3q) - i \left( 8p^4 + 8p^2q + q^2 \right) \right) z^3}{g_{\alpha,\gamma}(z)}.
\]

This completes the proof. \(\square\)
Theorem 3.9 For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with $(p,q)$-modified Pell numbers is given by
\[
\sum_{n=0}^{\infty} \hat{T}_n(x) MP_{p,q,n} z^n = \frac{1 - pxz + \alpha (2p^2 + q) z^2 + \gamma p (4p^2 + 3q) z^3}{g_{\alpha,\gamma}(z)}.
\] (3.11)

Proof. Recall that, we have
\[ MP_{p,q,n} = S_n(e_1 + [-e_2]) - p S_{n-1}(e_1 + [-e_2]) \text{ (see[25]).} \]
We see that
\[
\sum_{n=0}^{\infty} \hat{T}_n(x) MP_{p,q,n} z^n = \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n
\]
\[ = \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n \]
\[ - p \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n. \]

Using the relationships (3.7) and (3.8), we obtain
\[
\sum_{n=0}^{\infty} \hat{T}_n(x) MP_{p,q,n} z^n = \frac{1 + \alpha qz^2 + 2\gamma pqz^3 - p (xz - 2\alpha p z^2 - \gamma (4p^2 + q) z^3)}{g_{\alpha,\gamma}(z)}
\]
\[ = \frac{1 - pxz + \alpha (2p^2 + q) z^2 + \gamma p (4p^2 + 3q) z^3}{g_{\alpha,\gamma}(z)}. \]

This completes the proof. 

By the relationships (3.9), (3.10) and (3.11), we have two cases.

Case 1. With $\alpha = 0$, we obtain
\[ g_{0,\gamma}(z) = 1 - 2pxz - qz^2 + 2\gamma p (4p^2 + 3q) z^3 + 2\gamma q x (2p^2 + q) z^4 - \gamma^2 q^3 z^6, \]
and we have the following corollaries.

Corollary 3.10 For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the second kind with Gaussian $(p,q)$-Pell numbers is given by
\[
\sum_{n=0}^{\infty} \hat{U}_n(x) GP_{p,q,n} z^n = \frac{i + x (1 - 2ip) z + \gamma (4ip (2p^2 + q) - (4p^2 + q)) z^3}{g_{0,\gamma}(z)}. \]
Corollary 3.11 For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the second kind with Gaussian $(p, q)$-Pell Lucas numbers is given by

$$\sum_{n=0}^{\infty} \hat{U}_n (x) GQ_{p,q,n} z^n = 2 - 2ip + 2x \left( i \left( 2p^2 + q \right) - p \right) z + 2\gamma \left( p(4p^2 + 3q) - i \left( 8p^4 + 8p^2q + q^2 \right) \right) z^3 \frac{g_{0,\gamma}(z)}{g_{3,\gamma}(z)}.$$

Corollary 3.12 For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the second kind with Gaussian $(p, q)$-modified Pell numbers is given by

$$\sum_{n=0}^{\infty} \hat{U}_n (x) MP_{p,q,n} z^n = \frac{1 - pxz + \gamma p \left( 4p^2 + 3q \right) z^3}{g_{0,\gamma}(z)}.$$

Case 2. With $\alpha = 3$ and $\gamma = -1$, we obtain

$$g_{3,-1}(z) = 1 - 2pxz + (12p^2 + 6q - qx^2) z^2 + 2p (3qx - 4p^2 - 3q) z^3 + q \left( 9q - 2x (2p^2 + q) \right) z^4 - 6pq^2 z^5 - q^3 z^6,$$

and we have the following corollaries.

Corollary 3.13 For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with Gaussian $(p, q)$-Pell numbers is given by

$$\sum_{n=0}^{\infty} \hat{T}_n (x) GP_{p,q,n} z^n = \frac{i + x \left( 1 - 2ip \right) z + 3 \left( i \left( 4p^2 + q \right) - 2p \right) z^2 - \left( 4ip \left( 2p^2 + q \right) - \left( 4p^2 + q \right) \right) z^3}{g_{3,-1}(z)}.$$

