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Geometrical objects associated with a multi-time least squares Hamiltonian

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Abstract In this paper we associate with a given multi-time dynamical system some geometrical objects (such as nonlinear connection or $d-$torsions) produced by the least squares Hamiltonian naturally attached to this initial PDE system of first order. Possible geometrical interpretations are still an open problem.

Keywords dual 1-jet space · multi-time dynamical system · least squares Hamiltonian · nonlinear connection · $d-$torsions

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1 Least squares Hamiltonian associated with a multi-time dynamical system and some Riemannian metrics

Let us consider that T and M are two smooth, real, connected manifolds of dimensions q and n, whose coordinates[1] are $(t^a)_{a=\overline{1,q}}$ and $(x^i)_{i=\overline{1,n}}$. Let $J^1(T,M)$ be the jet fibre bundle of first order, whose coordinates are (t^a,x^i,x^i_a), where x^i_a have the physical meaning of *partial velocities*.

Let us start with a given $d-$tensor field $X^{(i)}_{(a)}(t^c,x^k)$ on $J^1(T,M)$, which defines the PDEs system of order one (*multi-time dynamical system*)

$$x^i_a = X^{(i)}_{(a)}(t^c, x^k(t^c)), \tag{1.1}$$

where $x^i_a = \dfrac{\partial x^i}{\partial t^a}$. The complete integrability conditions

$$\frac{\partial X^{(i)}_{(a)}}{\partial t^b} + \frac{\partial X^{(i)}_{(a)}}{\partial x^m} X^{(m)}_{(b)} = \frac{\partial X^{(i)}_{(b)}}{\partial t^a} + \frac{\partial X^{(i)}_{(b)}}{\partial x^m} X^{(m)}_{(a)}$$

[1] Note that, throughout this paper, the letters $a,b,c,...$, respectively $i,j,k,...$, run from 1 to q, respectively from 1 to n, being used for indexing the components of geometrical objects attached to the manifold T, respectively M.

are required for the existence of solutions of the PDEs system (1.1).

Starting with $h = (h_{ab}(t^c))$ and $g = (g_{ij}(x^k))$ as some *a priori* given Riemannian metrics on T, respectively M, we consider that $\chi^c_{ab}(t)$ and $\gamma^i_{jk}(x)$ are their associated Christoffel symbols. Moreover, we use the the following notations:

$$X^{(i)}_{(a)//b} = \frac{\partial X^{(i)}_{(a)}}{\partial t^b} - X^{(i)}_{(c)}\chi^c_{ab}, \quad X^{(i)}_{(a)\|j} = \frac{\partial X^{(i)}_{(a)}}{\partial x^j} + X^{(m)}_{(a)}\gamma^i_{mj}.$$

In this context, supposing that T is a compact and orientable manifold, we construct the *multi-time least squares Lagrangian* [6]

$$\mathcal{L} = \|\mathbf{C} - \mathbf{X}\|^2_{J^1(T,M)}\sqrt{h} = \left\{h_{ab}(t^c)g_{ij}(x^k)\left[x^i_a - X^{(i)}_{(a)}\right]\left[x^j_b - X^{(j)}_{(b)}\right]\right\}\sqrt{h} =$$
$$= \left\{h^{ab}(t^c)g_{ij}(x^k)x^i_a x^j_b + U^{(a)}_{(i)}(t^c, x^k)x^i_a + \mathcal{P}(t^c, x^k)\right\}\sqrt{h},$$

where $\mathbf{C} = x^i_a(\partial/\partial x^i_a)$, $\mathbf{X} = X^{(i)}_{(a)}(\partial/\partial x^i_a)$, $U^{(a)}_{(i)}(t^c, x^k) = -2h^{ac}g_{im}X^{(m)}_{(c)}$, $\mathcal{P}(t^c, x^k) = h^{cd}g_{rs}X^{(r)}_{(c)}X^{(s)}_{(d)}$. This produces the *least squares energy*

$$\mathcal{E}_{\mathcal{L}} : C^2(T, M) \to \mathbb{R}_+,$$

where

$$\mathcal{E}_{\mathcal{L}} = \int_T \mathcal{L}dt^1 \wedge dt^2 \wedge \ldots \wedge dt^q = \int_T \|\mathbf{C} - \mathbf{X}\|^2_{J^1(T,M)}\sqrt{h}dt^1 \wedge dt^2 \wedge \ldots \wedge dt^q \geq 0.$$

