2-Primal semimodule related to a Morita context

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Abstract In this paper we introduce the notion of 2-primal semimodule related to a Morita context \(< R, S, R_P S, Q, R, \theta, \phi >\). We obtain some characterizations of 2-primal semimodule.

Keywords the prime radical · 2-primal semimodule · Morita context

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1 Introduction

Though the notion of semiring was first introduced by Vandiver in 1934 [23], it has seen its tremendous growth in the last three decades of last century. This is evident from various monographs [7–10]. Recent work on semiring theory by Katsov [12, 13], Maity [16, 17], Bhuniya [1, 4], Sardar and Gupta [18, 19], Dey et al. [5, 20] (to name a few) indicates that this branch of mathematics has created a sustained research interest. One aspect of semiring theory is to investigate the validity of the ring theoretic analogues. One such generalization is the notion of Morita equivalence and Morita context of semirings. In this field Katsov et al. [12–15] initiated the study and Sardar et al. [18, 19] made some contribution. The purpose of this paper is to continue the study of Morita context of semirings of the present authors [5, 20].

Historically, some of the earliest results known to us about 2-primal rings (although not so called at the time) and prime ideals were due to Shin [21]. He showed that a ring \( R \) is 2-primal if and only if every minimal prime ideal of \( R \) is completely prime. Hirano [11] considered the 2-primal condition in the context of strongly \( \pi \)-regular rings. He used the term \( N \)-ring for what we call a 2-primal ring. The 2-primal condition was taken up independently by Sun [22], where in the setting of rings with identity he introduced a
condition called weakly symmetric, which is equivalent to the 2-primal condition for rings. Sun [22] showed that if $R$ is weakly symmetric, then each minimal prime ideal of $R$ is a completely prime ideal, and that the ring of $n$-by-$n$ upper triangular matrices over $R$ inherits the weakly symmetric condition. The name 2-primal rings originally and independently came from the context of left near rings by Birkenmeier, Heatherly and Lee in [2]. 2-primal semiring was first introduced by Das in [3]. Considering the recent development of Morita equivalence and Morita context of semiring [5,12,13,18,19], in this paper we make an attempt to study 2-primal semimodule in a Morita context of semiring. Firstly we introduce the notion of 2-primal semimodule in a Morita context (cf. Definition 3.1) and prove some results relating to this (cf. Propositions 3.5, 3.6, 3.7, Theorem 3.8, Lemma 3.16). Later we define some special subsets of $P$ (cf. Definition 3.9) and obtain some of their important properties (cf. Propositions 3.10, 3.11, 3.12, 3.13) in order to characterize the prime radical of $P$ (cf. Theorem 3.14) and also derived some important results for $P$ is to be 2-primal (cf. Theorem 3.17, 3.18, 3.19). In Theorem 3.26 we characterize the 2-primal semimodule $P$ and in 3.25, 3.28, 3.29 we came by the conditions for $P$ is to be 2-primal with the restriction that $P$ is strongly nil. Lastly we have shown that the property of being 2-primal is a Morita invariant property (cf. Theorem 3.30) in a Morita context $< R,S,R P S,S Q R,θ,ϕ >$ with some restriction. It has also been observed (cf. Concluding Remarks) that similar results also hold for the other component semimodule $Q$ of the Morita context $< R,S,R P S,S Q R,θ,ϕ >$.

2 Preliminaries

In this section we recall some basic definitions and results that are needed for our work.

Definition 2.1 [13] Two semirings $R$ and $S$ are said to be Morita equivalent if there exists a progenerator $R P \in |R M|$ for $R M$ such that $S \cong \text{End}(R P)$ as semirings; equivalently two semirings $R$ and $S$ are Morita equivalent if the categories $R M$ and $S M$ are equivalent categories.

Definition 2.2 [18] If $R$ and $S$ are two semirings $R P S,S Q R$ are $R$-$S$-bisseimodule and $S$-$R$-bisseimodule respectively, and $θ : P \otimes Q \mapsto R$ and $ϕ : Q \otimes P \mapsto S$ are respectively $R$-$R$-bisseimodule homomorphism and $S$-$S$-bisseimodule homomorphism such that $θ(p \otimes q)p' = pϕ(q \otimes p')$ and $ϕ(q \otimes p)q' = qθ(p \otimes q')$ for all $p,p' \in P$ and $q,q' \in Q$ then the sixtuple $< R,S,R P S,S Q R,θ,ϕ >$ is called a Morita context for semirings. Two semirings $R, S$ are Morita equivalent if and only if there exists a Morita context $< R,S,R P S,S Q R,θ,ϕ >$ with $θ$ and $ϕ$ surjective. Readers are referred to [9,12,13] for more notions of semirings, semimodules and Morita equivalence of semirings.

Definition 2.3 [19] Let $R$ and $S$ be Morita equivalent semirings via Morita context $< R,S,R P S,S Q R,θ,ϕ >$. Then we see that the lattices of ideals of
R and the lattices of subsemimodules of P are isomorphic. Moreover this isomorphism takes finitely generated ideals to finitely generated subsemimodules and vice-versa. Similar isomorphism can be defined for other pairs of the Morita context as follows.

\[ f_1 : \text{Id}(R) \to \text{Sub}(P) \text{ and } g_1 : \text{Sub}(P) \to \text{Id}(R) \text{ are defined by } \]
\[ f_1(I) := \{ \sum_{k=1}^{n} i_k p_k \mid p_k \in P, i_k \in I \text{ for all } k, n \in \mathbb{Z}^+ \}, \]
\[ \text{and } g_1(N) := \{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in N, q_k \in Q \text{ for all } k, n \in \mathbb{Z}^+ \} \]
\[ f_1 g_1 \text{ is the identity mapping on Sub}(P) \text{ and } g_1 f_1 \text{ is the identity mapping on Id}(R). \]

**Definition 2.4** [9] A semiring S is called regular if for every a ∈ S there exist b ∈ S such that a = aba.

**Definition 2.5** [5] Let R, S be two Morita equivalent semirings via Morita context \( < R, S, P, S, Q_R, \theta, \phi > \). An element a of P is said to be nilpotent if for any q ∈ Q there exists a positive integer \( n = n(q, a) \), depending on q and a, such that \( \theta(a \otimes q)^{n-1}a = 0 \).

**Definition 2.6** [5] Let R, S be two Morita equivalent semirings via Morita context \( < R, S, P, S, Q_R, \theta, \phi > \). An element a of P is said to be strongly nilpotent if there exists a positive integer \( n \) such that \( \theta(a \otimes Q)^{n-1}a = 0 \).

**Definition 2.7** [5] Let R, S be two Morita equivalent semirings via Morita context \( < R, S, P, S, Q_R, \theta, \phi > \). A submodule K of P is said to be prime if for subsemimodules M, N of P, \( g_1(M)N \subseteq K \) implies either \( M \subseteq K \) or \( N \subseteq K \).

**Definition 2.8** [5] Let R, S be two Morita equivalent semirings via Morita context \( < R, S, P, S, Q_R, \theta, \phi > \). P is said to be a prime semimodule if for subsemimodules M, N of P, \( g_1(M)N = 0 \) implies either \( M = 0 \) or \( N = 0 \).

**Definition 2.9** [6] Let R, S be two Morita equivalent semirings via Morita context \( < R, S, P, S, Q_R, \theta, \phi > \). A submodule M of the semimodule P is said to be completely prime if \( \theta(a \otimes Q)b \subseteq M \) implies that \( a \in M \) or \( b \in M \) for \( a, b \in P \).