Corollary 3.14 For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with Gaussian $(p, q)$-Pell Lucas numbers is given by

$$\sum_{n=0}^{\infty} \hat{T}_n (x) GQ_{p,q,n} z^n = \frac{2 - 2ip + 2x \left( i \left( 2p^2 + q \right) - p \right) z + 6 \left( 2p^2 + q - ip \left( 4p^2 + 3q \right) \right) z^2}{g_{3,-1}(z)} - \frac{2 \left( p(4p^2 + 3q) - i \left( 8p^4 + 8p^2q + q^2 \right) \right) z^3}{g_{3,-1}(z)}.$$

Corollary 3.15 For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with $(p, q)$-modified Pell numbers is given by

$$\sum_{n=0}^{\infty} \hat{T}_n (x) MP_{p,q,n} z^n = \frac{1 - pxz + 3 \left( 2p^2 + q \right) z^2 - p \left( 4p^2 + 3q \right) z^3}{g_{3,-1}(z)}.$$
4 Generating functions of the products of Gaussian 
\((p, q)\)-numbers with trivariate Fibonacci polynomials

Our next goal is to derive the new generating functions of the products of trivariate Fibonacci polynomials with Gaussian \((p, q)\)-Fibonacci numbers, Gaussian \((p, q)\)-Lucas numbers, Gaussian \((p, q)\)-Pell numbers, Gaussian \((p, q)\)-Pell Lucas numbers and \((p, q)\)-modified Pell numbers.

Replacing \(e_2\) by \((-e_2)\) in (2.7) and (2.9), we obtain

\[
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_n(e_1 + [-e_2])z^n =
\]

\[
\frac{(e_1 - e_2) z + e_1 e_2 (a_1 + a_2 + a_3) z^2 + e_1^2 e_2^2 a_1 a_2 a_3 z^4}{\prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 + a_i e_2 z)}.
\]

This case consists of two related parts. Firstly, the substitutions

\[
\begin{align*}
&\left\{ \begin{array}{l}
a_1 + a_2 + a_3 = x \\
a_1 a_2 + a_1 a_3 + a_2 a_3 = -y \\
a_1 a_2 a_3 = t
\end{array} \right.
\text{ and } \left\{ \begin{array}{l}
e_1 - e_2 = p \\
e_1 e_2 = q
\end{array} \right.
\]

in (4.1) and (4.2), we obtain

\[
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_n(e_1 + [-e_2])z^n =
\]

\[
\frac{z + e_1 e_2 (a_1 a_2 + a_1 a_3 + a_2 a_3) z^3 - e_1 e_2 (e_1 - e_2) a_1 a_2 a_3 z^4}{\prod_{i=1}^{3} (1 - a_i e_1 z) \prod_{i=1}^{3} (1 + a_i e_2 z)}.
\]

with

\[
f_{x,y,t}(z) = 1 - p x z - (y (p^2 + 2q) + q x z) z^2 - p (t (p^2 + 3q) + q y x) z^3 + q (q y^2 - x t (p^2 + 2q)) z^4 + p q y t z^5 - q^3 t^2 z^6,
\]

and we have the following results.
Theorem 4.1 For $n \in \mathbb{N}$, the new generating function of the product of trivariate Fibonacci polynomials with Gaussian $(p,q)$-Fibonacci numbers is given by

$$
\sum_{n=0}^{\infty} H_n(x,y,t) \text{GF}_{p,q,n} z^n = \frac{z + iqxz^2 + qy(ip - 1)z^3 + qt(i(p^2 + q) - p)z^4}{f_{x,y,t}(z)}.
$$

(4.5)

Proof. By referred to [25], we have

$$
\text{GF}_{p,q,n} = iS_n(e_1 + [-e_2]) + (1 - pi)S_{n-1}(e_1 + [-e_2]).
$$

We can easily see that

$$
\sum_{n=0}^{\infty} H_n(x,y,t) \text{GF}_{p,q,n} z^n = 
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)(iS_n(e_1 + [-e_2]) + (1 - pi)S_{n-1}(e_1 + [-e_2])) z^n
$$

$$
= i \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_n(e_1 + [-e_2])z^n
$$

$$
+ (1 - pi) \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_{n-1}(e_1 + [-e_2])z^n.
$$

According to (4.3) and (4.4) this gives the following equality

$$
\sum_{n=0}^{\infty} H_n(x,y,t) \text{GF}_{p,q,n} z^n = \frac{i(pz + qxz^2 + q^2tz^4)}{f_{x,y,t}(z)} + \frac{(1 - pi)(z - qy^3 - pqtz^4)}{f_{x,y,t}(z)}
$$

$$
= \frac{z + iqxz^2 + qy(ip - 1)z^3 + qt(i(p^2 + q) - p)z^4}{f_{x,y,t}(z)},
$$

then, the desired result.

