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Quotient near-rings involving P-homoderivations

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Abstract Let $\mathcal{N}$ be a near-ring and $\mathcal{P}$ be a 3-prime ideal of $\mathcal{N}$. This paper introduces a novel concept known as $\mathcal{P}$-homoderivation and investigates the commutativity of the quotient near-ring $\mathcal{N}/\mathcal{P}$ when $\mathcal{N}$ has a $\mathcal{P}$-homoderivation satisfying certain distinction identities. In addition, we extend several known results related to 3-prime near-rings with homoderivations.

Keywords near-rings · Jordan ideals · $\mathcal{P}$-homoderivations.

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1 Introduction

Throughout this paper, $\mathcal{N}$ will be zero-symmetric left near-ring. A near-ring $\mathcal{N}$ is 3-prime if for $a, b \in \mathcal{N}$ and $a\mathcal{N}b = \{0\}$ implies $a = 0$ or $b = 0$. Furthermore, $\mathcal{N}$ is 2-torsion free if for all $x \in \mathcal{N}$ and $2x = 0$ implies $x = 0$. For all pair of elements $a, b \in \mathcal{N}$, $a \circ b = ab + ba$ and $[a, b] = ab - ba$ will represent the Jordan and Lie products. A normal subgroup $\mathcal{P}$ of $(\mathcal{N}, +)$ is called a left ideal (resp. right ideal) if $\mathcal{N}\mathcal{P} \subseteq \mathcal{P}$ (resp. $(n+p)m-nm \in \mathcal{P}$ for all $n, m \in \mathcal{N}$ and $p \in \mathcal{P}$), and if $\mathcal{P}t$ is both a left and a right ideal, then $\mathcal{P}$ is said to be an ideal. We recall from [11] that an ideal $\mathcal{P}$ is a 3-prime ideal if for $a, b \in \mathcal{N}$ and $a\mathcal{N}b \subseteq \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$. A mapping $h : \mathcal{N} \to \mathcal{N}$ is $\mathcal{P}$-trivial if $h(\mathcal{N}) \subseteq \mathcal{P}$. If J is an additive subgroup of $\mathcal{N}$, then J is said to be Jordan left (resp. right) ideal of $\mathcal{N}$ if $n \circ j \in J$ (resp. $j \circ n \in J$) for all $j \in J$, $n \in \mathcal{N}$, and J is said to be a Jordan ideal of $\mathcal{N}$ if $j \circ n \in J$ and $n \circ j \in J$ for all $j \in J$, $n \in \mathcal{N}$.

An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$. Recently, the connection between the construction of the near-ring $\mathcal{N}$ and the action of certain special kinds of mappings on $\mathcal{N}$ has been studied by several authors (see for example ([1]-[12]), where further

references can be found. In [6] A. Boua (2019) defines a new notion in near-rings, called homoderivation, which is an additive mapping from $\mathcal{N}$ into itself such that $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in \mathcal{N}$, and obtains interesting results on 3-prime near-rings $\mathcal{N}$ when h satisfies certain differential identities. Furthermore, he studies the notion of homoderivation equipped with Jordan right ideal satisfying some algebraic conditions.

Motivated by the above, we will follow another direction of investigation, we can assume that for a homoderivation h of a certain near-ring $\mathcal{N}$, where $\mathcal{N}$ is not necessarily a 3-prime or semiprime near-ring, $h(xy) - (h(x)h(y) + h(x)y + xh(y))$ can be in an ideal of $\mathcal{N}$, for all $x, y \in \mathcal{N}$, obviously if the ideal is a zero ideal, then h is an ordinary homoderivation. So it is interesting to find a relative ideal that satisfies this condition.

In this paper, we will verify that the 3-prime ideal satisfies this condition.