**Definition 2.10** [6] Let R, S be two Morita equivalent semirings via Morita context \( < R, S, P, S, Q_R, \theta, \phi > \). A submodule M of the semimodule P is said to be completely semiprime if \( \theta(a \otimes Q)a \subseteq M \) implies that \( a \in M \) for \( a \in P \).

**Definition 2.11** [6] A submodule M of a semimodule P is called a k-subsemimodule if for \( x, y \in P, x + y \in M \) and \( y \in M \) implies that \( x \in M \).
Definition 2.12 [6] Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R, S, R P S, S Q R, \theta, \phi> \). Then a nonempty subset \( H \) of \( P \) is said to be an \( m \)–system of \( P \) if \( c, d \in H \) implies there exist \( p \in P \) and \( q, q' \in Q \) such that \( \theta(c \otimes q) \theta(p \otimes q') d \in H \).

Definition 2.13 [6] Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R, S, R P S, S Q R, \theta, \phi> \). Then \( P \) is \( 2 \)-primal if it is reduced.

3 Main results

We first formulate few definitions in order to obtain the main results.

Definition 3.1 Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R, S, R P S, S Q R, \theta, \phi> \). Then \( P \) is said to be a \( 2 \)-primal semimodule if and only if \( \mathcal{P}(P) = N(P) \), where \( \mathcal{P}(P) \) denotes the intersection of all prime subsemimodules of \( P \) i.e., the prime radical of \( P \) and \( N(P) \) denotes the set of all nilpotent elements of \( P \).

Definition 3.2 Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R, S, R P S, S Q R, \theta, \phi> \). Then \( P \) is said to be von Neumann regular if for \( a \in P \), there exist \( q_1, q_2 \in Q \) and \( p \in P \) such that \( a = a(\theta(q_1 \otimes p) \phi(q_2 \otimes a)) \).

Definition 3.3 Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R, S, R P S, S Q R, \theta, \phi> \). Then \( P \) is said to be \( NC1 \) if \( N(P) \) contains a nonzero subsemimodule of \( P \) whenever \( N(P) \) is nonzero.

Definition 3.4 Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R, S, R P S, S Q R, \theta, \phi> \). Then \( P \) is said to be reduced if it has no nonzero nilpotent elements.

Now we prove some results related to the prime radical \( \mathcal{P}(P) \) of \( P \).

Proposition 3.5 Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R, S, R P S, S Q R, \theta, \phi> \). Then \( \mathcal{P}(P) \subseteq N(P) \).

Proof. Let \( a \in \mathcal{P}(P) \). If possible, suppose for some \( q \in Q \) and for all positive integers \( n, \theta(qa \otimes q)^{n-1} a \neq 0 \). Let \( H = \{ \theta(qa \otimes q)^{n-1} a : n \text{ is any positive integer} \} \). Then \( H \) is an \( m \)–system not containing zero. Then there exists a prime subsemimodule \( M \) of \( P \) such that \( M \cap H = \phi \) (\( M \) is obtained by applying Zorn’s Lemma to the class of \( m \)-systems of \( P \)). Then \( a \notin M \). So \( a \notin \mathcal{P}(P) \), a contradiction. Hence \( \theta(qa \otimes q)^{n-1} a = 0 \) for all \( q \in Q \) and for some positive integer \( n \). Thus \( a \in N(P) \). So \( \mathcal{P}(P) \subseteq N(P) \).

\( \square \)

Proposition 3.6 Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R, S, R P S, S Q R, \theta, \phi> \). Then \( P \) is \( 2 \)-primal if it is reduced.
Proof. By Proposition 3.5, for $P$, $\mathcal{P}(P) \subseteq N(P)$. Again since $P$ is reduced, $N(P) = 0 \subseteq \mathcal{P}(P)$. Hence the result.

\begin{proposition}
Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R P S, S Q R, \theta, \phi >$ and $P$ be a strongly nil semimodule. Then $P$ is a 2-primal semimodule.
\end{proposition}

\begin{proof}
By Proposition 3.5, $\mathcal{P}(P) \subseteq N(P)$. Let $a \in N(P)$. Since $P$ is a strongly nil semimodule, then there exists a positive integer $n$ such that $\theta(a \otimes Q)^{n-1} a = 0$. If possible let $a \notin \mathcal{P}(P)$. Then $a \notin M$ for some prime subsemimodule $M$ of $P$ i.e., $a \in P - M$. Since $M$ is prime, $P - M$ is an $m$-system. Then there exist $p_1 \in P$, $\alpha_1, \beta_1 \in Q$ such that $\theta(a \otimes \alpha_1)\theta(p_1 \otimes \beta_1) a \in P - M$. Again since $\theta(a \otimes \alpha_1)\theta(p_1 \otimes \beta_1) a \in P - M$, applying $m$-system property $\theta(a \otimes \alpha_1)\theta(p_1 \otimes \beta_1) a \in P - M$, for some $\alpha_2, \beta_2 \in Q$ and $p_2 \in P$. Applying $m$-system property after finite step $\theta(a \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(p_2 \otimes \beta_2) a \in P - M$ for some $p_i \in P$ and $\alpha_i, \beta_i \in Q$, where $i = 1, 2, ... n - 1$. Since $Q$ is a right $R$ semimodule and $\theta(a \otimes Q)^{n-1} a = 0$, $\theta(a \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(p_n \otimes \beta_n) a = 0 \in M$, a contradiction. Hence $a \in \mathcal{P}(P)$. This completes the proof.
\end{proof}

\begin{theorem}
Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R P S, S Q R, \theta, \phi >$ and $P$ be a von Neumann regular semimodule. Then the following statements are equivalent.
\begin{enumerate}
\item $P$ is 2-primal.
\item $P$ is NI.
\item $P$ is NC1.
\item $P$ is reduced.
\end{enumerate}
\end{theorem}

\begin{proof}
(i) $\Rightarrow$ (ii)
Since $P$ is 2-primal. $N(P) = \mathcal{P}(P)$. So $\mathcal{P}(P)$ is a nil subsemimodule of $P$. Hence $\mathcal{P}(P) \subseteq N^*(P) \subseteq N(P)$. Therefore, $N^*(P) = N(P)$. So $P$ is NI.

(ii) $\Rightarrow$ (iii)
By (ii) $N^*(P) = N(P)$. Since $N^*(P)$ is nonzero, $P$ is NC1.

(iii) $\Rightarrow$ (iv)
Let $P$ be a NC1 semimodule. Suppose $P$ is not reduced i.e., $N(P) \neq 0$. Then there exists a nonzero subsemimodule, say $M$ such that $M \subseteq N(P)$. Let $a(\neq)0 \in M$. Since $P$ is von Neumann regular, there exists $q_1, q_2 \in Q$ and $b \in P$ such that
\begin{equation}
\theta(a \otimes q_1)\theta(b \otimes q_2) a = \theta(a \otimes q_1)\theta(b \otimes q_2)\theta(a \otimes q_1)\theta(b \otimes q_2) a = ... \tag{3.1}
\end{equation}
Then
\begin{equation}
a = \theta(a \otimes q_1)\theta(b \otimes q_2) a = \theta(a \otimes q_1)\theta(b \otimes q_2) a = ... \tag{3.2}
\end{equation}
Since $a \in M$, $\theta(a \otimes q_1)b \in M \subseteq N(P)$. So there exists a positive integer $n$ such that $\theta(\theta(a \otimes q_1)b \otimes q_2))^n\theta(a \otimes q_1)b = 0$. From 3.2, we have $a = (\theta(\theta(a \otimes q_1)b \otimes q_2))^n\theta(a \otimes q_1)b = 0$ a contradiction. Therefore $P$ is reduced.