Taking into account that the Riemannian metrics $h_{ab}(t)$ and $g_{ij}(x)$ are positive definite, it follows that a map $f \in C^2(T, M)$, locally expressed by $(t^c) \to (x^i(t^c))$, is a solution of the PDEs system (1.1) if and only if the map f is a global minimum point for the least squares energy functional $\mathcal{E}_{\mathcal{L}}$. Therefore, a solution f of class C^2 for the system (1.1) verifies the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial}{\partial t^a}\left(\frac{\partial \mathcal{L}}{\partial x^i_a}\right) = 0, \quad \forall\, i = \overline{1, n}. \tag{1.2}$$

But, the Euler-Lagrange equations (1.2) can be rearranged in the general Poisson form (equations of *generalized harmonic maps* [5])

$$h^{ab}\left\{x^i_{ab} + 2H^{(i)}_{(a)b} + 2G^{(i)}_{(a)b}\right\} = 0, \tag{1.3}$$

where

$$H^{(i)}_{(a)b} = -\frac{1}{2}\chi^c_{ab}x^i_c$$

and

$$G^{(i)}_{(a)b} = \frac{1}{2}\gamma^i_{jk}x^j_a x^k_b + \frac{h_{ab}g^{il}}{4q}\left[U^{(c)}_{(l)m}x^m_c + \frac{\partial U^{(c)}_{(l)}}{\partial t^c} + U^{(c)}_{(l)}\chi^d_{cd} - \frac{\partial \mathcal{P}}{\partial x^l}\right]$$

are endowed with the geometrical meaning of a *multi-time semispray* (see [6,5]). Here, we have $q = \dim T$ and

$$U^{(a)}_{(i)j} = \frac{\partial U^{(a)}_{(i)}}{\partial x^j} - \frac{\partial U^{(a)}_{(j)}}{\partial x^i}.$$

Remark 1.1 From above equations, it follows that the solutions of C^2 class of the initial PDEs system (1.1) are generalized harmonic maps naturally associated to the multi-time semispray

$$\mathcal{S} = \left(H^{(i)}_{(a)b}, G^{(i)}_{(a)b}\right).$$

Following the Lagrange geometrical ideas developed in the Miron-Anastasiei -Kirkovits' works [2,4], the preceding semispray allows us to construct an entire natural collection of Lagrangian jet geometrical objects, such as nonlinear connection, $d-$torsions, $d-$curvatures or some field-like theory (electromagnetic-like and gravitational-like) dependent on partial velocities. All details on these Lagrange geometrical ideas and constructions on 1-jet spaces find in the papers [6,5].

Let us return to the PDEs system of order one (1.1) whose solutions of C^2 class are global minimum points of the energy functional produced by the multi-time Lagrangian $\mathcal{L} = L\sqrt{h}$ defined by the jet least squares Lagrangian function

$$L = h^{ab}g_{ij}x^i_a x^j_b + U^{(a)}_{(i)}x^i_a + \mathcal{P}, \tag{1.4}$$

where $U^{(a)}_{(i)} = -2h^{ac}g_{im}X^{(m)}_{(c)}$ and $\mathcal{P} = h^{cd}g_{rs}X^{(r)}_{(c)}X^{(s)}_{(d)}$.