By the following definitions, we initiate our work:

Definition 1.1 *Let $\mathcal{N}$ be a near-ring and $\mathcal{P}$ be a subset of $\mathcal{N}$. An additive mapping $h : \mathcal{N} \to \mathcal{N}$ is a $\mathcal{P}$-homoderivation if $h(xy) - (h(x)h(y) + h(x)y + xh(y)) \in \mathcal{P}$, for all $x, y \in \mathcal{N}$.*

It is straightforward to observe that every homoderivation on $\mathcal{N}$ is also a $\mathcal{P}$-homoderivation on $\mathcal{N}$. However, the reverse is not necessarily true. The following example illustrates the existence of a $\mathcal{P}$-homoderivation that is not a homoderivation:

Example 1.1 Let $\mathcal{K}$ be a left near-ring. Define $\mathcal{N}$, $\mathcal{P}$ by

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} \mid u, v, w, 0 \in \mathcal{K}, \right\} \text{and } \mathcal{P} = \left\{ \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid s, 0 \in \mathcal{K} \right\}.$$

Then $\mathcal{N}$ is a left near-ring, and $\mathcal{P}$ is an ideal of $\mathcal{N}$. The map $h : \mathcal{N} \to \mathcal{N}$ given by:

$$h \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a $\mathcal{P}$-homoderivation, but not a homoderivation on $\mathcal{N}$.

In this paper, we will introduce a new area by examining algebraic differential identities involving $\mathcal{P}$-homoderivations acting on a near-ring $\mathcal{N}$, without assuming that the near-ring is 3-prime. Specifically, we extend the usual homoderivation of a near-ring to the broader notion of $\mathcal{P}$-homoderivation for near-rings.

2 Preliminaries

In order to develop our basic ideas, we need the following key lemmas.

Lemma 2.1 *If $\mathcal{P}$ is a 3-prime ideal of zero symmetric near-ring $\mathcal{N}$, then $\mathcal{N}/\mathcal{P}$ is zero symmetric 3-prime near-ring.*

Proof. It is easily to show that $\mathcal{N}/\mathcal{P}$ is zero-symmetric near-ring. Now, let $\bar{x}(\mathcal{N}/\mathcal{P})\bar{y} = \{\overline{0}\}$, then $\bar{x}\bar{n}\bar{y} = \overline{0}$ for all $n \in \mathcal{N}$. That is $x\mathcal{N}y \subseteq \mathcal{P}$, but $\mathcal{P}$ is 3-prime ideal, so either $x \in \mathcal{P}$ or $y \in \mathcal{P}$. i.e. either $\bar{x} = \overline{0}$ or $\bar{y} = \overline{0}$, which complete the proof. So, $\mathcal{N}/\mathcal{P}$ will be zero-symmetric throughout the rest of this paper.

□

Lemma 2.2 *Let $\mathcal{P}$ be a 3-prime ideal of a near-ring $\mathcal{N}$.*

(i) If $Z(\mathcal{N}/\mathcal{P})$ contains a nonzero element $\bar{z}$ for which $\bar{z} + \bar{z} \in Z(\mathcal{N}/\mathcal{P})$, then $(\mathcal{N}/\mathcal{P}, +)$ is abelian.
(ii) If $\bar{z} \in Z(\mathcal{N}/\mathcal{P})\backslash\{\overline{0}\}$ and $\bar{x} \in \mathcal{N}/\mathcal{P}$, such that $\bar{x}\bar{z} \in Z(\mathcal{N}/\mathcal{P})$ or $\bar{z}\bar{x} \in Z(\mathcal{N}/\mathcal{P})$ then $\bar{x} \in Z(\mathcal{N}/\mathcal{P})$.

Proof. Since $\{\overline{0}\}$ is 3-prime ideal of $\mathcal{N}/\mathcal{P}$, therefore (i) and (ii) are consequences of [10, Lemmas 1.2(ii) and 1.2(iii)].

□

Lemma 2.3 *Let $\mathcal{P}$ be a 3-prime ideal of a near-ring $\mathcal{N}$.*

(i) If h is a $\mathcal{P}$-homoderivation of $\mathcal{N}$ and $a \in \mathcal{N}$ such that $ah(\mathcal{N}) \subseteq \mathcal{P}$, then $a \in \mathcal{P}$ or h is $\mathcal{P}$-trivial.
(ii) If $\mathcal{N}/\mathcal{P}$ is 2-torsion free and h is $\mathcal{P}$-homderivation of $\mathcal{N}$ such that $h^2(\mathcal{N}) \subseteq \mathcal{P}$, then h is $\mathcal{P}$-trivial.