Theorem 4.2 For $n \in \mathbb{N}$, the new generating function of the product of trivariate Fibonacci polynomials with Gaussian $(p,q)$-Lucas numbers is given by

$$
\sum_{n=0}^{\infty} H_n(x,y,t) \text{GL}_{p,q,n} z^n = \frac{(p + 2q)z + qz(2 - ip)z^2 + qy(p - i(p^2 + 2q))z^3 + qt(p^2 + 2q - ip(p^2 + 3q))z^4}{f_{x,y,t}(z)}.
$$

(4.6)
Proof. By referred to [25], we have
\[ GL_{p,q,n} = (2 - ip) S_n(e_1 + [-e_2]) + (i (p^2 + 2q) - p) S_{n-1}(e_1 + [-e_2]). \]
This gives
\[ \sum_{n=0}^{\infty} H_n(x, y, t) GL_{p,q,n} z^n = \]
\[ \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3) ((2 - ip) S_n(e_1 + [-e_2]) + (i (p^2 + 2q) - p) S_{n-1}(e_1 + [-e_2])) z^n \]
\[ = (2 - ip) \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n \]
\[ + (i (p^2 + 2q) - p) \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n. \]
According to relationships (4.3) and (4.4), we obtain
\[ \sum_{n=0}^{\infty} H_n(x, y, t) GL_{p,q,n} z^n = (2 - ip) \left( p z + q x z^2 + q^2 t z^4 \right) + \]
\[ (i (p^2 + 2q) - p) \left( z - q y z^3 - q t z^4 \right) \]
\[ = \frac{(p + 2iq) z + qx (2 - ip) z^2 + qy (p - i (p^2 + 2q)) z^3 + \}
\[ qt (p^2 + 2q - ip (p^2 + 3q)) z^4}{f_{x,y,t}(z)}, \]
which completes the proof.

By the relationships (4.5) and (4.6), we have three cases.

Case 1. Writing \( x^2 \) instead of \( x \), \( x \) instead of \( y \) and taking \( t = 1 \), we obtain
\[ f_{x^2,x,1}(z) = 1 - px^2 z - x (p^2 + 2q + q x^3) z^2 - p (q x^3 + p^2 + 3q) z^3 - \]
\[ q x^2 (p^2 + q) z^4 + pq^2 x z^5 - q^3 z^6, \]
and we have the following corollaries.

Corollary 4.3 For \( n \in \mathbb{N} \), the new generating function of the product of Tribonacci polynomials with Gaussian \((p,q)\)-Fibonacci numbers is given by
\[ \sum_{n=0}^{\infty} T_n(x) GF_{p,q,n} z^n = \frac{z + iq x^2 z^2 + qx (ip - 1) z^3 + q (i (p^2 + q) - p) z^4}{f_{x^2,x,1}(z)}. \]
Corollary 4.4 For \( n \in \mathbb{N} \), the new generating function of the product of Tribonacci polynomials with Gaussian \((p,q)\)-Lucas numbers is given by

\[
\sum_{n=0}^{\infty} T_n(x) GL_{p,q,n} z^n = \frac{(p+2iq) z + qx^2 (2-ip) z^2 + qx (p - i (p^2 + 2q)) z^3 + q (p^2 + 2q - ip (p^2 + 3q)) z^4}{f_{x,z,1}(z)}.
\]

Case 2. With \( x = y = t = 1 \), we obtain

\[
f_{1,1,1}(z) = 1 - pqz - (p^2 + 3q) z^2 - p (p^2 + 4q) z^3 - q (p^2 + q) z^4 + pqz^5 - q^3z^6,
\]

and we have the following corollaries.