\end{proof}
\((iv) \Rightarrow (i)\). Follows from Proposition 3.6.

We now define some special subsets of \(P\) related to the Morita context \(< R, S, R P S, S Q R, \theta, \phi >\) and obtain some of their important properties (cf. Propositions 3.10, 3.11, 3.12, 3.13) in order to characterize the prime radical \(\mathcal{P}(P)\) of \(P\) (cf. Theorem 3.14).

**Definition 3.9** Let \(R, S\) be two Morita equivalent semirings via Morita context \(< R, S, R P S, S Q R, \theta, \phi >\) and \(M\) be a prime subsemimodule of \(P\). Then we define

1. \(N^*(M) = \{x \in P : \theta(x \otimes Q)y \subseteq \mathcal{P}(P) \text{ for some } y \in P - M\}\);
2. \(\overline{N^*(M)} = \{x \in P : \theta(x \otimes Q)y \subseteq N^*(M), \text{ for some positive integer } n\}\).

**Proposition 3.10** Let \(R, S\) be two Morita equivalent semirings via Morita context \(< R, S, R P S, S Q R, \theta, \phi >\). Then for any prime subsemimodule of \(P\), \(N^*(M) \subseteq M\) and \(N^*(M) \subseteq \overline{N^*(M)}\).

**Proof.** Let \(x \in N^*(M)\). Then \(\theta(x \otimes Q)y \subseteq \mathcal{P}(P)\) for some \(y \in P - M\). Since \(\mathcal{P}(P) \subseteq M\) for any prime subsemimodule \(x \in M\). (since \(M\) is prime [5]).

Hence \(N^*(M) \subseteq M\).

From definition it is obvious \(N^*(M) \subseteq \overline{N^*(M)}\).

\(\square\)

**Proposition 3.11** Let \(R, S\) be two Morita equivalent semirings via Morita context \(< R, S, R P S, S Q R, \theta, \phi >\). Then \(N^*(M) = \{x \in P : \theta(x \otimes Q)y \subseteq \mathcal{P}(P) \text{ for some } y \in P - M\}\), where \(< y >\) denotes the subsemimodule of \(P\) generated by \(y\).

**Proof.** Let \(A = \{x \in P : \theta(x \otimes Q)y \subseteq \mathcal{P}(P) \text{ for some } y \in P - M\}\). Since \(y \in < y >\), \(A \subseteq N^*(M)\). Let \(x \in N^*(M)\). Then \(\theta(x \otimes Q)y \subseteq \mathcal{P}(P)\) for some \(y \in P - M\). Now any element of \(< y >\) is of the form

\[ny + \sum_{i=1}^{m} y\phi(\alpha_i \otimes x_i) + \sum_{j=1}^{l} \theta(z_j \otimes \beta_j)y + \sum_{k=1}^{s} \theta(u_k \otimes \lambda_k)y\phi(\mu_k \otimes v_k),\]

where \(x_i, z_j, u_k, v_k \in P\), \(\alpha_i, \beta_j, \lambda_k, \mu_k \in Q\) and \(n, m, l, s\) are nonnegative integers.

Hence \(\theta(x \otimes Q)y \subseteq \mathcal{P}(P)\) as \(\mathcal{P}(P)\) is a subsemimodule of \(P\). Therefore \(x \in A\). Consequently \(N^*(M) = A\).

\(\square\)

**Proposition 3.12** Let \(R, S\) be two Morita equivalent semirings via Morita context \(< R, S, R P S, S Q R, \theta, \phi >\) and \(M\) be any prime subsemimodule of \(P\). Then \(N^*(M)\) is a subsemimodule of \(P\).

**Proof.** Since \(0 \in N^*(M)\), \(N^*(M)\) is a nonempty subset of \(P\). Let \(x_1, x_2 \in N^*(M)\). Then there exist \(y_1, y_2 \in P - M\) such that \(\theta(x_1 \otimes Q)y_1 \subseteq \mathcal{P}(P)\) and \(\theta(x_2 \otimes Q)y_2 \subseteq \mathcal{P}(P)\). Since \(M\) is a prime subsemimodule
of $P$, $P - M$ is an $m$-system. So there exist $p \in P, \alpha, \beta \in Q$ such that
$$\theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 \in P - M.$$ Now $< \theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 > \subseteq y_1 >$ and $< \theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 > < y_2 >$. Therefore, $\theta((x_1 + x_2) \otimes Q) < \theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 = \theta(x_1 \otimes Q) < \theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 + \theta(x_2 \otimes Q) < \theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 \subseteq \theta(x_1 \otimes Q) < y_1 > + \theta(x_2 \otimes Q) < y_2 > \subseteq \mathcal{P}(P)$ as $\mathcal{P}(P)$ is a subsemimodule of $P$. Thus $x_1 + x_2 \in N^*(M)$.

Let $x \in N^*(M)$. Then there exists $y \in P - M$ such that $\theta(x \otimes Q) < y > \subseteq \mathcal{P}(P)$. Therefore $\theta(Rx \otimes Q) < y > \subseteq \mathcal{P}(P)$ and $\theta(xS \otimes Q) < y > \subseteq \mathcal{P}(P)$. Thus $Rx, xS \subseteq N^*(M)$ for all $x \in N^*(M)$. Hence $N^*(M)$ is a subsemimodule of $P$.

\[ \square \]

**Proposition 3.13** Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R \text{Spec}_R S, S \text{Spec}_S R, \theta, \phi >$ and $M$ a prime subsemimodule of $P$. Then

(i) $N^*(M)$ is a subsemimodule of $P$;

(ii) $N^*(M)$ is a completely semiprime subsemimodule of $P$ if and only if $N^*(M) = N^*(N)$.

**Proof.** (i) Since $N^*(M)$ is a subsemimodule of $P$ and $N^*(M) \subseteq N^*(M)$, $N^*(M)$ is a nonempty subset of $P$. Let $x, y \in N^*(M)$. Then $\theta(x \otimes Q)^{n+1}x, \theta(y \otimes Q)^{n+1}y \subseteq N^*(M)$ for some $n, m \in \mathbb{Z}^+$. Therefore the elements of the form

$$\theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{k-1})\theta(p_{k-1} \otimes \beta_{k-1})x$$

$k \geq n$ and $\theta(y \otimes \gamma_1)\theta(p_1 \otimes \delta_1)\theta(y \otimes \gamma_2)\theta(p_2 \otimes \delta_2)\ldots\theta(y \otimes \gamma_{r-1})\theta(p_{r-1} \otimes \delta_{r-1})y \ (r \geq m) \in N^*(M)$ i.e. an expression containing at least $n$ $x$'s or $m$ $y$'s must belong to $N^*(M)$. Since $N^*(M)$ is a subsemimodule $\theta((x + y) \otimes Q)^{m+n}(x + y)$ is contained in $N^*(M)$. So $x + y \in N^*(M)$. Since $N^*(M)$ is a subsemimodule, $\theta(Rx \otimes Q)^{n+1}Rx, \theta(xS \otimes Q)^{n+1}xS \subseteq N^*(M)$ i.e. $Rx, xS \subseteq N^*(M)$. Therefore $N^*(M)$ is a subsemimodule of $P$.