Proof. (i) Since $ah(\mathcal{N}) \subseteq \mathcal{P}$, it follows that $\overline{0} = \bar{a}\overline{h(xy)} = \bar{a}\overline{h(x)h(y)} + \bar{a}\overline{h(x)}\bar{y} + \bar{a}\bar{x}\overline{h(y)}$ for all $x, y \in \mathcal{N}$. This implies that $\overline{0} = \bar{a}\bar{x}\overline{h(y)}$ for all $x, y \in \mathcal{N}$. Hence, we obtain $\bar{a}(\mathcal{N}/\mathcal{P})\overline{h(y)} = \overline{\{0\}}$ for all $y \in \mathcal{N}$. But $\mathcal{N}/\mathcal{P}$ is 3-prime, which shows that either $a \in \mathcal{P}$ or h is $\mathcal{P}$-trivial.

(ii) Assume that h is a $\mathcal{P}$-homoderivation with $h^2(\mathcal{N}) \subseteq \mathcal{P}$, then $\overline{0} = \overline{h^2(xy)}$. So, $\overline{0} = \overline{h(h(x)h(y) + h(x)y + xh(y))}$ which gives $\overline{0} = 2\overline{h(x)h(y)}$ for all $x, y \in \mathcal{N}$. Thus $\overline{h(x)h(y)} = \overline{0}$ for all $x, y \in \mathcal{N}$, because $\mathcal{N}/\mathcal{P}$ is 2-torsion free, using (i) forces h is $\mathcal{P}$-trivial.

□

Lemma 2.4 *Let $\mathcal{P}$ be an ideal of a near-ring $\mathcal{N}$. If h is a $\mathcal{P}$-homoderivation of $\mathcal{N}$, then for all $x, y, z \in \mathcal{N}$, we have*

$$h(xy)(h(z)+z)-(h(x)h(y)(h(z)+z)+h(x)y(h(z)+z)+xh(y)(h(z)+z)) \in \mathcal{P}$$

Proof. Since, $\overline{h(xy)} = \overline{h(x)h(y)} + \overline{h(x)}\bar{y} + \bar{x}\overline{h(y)}$ for all $x, y \in \mathcal{N}$. Then

$$\begin{aligned}\overline{h((xy)z)} &= \overline{h(xy)h(z)} + \overline{h(xy)}\bar{z} + \overline{xy}\overline{h(z)} \\ &= \overline{h(xy)}(\overline{h(z)} + \bar{z}) + \overline{xy}\overline{h(z)} \text{ for all } x, y, z \in \mathcal{N}\end{aligned}$$

Also,

$$\begin{aligned}\overline{h(x(yz))} &= \overline{h(x)h(yz)} + \overline{h(x)}\overline{yz} + \bar{x}\overline{h(yz)} \\ &= \overline{h(x)h(y)h(z)} + \overline{h(x)h(y)}\bar{z} + \overline{h(x)}\bar{y}\overline{h(z)} + \overline{h(x)}\overline{yz} + \bar{x}\overline{h(y)h(z)} \\ &+ \bar{x}\overline{h(y)}\bar{z} + \overline{xy}\overline{h(z)} \text{ for all } x, y, z \in \mathcal{N}\end{aligned}$$

Combining both equations, we get the required result.

□

3 *P*-homoderivation on a near-ring

In [6] A. Boua, showed that: if h is a homoderivation of a 3-prime near-ring $\mathcal{N}$ which satisfies any one of the properties: $h(\mathcal{N}) \subseteq Z(\mathcal{N})$, $h([x, y]) = 0$ or $h([x, y]) = [x, y]$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ must be commutative. Moreover, he proved that there is no homoderivtion $0 \neq h$ of $\mathcal{N}$ that satisfies any of the following hypotheses: $h(x \circ y) = 0, h(x \circ y) = (x \circ y)$, or $h[x, y] = (x \circ y)$ for all $x, y \in \mathcal{N}$. The main goal of this section is to discuss a new line by considering these algebraic identities with $\mathcal{P}$-homoderivations acting on a near-ring $\mathcal{N}$ without the 3-primeness assumption.