Corollary 4.5 For \( n \in \mathbb{N} \), the new generating function of the product of Tribonacci numbers with Gaussian \((p,q)\)-Fibonacci numbers is given by

\[
\sum_{n=0}^{\infty} T_n GF_{p,q,n} z^n = \frac{z + iqz^2 + q (ip - 1) z^3 + q (i (p^2 + q) - p) z^4}{f_{1,1,1}(z)}.
\]

Corollary 4.6 For \( n \in \mathbb{N} \), the new generating function of the product of Tribonacci numbers with Gaussian \((p,q)\)-Lucas numbers is given by

\[
\sum_{n=0}^{\infty} T_n GL_{p,q,n} z^n = \frac{(p+2iq) z + q (2-ip) z^2 + q (p - i (p^2 + 2q)) z^3 + q (p^2 + 2q - ip (p^2 + 3q)) z^4}{f_{1,1,1}(z)}.
\]

Case 3. With \( t = 0 \), we obtain

\[
f_{x,y,0}(z) = 1 - pxz - (y (p^2 + 2q) + qx^2) z^2 - pqxyz^3 + q^2 y^2 z^4,
\]

and we have the following corollaries.

Corollary 4.7 For \( n \in \mathbb{N} \), the new generating function of the product of bivariate Fibonacci polynomials with Gaussian \((p,q)\)-Fibonacci numbers is given by

\[
\sum_{n=0}^{\infty} F_n(x,y) GF_{p,q,n} z^n = \frac{z + iqxz^2 + qy (ip - 1) z^3}{f_{x,y,0}(z)}.
\]

Corollary 4.8 For \( n \in \mathbb{N} \), the new generating function of the product of bivariate Fibonacci polynomials with Gaussian \((p,q)\)-Lucas numbers is given by

\[
\sum_{n=0}^{\infty} F_n(x,y) GL_{p,q,n} z^n = \frac{(p+2iq) z + qx (2-ip) z^2 + qy (p - i (p^2 + 2q)) z^3}{f_{x,y,0}(z)}.
\]
Secondly, the substitutions
\[
\begin{aligned}
  a_1 + a_2 + a_3 &= x \\
a_1a_2 + a_1a_3 + a_2a_3 &= -y \quad \text{and} \quad e_1 - e_2 = 2p
\end{aligned}
\]

\[
\begin{aligned}
a_1a_2a_3 &= t \\
e_1e_2 &= q
\end{aligned}
\]

in (4.1) and (4.2), we obtain
\[
\begin{aligned}
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_n(e_1 + [-e_2])z^n &= \frac{2pz + qxz^2 + q^2tz^4}{g_{x,y,t}(z)}, \quad (4.7) \\
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_{n-1}(e_1 + [-e_2])z^n &= \frac{z - qyz^3 - 2pqtz^4}{g_{x,y,t}(z)}, \quad (4.8)
\end{aligned}
\]

with
\[
g_{x,y,t}(z) = 1 - 2pxz - (2y (2p^2 + q) + qx^2) z^2 - 2p (t (4p^2 + 3q) + qxy) z^3 + q (qy^2 - 2xt (2p^2 + q)) z^4 + 2pq^2ytz^5 - q^3t^2z^6,
\]

and we have the following results.

**Theorem 4.9** For \( n \in \mathbb{N} \), the new generating function of the product of trivariate Fibonacci polynomials with Gaussian \((p,q)\)-Pell numbers is given by
\[
\begin{aligned}
\sum_{n=0}^{\infty} H_n(x, y, t) GP_{p,q,n} z^n &= \sum_{n=0}^{\infty} H_n(x, y, t) GP_{p,q,n} z^n \\
&= \frac{z + qixz^2 + qy (2ip - 1) z^3 + qt (i (4p^2 + q) - 2p) z^4}{g_{x,y,t}(z)}. \quad (4.9)
\end{aligned}
\]

**Proof.** By [25], we have \( GP_{p,q,n} = iS_n(e_1 + [-e_2]) + (1 - 2ip) S_{n-1}(e_1 + [-e_2]) \).

Then, we see that
\[
\begin{aligned}
\sum_{n=0}^{\infty} H_n(x, y, t) GP_{p,q,n} z^n &= \\
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3) (iS_n(e_1 + [-e_2]) + (1 - 2ip) S_{n-1}(e_1 + [-e_2])) z^n \\
&= i \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_n(e_1 + [-e_2])z^n \\
&+ (1 - 2ip) \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_{n-1}(e_1 + [-e_2])z^n,
\end{aligned}
\]
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by using the relationships (4.7) and (4.8), we obtain

\[
\sum_{n=0}^{\infty} H_n(x,y,t) \cdot GP_{p,q,n} \cdot z^n = \\
\frac{i(2pz + qxz^2 + q^2tz^4)}{g_{x,y,t}(z)} + \frac{(1-2ip)(z - qyz^3 - 2pqtz^4)}{g_{x,y,t}(z)}
\]

\[
\begin{align*}
&= z + qixz^2 + qy(2ip - 1)z^3 + qt(i(4p^2 + q) - 2p)z^4.
\end{align*}
\]