(ii) Suppose that $N^*(M)$ is a completely semiprime subsemimodule of $P$. Clearly from the definition $N^*(M) \subseteq N^*(M)$. Let $a \in N^*(M)$. Then $\theta(a \otimes Q)^{n+1}a \subseteq N^*(M)$ for some positive integer $n$. As $N^*(M)$ is completely semiprime subsemimodule of $P$, $\theta(a \otimes Q)^{n+1}a \subseteq N^*(M)$ implies $a \in N^*(M)$. Therefore $N^*(M) = N^*(M)$.

\[ \square \]

**Notations 3.1** We use $PSpec(P)$ and $mPSpec(P)$ to denote the set of all prime subsemimodules and minimal prime subsemimodules of $P$ in a Morita context $< R, S, R \text{Spec}_R S, S \text{Spec}_S R, \theta, \phi >$.

Now we characterize the prime radical of $P$.

**Theorem 3.14** Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R \text{Spec}_R S, S \text{Spec}_S R, \theta, \phi >$.

Then $\mathcal{P}(P) = \bigcap_{M \in PSpec(P)} N^*(M) = \bigcap_{M' \in mPSpec(P)} N^*(M')$. 

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Proof. Let \( a \in \mathcal{P}(P) \). As \( \mathcal{P}(P) \) is a subsemimodule, \( \theta(a \otimes Q)P \subseteq \mathcal{P}(P) \). Since \( M \) is proper, \( P - M \) is nonempty. Let \( y \in P - M \). Then \( \theta(a \otimes Q)y \subseteq \mathcal{P}(P) \) for every prime subsemimodule \( M \) of \( P \). This implies \( a \in N^*(M) \) for every prime subsemimodule \( M \) of \( P \), that is, \( a \in \bigcap_{M \in \text{Spec}(P)} N^*(M) \). Hence \( \mathcal{P}(P) \subseteq \bigcap_{M \in \text{Spec}(P)} N^*(M) \). Again \( N^*(M) \subseteq M \) for any prime subsemimodule \( M \) of \( P \). This proves \( \bigcap_{M \in \text{Spec}(P)} N^*(M) \subseteq \bigcap_{M \in \text{Spec}(P)} M = \mathcal{P}(P) \).

Hence, \( \mathcal{P}(P) = \bigcap_{M \in \text{Spec}(P)} N^*(M) \). Also \( \mathcal{P}(P) = \bigcap_{M' \in \text{mSpec}(P)} N^*(M') \).

Hence \( \mathcal{P}(P) = \bigcap_{M \in \text{Spec}(P)} N^*(M) = \bigcap_{M' \in \text{mSpec}(P)} N^*(M') \). \qed

**Definition 3.15** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R P S, S Q R, \theta, \phi > \). \( P \) is said to be right symmetric if for \( a, b, c \in P \), \( \theta(a \otimes Q)\theta(b \otimes Q)c = 0 \) implies \( \theta(a \otimes Q)\theta(c \otimes Q)b = 0 \). A subsemimodule \( M \) of \( P \) is said to be right symmetric if \( \theta(a \otimes Q)\theta(b \otimes Q)c \subseteq M \) implies \( \theta(a \otimes Q)\theta(c \otimes Q)b \subseteq M \) for \( a, b, c \in P \).

Similarly we can define left symmetric semimodule and left symmetric subsemimodule.

**Lemma 3.16** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R P S, S Q R, \theta, \phi > \). \( P \) has no strongly nilpotent elements. Then \( P \) is right symmetric and also left symmetric.

Proof. Let \( P \) has no strongly nilpotent elements. Let \( a, b, c \in P \) be such that \( \theta(a \otimes Q)\theta(b \otimes Q)c = 0 \). Then \( \theta(a \otimes Q)\theta(b \otimes Q)\theta(c \otimes Q)\theta(a \otimes Q)b = \theta(c \otimes Q)(\theta(a \otimes Q)\theta(b \otimes Q)c)\phi(Q \otimes a)\phi(Q \otimes b) = 0 \). Since \( P \) has no strongly nilpotent elements, \( \theta(c \otimes Q)\theta(a \otimes Q)b = 0 \). Now \( \theta(a \otimes Q)\theta(b \otimes Q)\theta(c \otimes Q)\theta(a \otimes Q)c = \theta(a \otimes Q)\theta(b \otimes Q)\theta(c \otimes Q)\theta(a \otimes Q)c \). This implies \( \theta(a \otimes Q)\theta(b \otimes Q)\theta(c \otimes Q)c = 0 \). Again by using similar argument, we have

\[
\theta(a \otimes Q)\theta(b \otimes Q)\theta(c \otimes Q)c = 0 \\
\Rightarrow \theta(a \otimes Q)\theta(b \otimes Q)\theta(c \otimes Q)a = 0 \\
\Rightarrow \theta(a \otimes Q)\theta(b \otimes Q)\theta(c \otimes Q)b = 0
\]

and so \( \theta(a \otimes Q)\theta(c \otimes Q)\theta(b \otimes Q)c = 0 \), i.e., \( \theta(a \otimes Q)\theta(c \otimes Q)\theta(b \otimes Q)c = 0 \). Consequently \( \theta(a \otimes Q)\theta(c \otimes Q)b = 0 \). Hence \( P \) is right symmetric.

Similarly we have \( \theta(a \otimes Q)\theta(b \otimes Q)c = 0 \), i.e., \( \theta(a \otimes Q)\theta(b \otimes Q)c = 0 \).
Since $P$ is without strongly nilpotent, $\theta(b \otimes Q)\theta(a \otimes Q)c = 0$. Hence $P$ is left symmetric.

Now we prove some important results (cf. Theorem 3.17, 3.18, 3.19) that state the condition for $P$ is to be 2-primal with the restriction that $P$ is strongly nil.

**Theorem 3.17** Let $R, S$ be two Morita equivalent semirings via Morita context $<_R S, R P S, S Q, S, \theta, \phi >$ and $P$ be strongly nil semimodule. Then the following statements are equivalent:

(i) $P$ is a 2-primal semimodule.
(ii) $\mathcal{P}(P)$ is a completely semiprime subsemimodule of $P$.
(iii) $\mathcal{P}(P)$ is a left and right symmetric subsemimodule of $P$.
(iv) $\theta(x \otimes Q)y \subseteq \mathcal{P}(P)$ implies $\theta(y \otimes Q)x \subseteq \mathcal{P}(P)$ for $x, y \in P$.

**Proof.** (i) $\Rightarrow$ (ii)

Let $\theta(a \otimes Q)a \subseteq \mathcal{P}(P)$, where $a \in P$. Since $P$ is a strongly nil and 2-primal semimodule, $\theta(a \otimes Q)a \subseteq \mathcal{N}_Q(P)$. Then there exists a positive integer $n$ such that $\theta(\theta(a \otimes Q)a \otimes Q)\theta(a \otimes Q)a = 0$. This implies that $\theta(a \otimes Q)2^{n-1}a = 0$. So $a \in \mathcal{N}_Q(P) = N(P) = \mathcal{P}(P)$. Therefore $\mathcal{P}(P)$ is a completely semiprime subsemimodule of $P$.