Theorem 3.1 *Let $\mathcal{N}$ be a near-ring and $\mathcal{P}$ is a 3-prime ideal of $\mathcal{N}$ such that $\mathcal{N}/\mathcal{P}$ is 2-torsion free. If $\mathcal{N}$ admit a non $\mathcal{P}$-trivial $\mathcal{P}$-homoderivation h, then the following assertions are equivalent:*

(i) $\overline{h(\mathcal{N})} \subseteq Z(\mathcal{N}/\mathcal{P})$*;*
(ii) $h([x, y]) \in \mathcal{P}$ *for all* $x, y \in \mathcal{N}$*;*
(iii) $h([x, y]) - [x, y] \in \mathcal{P}$ *for all* $x, y \in \mathcal{N}$*;*
(iv) $\mathcal{N}/\mathcal{P}$ *is a commutative ring.*

Proof. The proof of $(iv) \Rightarrow (i)$, $(iv) \Rightarrow (ii)$ and $(iv) \Rightarrow (iii)$ are trivial.
$(i) \Rightarrow (iv)$ Since h is a non $\mathcal{P}$-trivial $\mathcal{P}$-homoderivation, then there exists $x \in \mathcal{N}$, such that $h(x) \notin \mathcal{P}$, it follows that $\overline{h(x)} \in Z(\mathcal{N}/\mathcal{P}) - \{\overline{0}\}$ and $\overline{h(x + x)} = \overline{h(x)} + \overline{h(x)} \in Z(\mathcal{N}/\mathcal{P})$, thus $(\mathcal{N}/\mathcal{P}, +)$ is abelian by Lemma 2.2 (i). Our assumption gives

$$\overline{h(xy)}(\overline{h(n)} + \bar{n}) = (\overline{h(n)} + \bar{n})\overline{h(xy)} \text{for all} x, y, n \in \mathcal{N} \tag{3.1}$$

Develop (3.1) and using Lemma 2.4 with hypothesis to get $\overline{h(x)}\bar{y}(\overline{h(n)}+\bar{n})+\bar{x}\overline{h(y)}(\overline{h(n)} + \bar{n}) = (\overline{h(n)} + \bar{n})\overline{h(x)}\bar{y} + (\overline{h(n)} + \bar{n})\bar{x}\overline{h(y)}$ for all $x, y, n \in \mathcal{N}$.

Replacing x by $h(x)$ in the last expression, and using the fact that $h(x) \in Z(\mathcal{N}/\mathcal{P})$, we get

$$\overline{h^2(x)}\bar{y}(\overline{h(n)} + \bar{n}) = \overline{h^2(x)}(\overline{h(n)} + \bar{n})\bar{y} \text{ for all } x, y, n \in \mathcal{N}$$

Therefore,

$$\overline{h^2(x)}(\mathcal{N}/\mathcal{P})[(\overline{h(n)} + \bar{n}), \bar{y}] = \{\overline{0}\} \text{ for all } x, y, n \in \mathcal{N}.$$

In light of 3-primeness of $\mathcal{N}/\mathcal{P}$ and Lemma 2.3 (ii), we find $\overline{h(n)} + \bar{n} \in Z(\mathcal{N}/\mathcal{P})$ for all $n \in \mathcal{N}$. It follows that

$$(\overline{h(n)} + \bar{n})(\overline{h(x)} + \bar{x}) = (\overline{h(x)} + \bar{x})(\overline{h(n)} + \bar{n}) \text{ for all } x, n \in \mathcal{N}.$$

While

$$\begin{aligned}(\overline{h(n)} + \bar{n})(\overline{h(x)} + \bar{x}) &= (\overline{h(n)} + \bar{n})\overline{h(x)} + (\overline{h(n)} + \bar{n})\bar{x} \\ &= \overline{h(x)}(\overline{h(n)} + \bar{n}) + \bar{x}(\overline{h(n)} + \bar{n}) \\ &= \overline{h(x)h(n)} + \overline{h(x)}\bar{n} + \bar{x}\overline{h(n)} + \bar{x}\bar{n} \text{ for all } x, n \in \mathcal{N}.\end{aligned}$$

And,

$$\begin{aligned}(\overline{h(x)} + \bar{x})(\overline{h(n)} + \bar{n}) &= (\overline{h(x)} + \bar{x})\overline{h(n)} + (\overline{h(x)} + \bar{x})\bar{n} \\ &= \overline{h(n)}(\overline{h(x)} + \bar{x}) + \bar{n}(\overline{h(x)} + \bar{x}) \\ &= \overline{h(n)h(x)} + \overline{h(n)}\bar{x} + \bar{n}\overline{h(x)} + \bar{n}\bar{x} \text{ for all } x, n \in \mathcal{N}.\end{aligned}$$

By equaling both equations, and since $(\mathcal{N}/\mathcal{P}, +\ )$ is abelian, we conclude that $\mathcal{N}/\mathcal{P}$ is a commutative ring.