This completes the proof. \(\square\)

**Theorem 4.10** For \(n \in \mathbb{N}\), the new generating function of the product of trivariate Fibonacci polynomials with Gaussian \((p,q)\)-Pell Lucas numbers is given by

\[
\sum_{n=0}^{\infty} H_n(x,y,t) \cdot GQ_{p,q,n} \cdot z^n = \\
\frac{2(p+ip)z + 2q(1-ip)z^2 + 2qy(p - i(2p^2 + q))z^3 + 2qt(2p^2 + q - ip(4p^2 + 3q))z^4}{g_{x,y,t}(z)}
\]

(4.10)

**Proof.** By [25], we have

\[
GQ_{p,q,n} = (2-2ip)S_n(e_1 + [-e_2]) + (i(4p^2 + 2q) - 2p)S_{n-1}(e_1 + [-e_2]).
\]

Then, we see that

\[
\sum_{n=0}^{\infty} H_n(x,y,t) \cdot GQ_{p,q,n} \cdot z^n = \\
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)((2-2ip)S_n(e_1 + [-e_2]) + (i(4p^2 + 2q) - 2p)S_{n-1}(e_1 + [-e_2])) z^n
\]

\[
= (2-2ip)\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_n(e_1 + [-e_2])z^n
\]

\[
+ (i(4p^2 + 2q) - 2p)\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3)S_{n-1}(e_1 + [-e_2])z^n,
\]

by using the relationships (4.7) and (4.8), we obtain

\[
\sum_{n=0}^{\infty} H_n(x,y,t) \cdot GQ_{p,q,n} \cdot z^n = \\
\frac{(2-2ip)(2pz + qxz^2 + q^2tz^4)}{g_{x,y,t}(z)}
\]

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\[
\begin{align*}
+ (i (4p^2 + 2q) - 2p) (z - qyz^3 - 2pqtz^4) \\
= 2 (p + iq) z + 2qx (1 - ip) z^2 + 2qy (p - i (2p^2 + q)) z^3 + \\
2qt (2p^2 + q - ip (4p^2 + 3q)) z^4
\end{align*}
\]
\[
g_{x,y,t}(z).
\]

This completes the proof.

\( \square \)

**Theorem 4.11** For \( n \in \mathbb{N} \), the new generating function of the product of trivariate Fibonacci polynomials with \((p, q)\)-modified Pell numbers is given by

\[
\sum_{n=0}^{\infty} H_n(x, y, t) MP_{p,q,n} z^n = \frac{pz + qx z^2 + pqyz^3 + qt (2p^2 + q) z^4}{g_{x,y,t}(z)}. \tag{4.11}
\]

**Proof.** By [25], we have

\[
MP_{p,q,n} = S_n(e_1 + [-e_2]) - pS_{n-1}(e_1 + [-e_2]).
\]

Then, we get

\[
\sum_{n=0}^{\infty} H_n(x, y, t) MP_{p,q,n} z^n =
\]

\[
\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3) (S_n(e_1 + [-e_2]) - pS_{n-1}(e_1 + [-e_2])) z^n
\]

\[
= \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3) S_n(e_1 + [-e_2]) z^n - p \sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2 + a_3) S_{n-1}(e_1 + [-e_2]) z^n,
\]

by using the relationships (4.7) and (4.8), we obtain

\[
\sum_{n=0}^{\infty} H_n(x, y, t) MP_{p,q,n} z^n = \frac{pz + qx z^2 + pqyz^3 + qt (2p^2 + q) z^4}{g_{x,y,t}(z)}
\]

\[
= \frac{pz + qx z^2 + pqyz^3 + qt (2p^2 + q) z^4}{g_{x,y,t}(z)}.
\]

This completes the proof.

\( \square \)

– By the relationships (4.9), (4.10) and (4.11), we have three cases.
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Case 1. Writing \(x^2\) instead of \(x\), \(x\) instead of \(y\) and taking \(t = 1\), we obtain

\[
g_{x^2,x,1}(z) = 1 - 2px^2 z - x(4p^2 + 2q + qx^3) z^2 - 2p(qx^5 + 4p^2 + 3q) z^3 - \]
\[
q x^2 (4p^2 + q) z^4 + 2pq^2 x^2 z^5 - q^3 z^6,
\]
and we have the following corollaries.