(ii) $\Rightarrow$ (iii)

Let $\theta(a \otimes Q)\theta(b \otimes Q)c \subseteq \mathcal{P}(P)$, where $a, b, c \in P$. Then $\theta(\theta(c \otimes Q)\theta(a \otimes Q)b \otimes Q)\theta(c \otimes Q)\theta(a \otimes Q)b = \theta(c \otimes Q)\theta(b \otimes Q)c \phi(Q \otimes a) \phi(Q \otimes b) \subseteq \mathcal{P}(P)$. Since $\mathcal{P}(P)$ is completely semiprime, $\theta(c \otimes Q)\theta(b \otimes Q)c \subseteq \mathcal{P}(P)$. Now $\theta(\theta(a \otimes Q)\theta(b \otimes Q)\theta(a \otimes Q)c \phi(Q \otimes a) \phi(Q \otimes b) \subseteq \mathcal{P}(P)$ as $\mathcal{P}(P)$ is a subsemimodule of $P$. This implies $\theta(a \otimes Q)\theta(b \otimes Q)\theta(a \otimes Q)c \subseteq \mathcal{P}(P)$.

Again by using similar argument, we have

$$\theta(\theta(b \otimes Q)\theta(a \otimes Q)\theta(c \otimes Q)\theta(b \otimes Q)a \otimes Q)\theta(b \otimes Q)\theta(a \otimes Q)\theta(c \otimes Q)\theta(b \otimes Q)a \subseteq \mathcal{P}(P) \Rightarrow \theta(b \otimes Q)\theta(a \otimes Q)\theta(c \otimes Q)\theta(b \otimes Q)a \subseteq \mathcal{P}(P).$$

This implies that

$$\theta(a \otimes Q)\theta(c \otimes Q)\theta(b \otimes Q)\theta(a \otimes Q)\phi(Q \otimes c) \phi(Q \otimes b) \subseteq \mathcal{P}(P) \Rightarrow \theta(a \otimes Q)\theta(c \otimes Q)\phi(Q \otimes b) \subseteq \mathcal{P}(P)$$

as $\mathcal{P}(P)$ is completely semiprime. Hence, $\mathcal{P}(P)$ is a right symmetric subsemimodule of $P$. Also $\theta(\theta(b \otimes Q)\theta(a \otimes Q)c \otimes Q)\theta(b \otimes Q)\theta(a \otimes Q)c = \theta(b \otimes Q)\theta(c \otimes Q)\phi(Q \otimes a) \phi(Q \otimes c) \subseteq \mathcal{P}(P) \Rightarrow \theta(b \otimes Q)\theta(a \otimes Q)c \subseteq \mathcal{P}(P)$.

This shows that $\mathcal{P}(P)$ is a left symmetric subsemimodule of $P$. Therefore, $\mathcal{P}(P)$ is a left and right symmetric subsemimodule of $P$. 

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Let \( \theta(x \otimes Q)y \in \mathcal{P}(P) \), where \( x, y \in P \). Suppose \( p \in P \), then \( \theta(p \otimes Q)\theta(x \otimes Q)y \subseteq \mathcal{P}(P) \). As \( \mathcal{P}(P) \) is right symmetric, \( \theta(p \otimes Q)\theta(y \otimes Q)x \subseteq \mathcal{P}(P) \). Also since \( \mathcal{P}(P) \) is left symmetric, \( \theta(y \otimes Q)\theta(p \otimes Q)y \subseteq \mathcal{P}(P) \). Therefore \( \theta(y \otimes Q)\theta(P \otimes Q)x \subseteq \mathcal{P}(P) \) implies \( \theta(y \otimes Q)x \subseteq \mathcal{P}(P) \) (since \( \theta(P \otimes Q) = R \) has identity).

(iii) \( \Rightarrow \) (iv)

It is clear that \( \mathcal{P}(P) \subseteq N(P) \) always. Let \( x \in N(P) \). Since \( P \) is a strongly nil semimodule, \( x \in N_Q(P) \). Hence, exists a positive integer \( n \) such that \( \theta(x \otimes Q)^{n-1}x = 0 \). If possible, let \( x \notin \mathcal{P}(P) \). Then \( x \notin M \) for some prime subsemimodule \( M \) of \( P \). Hence \( x \in P - M \). Since \( M \) is prime, \( P - M \) is an \( m \)-system. Clearly, there exists \( p_1 \in P \), \( \alpha_1, \beta_1 \in Q \) such that \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)x \in P - M \). Again since \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)x \in P - M \), by the property of \( m \)-system, we have \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)x \in P - M \), for some \( \alpha_2, \beta_2 \in Q \) and \( p_2 \in P \). Applying the property of \( m \)-system again after a number of finite steps, we have \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in P - M \) for some \( p_i \in P \), \( \alpha_i, \beta_i \in Q \), where \( i = 1, 2, \ldots, n - 1 \). Now by (iv) as \( \theta(x \otimes Q)^{n-1}x = 0 \subseteq \mathcal{P}(P) \) implies \( \theta(x \otimes Q)R\theta(x \otimes Q)R\ldots\theta(x \otimes Q)Rx \subseteq \mathcal{P}(P) \) implies \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in \mathcal{P}(P) \) i.e. \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in M \), a contradiction. Thus \( x \in \mathcal{P}(P) \) and so \( \mathcal{P}(P) = N(P) \). Hence \( P \) is a 2-primal semimodule.

\[ \square \]

**Theorem 3.18** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R \mathcal{P}_S, S \mathcal{Q}_R, \theta, \phi > \) and \( P \) be strongly nil semimodule. Then the following statements are equivalent:

(i) \( P \) is a 2-primal semimodule.

(ii) \( N^*(M) = N^*(\overline{M}) \) for each prime subsemimodule \( M \) of \( P \).

Proof. (i) \( \Rightarrow \) (ii)

Let \( M \) be a prime subsemimodule of \( P \). Let \( a \in N^*(\overline{M}) \). Then \( \theta(a \otimes Q)^{n-1}a \subseteq N^*(M) \), for some positive integer \( n \). Hence there exists \( b \in P - M \) such that \( \theta(a \otimes Q)^{n-1}\theta(a \otimes Q)b \subseteq \mathcal{P}(P) \). By Theorem 3.17, we see that \( \mathcal{P}(P) \) is a left and right symmetric subsemimodule of \( P \). Thus \( \theta(\theta(a \otimes Q)b \otimes Q)\theta(\theta(a \otimes Q)b \otimes Q)\ldots\theta(\theta(a \otimes Q)b \otimes Q) \) (\( n - times \)) \( \subseteq \mathcal{P}(P) \). Again by Theorem 3.17 \( \mathcal{P}(P) \) is a completely semiprime subsemimodule of \( P \). Then we have \( \theta(a \otimes Q)b \subseteq \mathcal{P}(P) \). Therefore \( a \in N^*(M) \). Consequently, we deduce that \( N^*(\overline{M}) \subseteq N^*(M) \).

By Proposition 3.10, \( N^*(M) \subseteq N^*(\overline{M}) \) for each prime subsemimodule \( M \) of \( P \). Therefore, \( N^*(M) = N^*(\overline{M}) \).

(ii) \( \Rightarrow \) (i)

Let \( N^*(M) = N^*(\overline{M}) \) for each prime subsemimodule \( M \) of \( P \). To prove the result it is sufficient to prove that \( N(P) \subseteq \mathcal{P}(P) \). Let \( a \in N(P) \). Since \( P \) is an strongly nil-semimodule, \( a \in N_Q(P) \). This implies that \( \theta(a \otimes Q)^{n-1}a = 0 \)

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for some positive integer \( n \) and so \( \theta(a \otimes Q)^{n-1}a \subseteq N^*(M) \Rightarrow a \in \overline{N^*(M)} \Rightarrow a \in N^*(M) \Rightarrow a \in M \) (as \( N^*(M) \subseteq M \) for each prime subsemimodule \( M \) of \( P \)) (cf. Proposition 3.10), \( a \in \mathcal{P}(P) \). Thus, \( N(P) \subseteq \mathcal{P}(P) \). This shows that
\( P \) is an \( \mathbf{2} \)-primal semimodule.