$(ii) \Rightarrow (iv)$ Since $h([x, y]) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$, thus

$$\overline{h([x, y])} = \overline{0} \text{ for all } x, y \in \mathcal{N}. \tag{3.2}$$

Replace y by xy in (3.2) to get $\overline{h(x)[x, y]} = \overline{0}$ for all $x, y \in \mathcal{N}$, thus

$$\overline{h(x)}\bar{x}\bar{y} = \overline{h(x)}\bar{y}\bar{x} \text{ for all } x, y \in \mathcal{N}.$$

If we change y by ty in (3.2), we get $\overline{h(x)}\bar{t}\bar{y}\bar{x} = \overline{h(x)}\bar{x}\bar{t}\bar{y} = \overline{h(x)}\bar{t}\bar{x}\bar{y}$ for all $x, y \in \mathcal{N}$, hence we conclude that $\overline{h(x)}(\mathcal{N}/\mathcal{P})[\bar{x} \cdot \bar{y}] = \{\overline{0}\}$ for all $x, y \in \mathcal{N}$, but $\mathcal{N}/\mathcal{P}$ is 3-prime which gives

$$\bar{x} \in Z(\mathcal{N}/\mathcal{P}) \text{ or } \overline{h(x)} = \overline{0} \text{ for all } x \in \mathcal{N}. \tag{3.3}$$

If there is $x_0 \in \mathcal{N}$ such that $\overline{h(x_0)} = \overline{0}$ return to hypothesis, we get $\overline{h(x_0 y)} = \overline{h(yx_0)}$ for all $y \in \mathcal{N}$. Therefore, $\overline{x_0}\overline{h(y)} = \overline{h(y)}\overline{x_0}$ for all $y \in \mathcal{N}$. Which can be written as $(h(x_0) + x_0)h(y) = h(y)(h(x_0) + x_0)$ for all $y \in \mathcal{N}$. Put $h(y)t$ in place of y and using Lemma 2.4, then for all $y, t \in \mathcal{N}$, we have

$$\overline{x_0}\overline{h^2(y)h(t)} + \overline{x_0}\overline{h^2(y)}\bar{t} + \overline{x_0}\overline{h(y)h(t)} = \overline{h^2(y)h(t)}\overline{x_0} + \overline{h^2(y)}\bar{t}\overline{x_0} + \overline{h(y)h(t)}\overline{x_0}.$$

The last equations force that $\overline{h^2(y)}\bar{t}\overline{x_0} = \overline{h^2(y)}\overline{x_0}\bar{t}$ for all $y, t \in \mathcal{N}$. Letting nt instead of t and using it again to get $\overline{h^2(y)}(\mathcal{N}/\mathcal{P})\,[\overline{x_0}, \bar{t}] = \{\overline{0}\}$ for all $y, t \in \mathcal{N}$. Using the 3-primeness of $\mathcal{N}/\mathcal{P}$ and Lemma 2.3(ii), we obtain $\overline{x_0} \in Z(\mathcal{N}/\mathcal{P})$. In view of (3.3), we get $\mathcal{N}/\mathcal{P} \subseteq Z(\mathcal{N}/\mathcal{P})$. Hence, $\mathcal{N}/\mathcal{P}$ is a commutative ring.
$(iii) \Rightarrow (iv)$ Consider that $h([x, y]) - [x, y] \in \mathcal{P}$ for all $x, y \in \mathcal{N}$, that is

$$\overline{h([x, y])} = \overline{[x, y]} \text{ for all } x, y \in \mathcal{N}. \tag{3.4}$$

If we replace y by xy in (3.4) and use it to get $2\overline{h(x)[x, y]} = \overline{0}$ for all $x, y \in \mathcal{N}$, again if we put $[u, v]$ instead of x and use hypothesis, the last expression can be reduced to $2\overline{[u, v][u, v], y]} = \overline{0}$ for all $u, v, y \in \mathcal{N}$, but $\mathcal{N}/\mathcal{P}$ is 2-torsion free, so $\overline{[u, v]}\overline{[[u, v], y]} = \overline{0}$ for all $u, v, y \in \mathcal{N}$. Which can be rewritten as

$$\overline{[u, v][u, v]}\bar{y} = \overline{[u, v]}\bar{y}\overline{[u, v]} \text{ for all } u, v, y \in \mathcal{N}. \tag{3.5}$$