Corollary 4.12 For \(n \in \mathbb{N}\), the new generating function of the product of Tribonacci polynomials with Gaussian \((p,q)\)-Pell numbers is given by

\[
\sum_{n=0}^{\infty} T_n(x) GP_{p,q,n} z^n = \frac{z + qix^2 z^2 + qx (2ip - 1) z^3 + q (i (4p^2 + q) - 2p) z^4}{g_{x^2,x,1}(z)}.
\]

Corollary 4.13 For \(n \in \mathbb{N}\), the new generating function of the product of Tribonacci polynomials with Gaussian \((p,q)\)-Pell Lucas numbers is given by

\[
\sum_{n=0}^{\infty} T_n(x) GQ_{p,q,n} z^n = \frac{2 (p+ iq) z + 2q x^2 (1 - ip) z^2 + 2q x (p - i (2p^2 + q)) z^3}{g_{x^2,x,1}(z)} + \frac{2q (2p^2 + q - ip (4p^2 + 3q)) z^4}{g_{x^2,x,1}(z)}.
\]

Corollary 4.14 For \(n \in \mathbb{N}\), the new generating function of the product of Tribonacci numbers with \((p,q)\)-modified Pell numbers is given by

\[
\sum_{n=0}^{\infty} T_n(x) MP_{p,q,n} z^n = \frac{pz + qx^2 z^2 + pq x^3 + q (2p^2 + q) z^4}{g_{x^2,x,1}(z)}.
\]

Case 2. With \(x = y = t = 1\), we obtain

\[
g_{1,1,1}(z) = 1 - 2pz - (4p^2 + 3q) z^2 - 8p (p^2 + q) z^3 - \]
\[
q (4p^2 + q) z^4 + 2pq^2 z^5 - q^3 z^6,
\]
and we have the following corollaries.

Corollary 4.15 For \(n \in \mathbb{N}\), the new generating function of the product of Tribonacci numbers with Gaussian \((p,q)\)-Pell numbers is given by

\[
\sum_{n=0}^{\infty} T_n GP_{p,q,n} z^n = \frac{z + qiz^2 + q (2ip - 1) z^3 + q (i (4p^2 + q) - 2p) z^4}{g_{1,1,1}(z)}.
\]
Corollary 4.16 For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with Gaussian $(p, q)$-Pell Lucas numbers is given by
\[
\sum_{n=0}^{\infty} T_n G_{p,q,n}z^n = \frac{2(p + iq)z + 2q(1 - ip)z^2 + 2q(p - i(2p^2 + q))z^3}{g_{1,1,1}(z)} + \frac{2q(2p^2 + q - ip(4p^2 + 3q))z^4}{g_{1,1,1}(z)}.
\]

Corollary 4.17 For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with $(p, q)$-modified Pell numbers is given by
\[
\sum_{n=0}^{\infty} T_n M_{p,q,n}z^n = \frac{pz + qz^2 + pqz^3 + q(2p^2 + q)z^4}{g_{1,1,1}(z)}.
\]

Case 3. With $t = 0$, we obtain
\[
g_{x,y,0}(z) = 1 - 2pxz - (2y(2p^2 + q) + qx^2)z^2 - 2pqxyz^3 + q^2y^2z^4,
\]
and we have the following corollaries.

Corollary 4.18 For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with Gaussian $(p, q)$-Pell numbers is given by
\[
\sum_{n=0}^{\infty} F_n(x,y)GP_{p,q,n}z^n = \frac{z + iqxz^2 + qy(2ip - 1)z^3}{g_{x,y,0}(z)}.
\]

Corollary 4.19 For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with Gaussian $(p, q)$-Pell Lucas numbers is given by
\[
\sum_{n=0}^{\infty} F_n(x,y)GQ_{p,q,n}z^n = \frac{2(p + iq)z + 2q(x(1 - ip)z^2 + 2qy(p - i(2p^2 + q)))z^3}{g_{x,y,0}(z)}.
\]

Corollary 4.20 For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with $(p, q)$-modified Pell numbers is given by
\[
\sum_{n=0}^{\infty} F_n(x,y)MP_{p,q,n}z^n = \frac{pz + qxz^2 + pqyz^3}{g_{x,y,0}(z)}.
\]

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