By direct computations, via the classical Legendre transformation, the jet multi-time Lagrangian function of electrodynamics (1.4) produces the Hamiltonian function of *polymomenta* (that is the *least squares Hamiltonian*)

$$H : E^* = J^{1*}(T, M) \to \mathbb{R},$$

given by

$$H = \frac{1}{4}h_{ab}g^{ij}p^a_i p^b_j + X^{(i)}_{(a)}p^a_i, \tag{1.5}$$

where $p^a_i = \partial L/\partial x^i_a$ and $H = p^a_i x^i_a - L$.

Remark 1.2 Such kind of Hamiltonians as the least squares Hamiltonian (1.5), which are of electrodynamics type (see [7]), produces an entire collection of geometrical objects on the dual 1-jet space $E^* = J^{1*}(T, M)$, such as nonlinear connection, $d-$torsions, $d-$curvatures and some field theories (electromagnetic-like or gravitational like) dependent on polymomenta. For the complete description of such a dual jet Hamilton geometry, consult the works [7, 1].

2 From PDEs system of order one and metrics to Hamiltonian geometrical objects

Thus, following the Hamilton geometrical ideas initiated in the book Miron et al. [3], and using some general results from the geometrical theory of dual jet multi-time Hamilton spaces exposed in the works [7, 1], we can construct the *canonical nonlinear connection*

$$N = \left(\underset{1}{N}{}^{(a)}_{(i)b},\ \underset{2}{N}{}^{(a)}_{(i)j} \right)$$

on the dual 1-jet space $E^* = J^{1*}(T, M)$, which is given by local components

$$\underset{1}{N}{}^{(a)}_{(i)b} = \chi^a_{bf} p^f_i, \quad \underset{2}{N}{}^{(a)}_{(i)j} = \gamma^r_{ij} \left[2A^{(a)}_{(r)} - p^a_r \right] - \left[\frac{\partial A^{(a)}_{(i)}}{\partial x^j} + \frac{\partial A^{(a)}_{(j)}}{\partial x^i} \right], \tag{2.1}$$

where $A^{(a)}_{(i)} = -h^{ac} g_{im} X^{(m)}_{(c)}$. The nonlinear connection (2.1) determines the following *adapted bases*

$$\left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p^a_i} \right\} \subset \mathcal{X}(E^*), \qquad \left\{ dt^a, dx^i, \delta p^a_i \right\} \subset \mathcal{X}^*(E^*)$$

where

$$\begin{gathered} \frac{\delta}{\delta t^a} = \frac{\partial}{\partial t^a} - \underset{1}{N}{}^{(f)}_{(r)a} \frac{\partial}{\partial p^f_r}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{2}{N}{}^{(f)}_{(r)i} \frac{\partial}{\partial p^f_r}, \\ \delta p^a_i = dp^a_i + \underset{1}{N}{}^{(a)}_{(i)f} dt^f + \underset{2}{N}{}^{(a)}_{(i)r} dx^r. \end{gathered} \tag{2.2}$$

By direct computations, we can determine the adapted components of the *generalized Cartan canonical connection*, namely (see, for details, [7, 1])

$$C\Gamma(N) = \left(H^c_{ab} = \chi^c_{ab},\ A^i_{jc} = 0,\ H^i_{jk} = \gamma^i_{jk},\ C^{i(k)}_{j(c)} = 0 \right). \tag{2.3}$$

The torsion $\mathbf{T}$ of the generalized Cartan canonical linear connection produced by the Hamiltonian (1.5) is characterized by *three* effective adapted

components [7, 1]:

$$R^{(f)}_{(r)ab} = \chi^f_{gab} p^g_r, \quad R^{(f)}_{(r)aj} = -2\gamma^s_{rj} A^{(f)}_{(s);a} + \left[\frac{\partial A^{(f)}_{(r)}}{\partial x^j} + \frac{\partial A^{(f)}_{(j)}}{\partial x^r} \right]_{;a},$$
$$R^{(f)}_{(r)ij} = \mathfrak{R}^s_{rij} \left[2A^{(f)}_{(s)} - p^f_s \right] - \left[\frac{\partial A^{(f)}_{(i)}}{\partial x^j} - \frac{\partial A^{(f)}_{(j)}}{\partial x^i} \right]_{:r}, \tag{2.4}$$

where $\chi^c_{dab}(t)$ (respectively $\mathfrak{R}^k_{rij}(x)$) are the classical local curvature tensors of the Riemannian metric $h_{ab}(t)$ (respectively $g_{ij}(x)$), and "$_{;a}$" and "$_{:k}$" represent the following *generalized Levi-Civita covariant derivatives*:

– the T-generalized Levi-Civita covariant derivative:

$$T^{bi(d)(r)...}_{cj(l)(f)...;a} \overset{def}{=} \frac{\partial T^{bi(d)(r)...}_{cj(l)(f)...}}{\partial t^a} + T^{gi(d)(r)...}_{cj(l)(f)...} \chi^b_{ga} + T^{bi(g)(r)...}_{cj(l)(f)...} \chi^d_{ga} + ...$$
$$... - T^{bi(d)(r)...}_{gj(l)(f)...} \chi^g_{ca} - T^{bi(d)(r)...}_{cj(l)(g)...} \chi^g_{fa} - ...$$

– the M-generalized Levi-Civita covariant derivative:

$$T^{bi(d)(r)...}_{cj(l)(f)...:k} \overset{def}{=} \frac{\partial T^{bi(d)(r)...}_{cj(l)(f)...}}{\partial x^k} + T^{bs(d)(r)...}_{cj(l)(f)...} \gamma^i_{sk} + T^{bi(d)(s)...}_{cj(l)(f)...} \gamma^r_{sk} + ...$$
$$... - T^{bi(d)(r)...}_{cs(l)(f)...} \gamma^s_{jk} - T^{bi(d)(r)...}_{cj(s)(f)...} \gamma^s_{lk} - ... .$$

Afterward, the curvature $\mathbf{R}$ of the Cartan canonical connection produced by the Hamiltonian (1.5) is characterized by the following *four* effective adapted components [7, 1]:

$$H^d_{abc} = \chi^d_{abc}, \quad R^l_{ijk} = \mathfrak{R}^l_{ijk}, \quad -R^{(d)(i)}_{(l)(a)bc} = \delta^i_l \chi^d_{abc}, \quad -R^{(d)(l)}_{(i)(a)jk} = -\delta^d_a \mathfrak{R}^l_{ijk}.$$

3 Field-like theories depending on polymomenta produced by a multi-time dynamical system and some Riemannian metrics

3.1 The electromagnetic-like geometrical model

In order to describe our geometrical electromagnetic-like theory (depending on polymomenta) produced by the Hamiltonian (1.5), we underline that it is characterized by the distinguished 2-form on $J^{1*}(T, M)$, locally defined by

$$\mathbb{F} = F^{(i)}_{(a)j} \delta p^a_i \wedge dx^j,$$

where

$$F^{(i)}_{(a)j} = \underset{\{i,j\}}{\mathcal{A}} \left\{ h_{af} g^{ir} \left[A^{(f)}_{(r):j} + A^{(f)}_{(j):r} \right] \right\}. \tag{3.1}$$

We recall that we have $A_{(i)}^{(a)} = -h^{ac} g_{im} X_{(c)}^{(m)}$.

The electromagnetic field (3.1) obeys the following *generalized Maxwell-like equations*:

$$\begin{cases} F_{(a)j/b}^{(i)} = h_{af} \cdot \underset{\{i,j\}}{\mathcal{A}} \left\{ g^{ir} \left[\dfrac{\partial A_{(r)}^{(f)}}{\partial x^j} + \dfrac{\partial A_{(j)}^{(f)}}{\partial x^r} \right]_{;b} - 2 g^{ir} \gamma_{rj}^{s} A_{(s);b}^{(f)} \right\} \\ \\ \underset{\{i,j,k\}}{\sum} F_{(a)j|k}^{(i)} = -h_{af} \cdot \underset{\{i,j,k\}}{\sum} \left\{ \left[g^{sr} \mathfrak{R}_{rjk}^{i} - g^{ir} \mathfrak{R}_{rjk}^{s} \right] p_s^f + \right. \\ \left. \qquad + g^{ir} \left[2 \mathfrak{R}_{rjk}^{s} A_{(s)}^{(f)} - \left(\dfrac{\partial A_{(j)}^{(f)}}{\partial x^k} - \dfrac{\partial A_{(k)}^{(f)}}{\partial x^j} \right)_{:r} \right] \right\} \\ \\ \underset{\{i,j,k\}}{\sum} F_{(a)j}^{(i)}{}_{|(c)}^{(k)} = 0, \end{cases} \tag{3.2}$$