\( \square \)

**Theorem 3.19** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, I, P, \psi, Q, I, \theta, \phi > \) and \( P \) be strongly nil semimodule. Then the following statements are equivalent:

(i) \( P \) is a \( \mathbf{2} \)-primal semimodule.

(ii) \( N^*(M) \) is a completely semiprime subsemimodule of \( P \) for each prime subsemimodule \( M \) of \( P \).

(iii) \( N^*(M) \) is a left and right symmetric subsemimodule of \( P \) for each prime subsemimodule \( M \) of \( P \).

(iv) \( \theta(x \otimes Q)y \subseteq N^*(M) \) implies \( \theta(y \otimes Q)x \subseteq N^*(M) \) for \( x, y \in P \) for each prime subsemimodule \( M \) of \( P \).

**Proof.** (i) \( \Rightarrow \) (ii)

Let \( P \) be a \( \mathbf{2} \)-primal semimodule. Then by Theorem 3.18 \( N^*(M) = N^*(M) \) for each prime subsemimodule \( M \) of \( P \). Let \( x \in P \) be such that \( \theta(x \otimes Q)x \subseteq N^*(M) \). Then \( \theta(x \otimes Q)x \subseteq N^*(M) \). Hence there exists a positive integer \( n \) such that \( \theta((\theta(x \otimes Q)x \otimes Q)^{n-1}) \theta(x \otimes Q)x \subseteq N^*(M) \). Therefore \( \theta(x \otimes Q)^{2n-1}x \subseteq N^*(M) \). So \( x \in N^*(M) \subseteq N^*(M) \). Hence \( N^*(M) \) is a completely semiprime subsemimodule of \( P \) for each prime subsemimodule \( M \) of \( P \).

(ii) \( \Rightarrow \) (iii)

Let \( \theta(a \otimes Q)\theta(b \otimes Q)c \subseteq N^*(M) \), where \( a, b, c \in P \). Then \( \theta(\theta(c \otimes Q)\theta(a \otimes Q)b \otimes Q)\theta(c \otimes Q)\theta(a \otimes Q)\theta(b \otimes Q)c \phi(Q \otimes c) \subseteq N^*(M) \). Since \( N^*(M) \) is completely semiprime, \( \theta(c \otimes Q)\theta(a \otimes Q)b \subseteq N^*(M) \). Now \( \theta(\theta(a \otimes Q)\theta(b \otimes Q)\theta(a \otimes Q)c \otimes Q)\theta(a \otimes Q)\theta(b \otimes Q)c \theta(a \otimes Q)c \phi(Q \otimes c) \subseteq N^*(M) \) as \( N^*(M) \) is a subsemimodule of \( P \). This implies \( \theta(a \otimes Q)\theta(b \otimes Q)\theta(a \otimes Q)c \subseteq N^*(M) \).

Again by using similar argument, we have

\[
\theta(\theta(b \otimes Q)\theta(a \otimes Q)c \otimes Q)\theta(b \otimes Q)a \phi(Q \otimes c) \subseteq N^*(M) \Rightarrow \theta(b \otimes Q)\theta(a \otimes Q)c \phi(Q \otimes c) \subseteq N^*(M).
\]

This implies that

\[
\theta(a \otimes Q)\theta(c \otimes Q)\phi(Q \otimes c) \subseteq N^*(M) \Rightarrow \theta(a \otimes Q)\theta(c \otimes Q)b \subseteq N^*(M)
\]
as \( N^*(M) \) is completely semiprime. Hence, \( N^*(M) \) is a right symmetric subsemimodule of \( P \). Also \( \theta(\theta(b \otimes Q)\theta(a \otimes Q)c \otimes Q)\theta(b \otimes Q)\theta(a \otimes Q)c = \)

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\[ \theta(b \otimes Q)(\theta(a \otimes Q)\theta(c \otimes Q)b)\phi(Q \otimes a)\phi(Q \otimes c) \subseteq N^*(M) \Rightarrow \theta(b \otimes Q)\theta(a \otimes Q)c \subseteq N^*(M). \]

This shows that \( N^*(M) \) is a left symmetric subsemimodule of \( P \).

\( \text{(iii)} \Rightarrow (iv) \)

Let \( \theta(x \otimes Q)y \in N^*(M) \), where \( x, y \in P \). Suppose \( p \in P \), then \( \theta(p \otimes Q)\theta(x \otimes Q)y \in N^*(M) \). As \( N^*(M) \) is right symmetric, \( \theta(p \otimes Q)\theta(y \otimes Q)x \subseteq N^*(M) \). Also since \( N^*(M) \) is left symmetric, \( \theta(y \otimes Q)\theta(p \otimes Q)y \subseteq N^*(M) \). Therefore \( \theta(y \otimes Q)\theta(p \otimes Q)x \subseteq N^*(M) \) implies \( \theta(y \otimes Q)x \subseteq N^*(M) \) (as \( \theta(P \otimes Q) = R \) has identity).

\( (iv) \Rightarrow (i) \)

It is clear that \( \mathcal{P}(P) \subseteq N(P) \) always. Let \( x \in N(P) \). Since \( P \) is a strongly nil semimodule, \( x \in N_Q(P) \). Hence, exists a positive integer \( n \) such that \( \theta(x \otimes Q)^{n-1}x = 0 \). Suppose if possible, \( x \notin \mathcal{P}(P) \). Then \( x \notin M \) for some prime subsemimodule \( M \) of \( P \). Hence \( x \in P - M \). Since \( M \) is prime, \( P - M \) is an \( m - \text{system} \). Clearly, there exists \( p_1 \in P, \alpha_1, \beta_1 \in Q \) such that \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)x \in P - M \). Again since \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)x, x \in P - M \), by the property of \( m - \text{system} \), we have \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)x \in P - M \), for some \( \alpha_2, \beta_2 \in Q \) and \( p_2 \in P \). Applying the property of \( m - \text{system} \) again after a number of finite steps, we have \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in P - M \) for some \( p_i \in P, \alpha_i, \beta_i \in Q \), where \( i = 1, 2, \ldots, n - 1 \). Now by \( (iv) \) \( \theta(x \otimes Q)^{n-1}x = 0 \subseteq N^*(M) \) implies \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in N^*(M) \) i.e. \( \theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in M \) (as \( N^*(M) \subseteq M \)), a contradiction. Thus \( x \in \mathcal{P}(P) \) and so \( \mathcal{P}(P) = N(P) \).

Hence \( P \) is a 2-primal semimodule.

Now we define another special subsets of \( P \) that will help us to characterize the 2-primal semimodule \( P \).

**Definition 3.20** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R, P, S, Q, R, \theta, \phi > \). For a prime subsemimodule \( M \) of \( P \), we define the following subsets of \( P \):

1. \( \theta_0^*(M) = \{ x \in P : \theta(x \otimes Q)y = 0 \text{ for some } y \in P - M \} \);
2. \( \bar{\theta}_0^*(M) = \{ x \in P : \theta(x \otimes Q)^{n-1}x \subseteq \theta_0^*(M), \text{ for some positive integer } n \} \).