where $\mathcal{A}_{\{i,j\}}$ represents an alternate sum and $\sum_{\{i,j,k\}}$ represents a cyclic sum.

3.2 The gravitational-like geometrical model

To expose our geometrical Hamiltonian polymomentum gravitational theory produced by the Hamiltonian (1.5), we introduce the polymomentum gravitational h^*-potential $\mathbb{G}$ on $E^* = J^{1*}(T, M)$, locally given by

$$\mathbb{G} = h_{ab}^* dt^a \otimes dt^b + g_{ij} dx^i \otimes dx^j + h_{ab}^* g^{ij} \delta p_i^a \otimes \delta p_j^b,$$

where $h_{ab}^* = (1/4) h_{ab}$.

We postulate that the *geometrical Einstein-like equations*, which govern the multi-time gravitational potential $\mathbb{G}$ are the abstract geometrical Einstein equations

$$\mathrm{Ric}(C\Gamma) - \frac{\mathrm{Sc}(C\Gamma)}{2} \mathbb{G} = \mathcal{K} \mathbb{T}, \tag{3.3}$$

where $\mathrm{Ric}(C\Gamma)$ represents the *Ricci tensor* of the Cartan canonical connection, $\mathrm{Sc}(C\Gamma)$ is the *scalar curvature*, $\mathcal{K}$ is the *Einstein constant* and $\mathbb{T}$ is an intrinsic $d-$tensor of matter, which is called the *stress-energy $d-$tensor of polymomenta*.

Following general results from [7, 1], we infer

Theorem 3.1 *The Ricci tensor* $\mathrm{Ric}(C\Gamma)$ *of the Cartan canonical connection of the Hamiltonian (1.5) is characterized by* ***two*** *effective local Ricci $d-$tensors (i.e. the classical Ricci tensors of the Riemannian temporal metric $h_{ab}(t)$ and spatial metric $g_{ij}(x)$): $\chi_{ab} = \chi_{abf}^{f}$, $\mathfrak{R}_{ij} = \mathfrak{R}_{ijr}^{r}$.*

It follows that, using the notations $\chi = h^{ab}\chi_{ab}$ and $\mathfrak{R} = g^{ij}\mathfrak{R}_{ij}$, we find

Theorem 3.2 *The scalar curvature* $\mathrm{Sc}(C\Gamma)$ *of the Cartan canonical connection produced by the Hamiltonian (1.5) has the expression*

$$Sc(C\Gamma) = 4\chi + \mathfrak{R}.$$

In conclusion, the *generalized Einstein-like equations*, that govern the polymomentum gravitational potential $\mathbb{G}$ produced by the Hamiltonian (1.5), have the form [7,1]

$$\begin{cases} \chi_{ab} - \dfrac{4\chi + \mathfrak{R}}{8} h_{ab} = \mathcal{K}\mathbb{T}_{ab} \\ \mathfrak{R}_{ij} - \dfrac{4\chi + \mathfrak{R}}{2} g_{ij} = \mathcal{K}\mathbb{T}_{ij} \\ -\dfrac{4\chi + \mathfrak{R}}{8} h_{ab} g^{ij} = \mathcal{K}\mathbb{T}^{(i)(j)}_{(a)(b)}, \end{cases} \tag{3.4}$$

$$\begin{cases} 0 = \mathbb{T}_{ai}, \quad 0 = \mathbb{T}_{ia}, \quad 0 = \mathbb{T}^{(i)}_{(a)b} \\ 0 = \mathbb{T}^{\ (j)}_{a(b)}, \, 0 = \mathbb{T}^{(j)}_{i(b)}, \, 0 = \mathbb{T}^{(i)}_{(a)j}, \end{cases} \tag{3.5}$$

where $\mathbb{T}_{AB},\ A, B \in \left\{a, i, {}^{(i)}_{(a)}\right\}$, are the adapted components of the polymomentum stress-energy $d-$tensor of matter $\mathbb{T}$.

The *polymomentum conservation laws* of the geometrical Einstein-like equations produced by the Hamiltonian (1.5) are expressed by the formulas [7,1]

$$\begin{cases} \left[4\chi^f_b - \dfrac{4\chi + \mathfrak{R}}{2}\delta^f_b\right]_{/f} = 0 \\ \left[\mathfrak{R}^r_j - \dfrac{4\chi + \mathfrak{R}}{2}\delta^r_j\right]_{|r} = 0, \end{cases} \tag{3.6}$$

where $\chi^f_b = h^{fd}\chi_{db}$ and $\mathfrak{R}^r_j = g^{rs}\mathfrak{R}_{sj}$.

4 Hamiltonian geometrical objects induced only by PDE systems of first order

In this section we will consider some important particular PDE systems of order one and we will apply them the Hamilton geometrical results from preceding sections by considering the particular flat Riemannian manifolds $(T = \mathbb{R}^q, \delta_{ab})$ and $(M = \mathbb{R}^n, \delta_{ij})$. Obviously, we will study only the nonzero geometrical and physical objects which are *effectively* determined by these systems (i.e., the nonlinear connection, the torsion $d-$tensors). From this point of view, we remark that, in this particular situation, the least squares

Hamiltonian associated with the PDEs system of order one (1.1) has the form

$$H = \frac{1}{4}\delta_{ab}\delta^{ij}p_i^a p_j^b + X_{(a)}^{(i)}p_i^a. \tag{4.1}$$

Consequently, the nonlinear connection on the dual 1-jet space $J^{1*}(\mathbb{R}^q, \mathbb{R}^n)$, produced by the Hamiltonian (4.1), has the local components

$$N_{1\ (i)b}^{(a)} = 0, \quad N_{2\ (i)j}^{(a)} = \frac{\partial X_{(a)}^{(i)}}{\partial x^j} + \frac{\partial X_{(a)}^{(j)}}{\partial x^i}, \tag{4.2}$$

because we have $A_{(i)}^{(a)} = -X_{(a)}^{(i)}$ (see equations (2.1)). The nonzero local $d-$torsions of the generalized Cartan canonical linear connection produced by the Hamiltonian (4.1) are given by (see equations (2.4))

$$R_{(r)aj}^{(f)} = -\left[\frac{\partial^2 X_{(f)}^{(r)}}{\partial t^a \partial x^j} + \frac{\partial^2 X_{(f)}^{(j)}}{\partial t^a \partial x^r}\right], \quad R_{(r)ij}^{(f)} = \frac{\partial^2 X_{(f)}^{(i)}}{\partial x^r \partial x^j} - \frac{\partial^2 X_{(f)}^{(j)}}{\partial x^r \partial x^i}. \tag{4.3}$$

Open problem What is the real meaning in the phenomenon studied by the multi-time dynamical system (1.1) for the above Hamilton geometrical objects naturally associated with the least squares Hamiltonian (4.1)?