We obtain the following propositions involving the properties of \( \theta_0^*(M) \) and \( \bar{\theta}_0^*(M) \) which are also used to establish Theorem 3.25, 3.26, 3.27, 3.28, 3.29. Since the proofs of these propositions are similar to those of Propositions 3.10, 3.11, 3.12, 3.13 we omit them.

**Proposition 3.21** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R, P, S, Q, R, \theta, \phi > \). Then for any prime subsemimodule \( M \) of \( P \), \( \theta_0^*(M) \subseteq M \) and \( \bar{\theta}_0^*(M) \subseteq \theta_0^*(M) \).

**Proposition 3.22** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R, P, S, Q, R, \theta, \phi > \) and \( M \) be a prime subsemimodule of \( P \).
Then $0^*(M) = \{ x \in P : \theta(x \otimes Q) < y > = 0 \text{ for some } y \in P - M \}$, where $< y >$ denotes the subsemimodule of $P$ generated by $y$.

**Proposition 3.23** Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R P S, S Q R, \theta, \phi >$ and $M$ be a prime subsemimodule of $P$. Then $0^*(M)$ is a two sided $k$-subsemimodule of $P$.

**Proof.** Here we only prove that $0^*(M)$ is a two sided $k$-subsemimodule of $P$.

Let $x_1 + x_2 \in 0^*(M)$ and $x_1 \in 0^*(M)$. Then there exists $y_1, y_2 \in P - M$ such that $\theta((x_1 + x_2) \otimes Q) < y_1 >= 0$ and $\theta(x_1 \otimes Q) < y_2 >= 0$. Since $P$ is a strongly nil semimodule, $a \in NQ(P)$. So $\theta(a \otimes Q)^n - 1 a = 0$, for some positive integer $n$. Then $\theta(a \otimes Q)^n - 1 \theta(a \otimes Q)y = 0$.

**Theorem 3.25** Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R P S, S Q R, \theta, \phi >$ and $M$ be a prime subsemimodule of $P$. Then

(i) $0^*(M)$ is a subsemimodule of $P$;

(ii) $0^*(M)$ is a completely semiprime subsemimodule of $P$ if and only if $0^*(M) = 0^*(M')$.

**Proof.** We first notice that if $M_1$ and $M_2$ are two prime subsemimodules of $P$ such that $M_1 \subseteq M_2$, then $0^*(M_2) \subseteq 0^*(M_1)$. Let $a \in 0^*(M_2)$. Then $\theta(a \otimes Q)^n - 1 a \subseteq 0^*(M_2)$, for some positive integer $n$, which implies that $\theta(a \otimes Q)^n - 1 \theta(a \otimes Q)b = 0$, for some $b \in P - M_2$. Since $M_1 \subseteq M_2$, $b \in P - M_1$. So $\theta(a \otimes Q)^n - 1 a \subseteq 0^*(M_1)$, for some positive integer $n$. Thus $a \in 0^*(M_1)$. Hence $0^*(M_2) \subseteq 0^*(M_1)$.

Let $M$ be a prime subsemimodule of $P$, then there exists a minimal prime subsemimodule $M'$ of $P$ such that $M' \subseteq M$. Therefore $\bigcap_{M \in P \text{Spec}(P)} 0^*(M') \subseteq \bigcup_{M' \in mP \text{Spec}(P)} 0^*(M')$.

Let $a \in N(P)$. Since $P$ is a strongly nil semimodule, $a \in NQ(P)$. So $\theta(a \otimes Q)^n - 1 a = 0$, for some positive integer $n$. Then $\theta(a \otimes Q)^n - 1 \theta(a \otimes Q)y = 0$.
for each \( y \in P - M \), where \( M \) is a prime subsemimodule of \( P \). So \( \theta(a \otimes Q)^{n-1} a \subseteq 0^*(M) \), for each prime subsemimodule \( M \) of \( P \). i.e. \( a \in 0^*(M) \) for each prime subsemimodule of \( P \), which implies that \( a \in \bigcap_{M \in \text{PSpec}(P)} 0^*(M) \).

Hence \( N(P) \subseteq \bigcap_{M \in \text{PSpec}(P)} 0^*(M) \subseteq \bigcap_{M \in \text{PSpec}(P)} F^*(M) \).

\( \square \)

Now we characterize the 2-primal semimodule \( P \).

**Theorem 3.26** Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R,S,R \text{PS}_R,S Q_R,\theta,\phi> \). Then the following statements are equivalent:

1. \( P \) is a 2-primal semimodule.
2. \( 0^*(M) \subseteq M \) for each \( M \in \text{PSpec}(P) \).
3. \( N(P) = \bigcap_{M \in \text{PSpec}(P)} 0^*(M) = \mathcal{P}(P) \).

**Proof.** (1) \( \Rightarrow \) (2)

Let \( a \in 0^*(M) \). Then there exists a positive integer \( n \) such that \( \theta(a \otimes Q)^{n-1} a \subseteq 0^*(M) \). Hence \( \theta(a \otimes Q)^{n-1} \theta(a \otimes Q)b = 0 \) for some \( b \in P - M \) i.e. \( \theta(a \otimes Q)^{n-1} \theta(a \otimes Q)b \subseteq \mathcal{P}(P) \), for some \( b \in P - M \), which implies \( \theta(a \otimes Q)^{n-1} a \subseteq N^*(M) \) i.e., \( a \in N^*(M) \). Hence, \( 0^*(M) \subseteq N^*(M) \) for each prime subsemimodule. Also by Theorem 3.18, \( N^*(M) = N^*(M) \) for each prime subsemimodule \( M \) of \( P \). Again since \( N^*(M) \subseteq M \) for each prime subsemimodule \( M \) of \( P \). Thus \( 0^*(M) \subseteq N^*(M) = N^*(M) \subseteq M \) for each prime subsemimodule \( M \) of \( P \).

(2) \( \Rightarrow \) (3)

Since \( 0^*(M) \subseteq M \) for each prime subsemimodule \( M \) of \( P \),

\[
\bigcap_{M \in \text{PSpec}(P)} 0^*(M) \subseteq \bigcap_{M \in \text{PSpec}(P)} M = \mathcal{P}(P)
\]

Now by theorem 3.25 \( N(P) \subseteq \bigcap_{M \in \text{PSpec}(P)} 0^*(M) \subseteq \mathcal{P}(P) \). Also \( \mathcal{P}(P) \subseteq N(P) \). Therefore \( N(P) = \bigcap_{M \in \text{PSpec}(P)} 0^*(M) = \mathcal{P}(P) \).

(3) \( \Rightarrow \) (1)

This part is obvious.

\( \square \)

**Theorem 3.27** Let \( R, S \) be two Morita equivalent semirings via Morita context \( <R,S,R \text{PS}_R,S Q_R,\theta,\phi> \) and \( P \) is strongly nil semimodule. Suppose \( 0^*(M) = M \) for each \( M \in \text{PSpec}(P) \). Then
(1) \( P \) is 2-primal semimodule.

(2) \( 0^*(M) = N^*(M) \) for each \( M \in \mathcal{PSpec}(P) \).

(3) Every prime subsemimodule of \( P \) is minimal.

Proof. (1) Since \( 0^*(M) = M, 0^*(M) \subseteq M \) for each \( M \in \mathcal{PSpec}(P) \). Hence by Theorem 3.26, \( \mathcal{P}(P) = \mathcal{N}(P) \). So \( P \) is 2-primal semimodule.

(2) Since \( N^*(M) \subseteq M \) and \( 0^*(M) = M \) for each prime subsemimodule \( M \) of \( P \), we have \( N^*(M) \subseteq 0^*(M) \) for each prime subsemimodule \( M \) of \( P \). Since \( P \) is 2-primal, by Theorem 3.18, \( N^*(M) = 0^*(M) \) for each prime subsemimodule \( M \) of \( P \). Thus \( 0^*(M) \subseteq N^*(M) \) for each prime subsemimodule \( M \) of \( P \). Therefore \( 0^*(M) = N^*(M) \) for each prime subsemimodule \( M \) of \( P \).

(3) Let \( M \) be a prime subsemimodule of \( P \). From (2) and the given condition \( 0^*(M) = M \), we get \( N^*(M) = M \) for each prime subsemimodule of \( M \) of \( P \). If \( M' \) is a minimal prime subsemimodule of \( P \) contained in \( M \), then \( N^*(M) \subseteq N^*(M') \subseteq M' \subseteq M = N^*(M) \). Thus \( M = M' \) that is \( M \) is a minimal prime subsemimodule of \( P \).

\( \square \)

Theorem 3.28 Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, \mathcal{S}_R, P_{\mathcal{S}_R}, Q_R, \theta, \phi > \) and \( 0^*(M) \) be a prime subsemimodule of the strongly nil semimodule \( P \) for each minimal prime subsemimodule \( M \) of \( P \). Then the following statements are equivalent:

(1) \( P \) is a 2-primal semimodule.

(2) \( 0^*(M) \) is a completely semiprime subsemimodule for each minimal prime subsemimodule \( M \) of \( P \).

(3) \( 0^*(M) \) is a left right symmetric subsemimodule for each minimal prime subsemimodule \( M \) of \( P \).

(4) \( \theta(x \otimes Q)y \subseteq 0^*(M) \) implies \( \theta(y \otimes Q)x \subseteq 0^*(M) \) for \( x, y \in P \) and for each prime subsemimodule \( M \) of \( P \).

Proof. (1) \( \Rightarrow \) (2)

Let \( \theta(x \otimes Q)x \subseteq 0^*(M) \). Then \( x \in 0^*(M) \), as \( 0^*(M) \) is a prime subsemimodule of \( P \). Hence \( 0^*(M) \) is a completely semiprime subsemimodule for each minimal prime subsemimodule \( M \) of \( P \).

The proofs of (2) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (4) are similar to the proofs (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (iv) of Theorem 3.17 respectively.

(4) \( \Rightarrow \) (1)

Let \( \theta(x \otimes Q)y \subseteq \mathcal{P}(P) \). Then \( \theta(x \otimes Q)y \subseteq M \) for each minimal prime subsemimodule \( M \) of \( P \). Since \( 0^*(M) \) is a prime sumsemimodule of \( P \) for each prime subsemimodule of \( P \), \( 0^*(M) = M \), for each minimal prime subsemimodule of \( P \). Thus \( \theta(x \otimes Q)y \subseteq 0^*(M) \) for each minimal prime subsemimodule \( M \) of \( P \). So by (4), \( \theta(y \otimes Q)x \subseteq 0^*(M) = M \) for each minimal prime
subsemimodule $M$ of $P$. Therefore $\theta(y \otimes Q)x \subseteq P(P)$. Hence by (iv) $\Rightarrow$ (i) of Theorem 3.17, $P$ is a 2-primal semimodule.

\[ \square \]

**Theorem 3.29** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho, \gamma, \phi >$ and $P$ is strongly nil semimodule. Let $M = 0^*(M) = 0^*(M)$ for each minimal prime subsemimodule $M$ of $P$. Then $P$ is 2-primal.

**Proof.** Let $M = 0^*(M) = 0^*(M)$ for each minimal prime subsemimodule $M$ of $P$. Then by Proposition 3.25, $N(P) \subseteq \bigcap_{M \in \text{Spec}(P)} 0^*(M) \subseteq \bigcap_{M' \in \text{Spec}(P)} M' = \mathcal{P}(P)$. Also $\mathcal{P}(P) \subseteq N(P)$. So $\mathcal{P}(P) = N(P)$. Hence $P$ is a 2-primal semimodule.

\[ \square \]

In order to conclude the paper we deduce the following result which shows that the property of being 2-primal is a Morita invariant property with some restriction.

**Theorem 3.30** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho, \gamma, \phi >$ and $R$, $S$ are Von-neumann regular semiring. Let $P$ is Von-neumann regular semimodule. Then the following are equivalent.

1. $P$ is 2-primal semimodule.
2. $R$ is a 2-primal semiring.
3. $S$ is a 2-primal semiring.

**Proof.** (1) $\Rightarrow$ (2)

Since $R$ is regular semiring so Theorem 3.8 holds for $R$ also (proof can be done in a similar manner as in Theorem 3.8). Now let $P$ is 2-primal. This implies that $P$ is NI (cf. Theorem 3.8). Now we prove $R$ is NI semiring i.e., $N^*(R) = N(R)$, where $N^*(R)$ denotes the unique maximal nil ideal of $R$ and $N(R)$ is the set of all nilpotent elements of $R$. Here we show that $g_1(N^*(P)) = N^*(R)$. Let $\sum_{i=1}^{m} \theta(p_i \otimes q_i) \in g_1(N^*(P))$ where $p_i \in (N^*(P))$ and for each $p_i$ and $q_i$ there exist $n_i$ such that $\theta(p_i \otimes q_i)^{n_i-1}p_i = 0$, $i = 1, 2, \ldots, m$. Now we can easily check that $\sum_{i=1}^{m} \theta(p_i \otimes q_i)^{n_1+n_2+\cdots+n_m+1} = 0$.

So $g_1(N^*(P))$ is a nil subsemimodule. The uniqueness and maximalness can be obtained easily from the lattice isomorphism $g_1$. So $g_1(N^*(P))$ is an unique maximal nil ideal of $R$ i.e., $g_1(N^*(P)) = N^*(R)$ and it must be equal to $N(R)$. If not let $r \in R$ be a nilpotent element $\notin N(R)$. Then the
nil ideal containing \( N(R) \) and \( r \) contains \( N^*(R) \) (since \( N^*(R) \subseteq N(R) \)) contradicting the maximality of \( N^*(R) \). So \( N^*(R) = N(R) \) i.e., \( R \) is a NI—semiring consequently \( R \) is 2-primal.

(2) \( \Rightarrow \) (1)

Let \( R \) is 2-primal semiring. Then \( R \) is NI-semiring (cf. Theorem 3.8). Now we prove \( P \) is NI-semimodule. We first show that \( f_1(N^*(R)) = N^*(P) \). Now using the above result \( f_1(N^*(R)) = f_1(g_1(N^*(P))) = N^*(P) \) (since \( f_1g_1 = \text{identity mapping} \)). Again by the similar argument as in (1) \( \Rightarrow \) (2) we have \( N^*(P) = N(P) \). Hence \( P \) is a NI—semimodule consequently \( P \) is 2-primal semimodule (cf. Theorem 3.8).

(1) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (1) follow in a similar manner.

\( \Box \)

4 Concluding remarks

All the results of this paper are related to the component \( P \) of a Morita context \( < R, S, R_P S, P_R, \theta, \phi > \). But similar results for the component \( Q \) can be established analogously which in turn provide us with alternative proofs for the relevant results discussed in this paper.

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References


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