4.1 Field lines

Consider $T = [a, b] \subset \mathbb{R}$, $h_{11}(t) = 1$ and $X_{(1)}^{(i)}(t, x^k) = \xi^i(x^k)$, where $\xi^i(x^k)$ is a $d-$vector field on $J^1([a, b], \mathbb{R}^n)$. The PDEs system (1.1) becomes

$$\frac{dx^i}{dt} = \xi^i(x^k(t)), \tag{4.4}$$

that is, the DEs system which gives the field lines of the spatial vector field ξ. In this context, the least squares Hamiltonian is

$$H = \frac{\delta^{ij}}{4}p_i p_j + \xi^j p_j.$$

1. The canonical spatial nonlinear connection, induced by the system (4.4) on $J^{1*}([a, b], M) \equiv T^*M$, is given by the components

$$N_{2\ (i)j}^{(1)} = \frac{\partial \xi^i}{\partial x^j} + \frac{\partial \xi^j}{\partial x^i};$$

2. The Hamilton $d-$torsions induced by the differential system (4.4), are determined only by the local components

$$R_{(r)ij}^{(1)} = \frac{\partial^2 \xi^i}{\partial x^r \partial x^j} - \frac{\partial^2 \xi^j}{\partial x^r \partial x^i}.$$

4.2 Pfaffian systems

Consider $M = \mathbb{R}$, $g_{11}(x) = 1$ and $X^{(1)}_{(a)}(t^c, x) = A_a(t^c)$, where $A_a(t^c)$ is a distinguished 1-form on $J^1(\mathbb{R}^q, \mathbb{R})$. We have the Pfaffian system

$$x_a = A_a(t^c). \tag{4.5}$$

In this context, the least squares Hamiltonian is

$$H = \frac{\delta_{ab}}{4} p^a p^b + A_a p^a,$$

and all corresponding Hamilton geometrical objects are zero.

4.3 Continuous groups of transformations

The fundamental PDEs system of a transformations group having the infinitesimal generators $\{\xi_\alpha(x^k)\}_{\alpha=\overline{1,\gamma}}$, as $d-$vector fields on $J^1(\mathbb{R}^q, \mathbb{R}^n)$, is given by

$$x^i_a = \sum_{\alpha=1}^{\gamma} \xi^i_\alpha(x^k(t^c)) A^\alpha_a(t^c), \tag{4.6}$$

where $\{A^\alpha(t^c)\}_{\alpha=\overline{1,\gamma}}$ are $d-$forms on $J^1(\mathbb{R}^q, \mathbb{R}^n)$. In this context, the least squares Hamiltonian is

$$H = \frac{\delta_{ab}\delta^{ij}}{4} p^a_i p^b_j + \xi^i_\alpha A^\alpha_a p^a_i. \tag{4.7}$$

1. The spatial nonlinear connection on the dual 1-jet space $J^{1*}(\mathbb{R}^q, \mathbb{R}^n)$, produced by the Hamiltonian (4.7), has the local components

$$N^{(a)}_{2\ (i)j} = \left(\frac{\partial \xi^i_\alpha}{\partial x^j} + \frac{\partial \xi^j_\alpha}{\partial x^i} \right) A^\alpha_a;$$

2. The nonzero local $d-$torsions of the generalized Cartan canonical linear connection produced by the Hamiltonian (4.7) are expressed by

$$R^{(f)}_{(r)aj} = -\left[\frac{\partial \xi^r_\alpha}{\partial x^j} + \frac{\partial \xi^j_\alpha}{\partial x^r} \right] \frac{\partial A^\alpha_f}{\partial t^a}, \quad R^{(f)}_{(r)ij} = \left[\frac{\partial^2 \xi^i_\alpha}{\partial x^r \partial x^j} - \frac{\partial^2 \xi^j_\alpha}{\partial x^r \partial x^i} \right] A^\alpha_f.$$

5 Conclusion

The novelty of this paper is that we associated to a given multi-time dynamical system some natural Hamilton geometrical objects, via the new least squares Hamiltonians (1.5) and (4.1). We talk about a spatial nonlinear connection and some local $d-$torsions. In our opinion these geometrical objects could lead to a classification of the multi-time dynamical systems. On the other hand, their real meaning in the phenomena studied by the initial multi-time dynamical system is still an open problem, which is in our attention, being part of our work in progress.

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