Placing ty instead of y in (3.5) and use it to get $\overline{[u, v]}\bar{t}\overline{[u, v], y]} = \overline{0}$ for all $u, v, y, t \in \mathcal{N}$. i.e. $\overline{[u, v]}(\mathcal{N}/\mathcal{P})\overline{[[u, v], y]} = \overline{0}$ for all $u, v, y \in \mathcal{N}$. But $\mathcal{N}/\mathcal{P}$ is 3-prime, hence, $\overline{[u, v]} \in Z(\mathcal{N}/\mathcal{P})$ for all $u, v \in \mathcal{N}$. So we find that $\overline{[u, uv]} = \bar{u}\overline{[u, v]} \in Z(\mathcal{N}/\mathcal{P})$ for all $u, v \in \mathcal{N}$. By Lemma 2.2(ii), we get $\bar{u} \in Z(\mathcal{N}/\mathcal{P})$ for all $u \in \mathcal{N}$. So, $\mathcal{N}/\mathcal{P}$ is a commutative ring.

□

Corollary 3.2 [6, Theorem 3.7] *Let N be 2-torsion free, 3-prime near-ring, if h is a non-zero homoderivation h of $\mathcal{N}$, then the following assertions are equivalent:*

(i) $h(\mathcal{N}) \subseteq Z(\mathcal{N})$;
(ii) $h([x, y]) = 0$ for all $x, y \in \mathcal{N}$;
(iii) $h([x, y]) = [x, y]$ for all $x, y \in \mathcal{N}$;
(iv) $\mathcal{N}$ is a commutative ring.

Theorem 3.3 *Let $\mathcal{P}$ is 3-prime ideal of a near-ring $\mathcal{N}$ with $\mathcal{N}/\mathcal{P}$ is 2-torsion free. Then there is no non $\mathcal{P}$-trivial $\mathcal{P}$-homoderivation h of $\mathcal{N}$ that satisfies any one of the following claims:*

(i) $h(x \circ y) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$
(ii) $h(x \circ y) - (x \circ y) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$.
(iii) $h([x, y]) - (x \circ y) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$.
(vi) $h(x \circ y) - [x, y] \in \mathcal{P}$ for all $x, y \in \mathcal{N}$.

Proof. (i) Suppose that $h(x \circ y) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$. So $\overline{h(x \circ y)} = \overline{0}$ for all $x, y \in \mathcal{N}$. Also, $\overline{h(x \circ xy)} = \overline{0}$ for all $x, y \in \mathcal{N}$. It follows $\overline{h(x(x \circ y))} = \overline{0}$ for all $x, y \in \mathcal{N}$. Applying the definition of h and using hypothesis, the last expression can be reduced to $\overline{h(x)(x \circ y)} = \overline{0}$ for all $x, y \in \mathcal{N}$. Thus

$$\overline{h(x)}\bar{y}\bar{x} = -\overline{h(x)}\bar{x}\bar{y} \text{ for all } x, y \in \mathcal{N}. \tag{3.6}$$

If we change y by ty in (3.6), we get $\overline{h(x)}\bar{t}\bar{y}\bar{x} = -\overline{h(x)}\bar{x}\bar{t}\bar{y} = \overline{h(x)}\bar{x}\overline{t(-y)} = \overline{h(x)t(-x)(-y)}$ for all $x, y \in \mathcal{N}$. Hence, we conclude that $\overline{h(x)}(\mathcal{N}/\mathcal{P})[-\bar{x} \cdot \bar{y}] = \overline{0}$ for all $x, y \in \mathcal{N}$, if we replace x by $-x$ then last equation will be in the form $\overline{(h(-x))}(\mathcal{N}/\mathcal{P})[\bar{x} \cdot \bar{y}] = \{\overline{0}\}$ for all $x, y \in \mathcal{N}$ but $\mathcal{N}/\mathcal{P}$ is 3-prime which gives

$$\bar{x} \in Z(\mathcal{N}/\mathcal{P}) \text{ or } \overline{h(x)} = \overline{0} \text{ for all } x \in \mathcal{N} \tag{3.7}$$

If there is $x_0 \in \mathcal{N}$ such that $\overline{x_0} \in Z(\mathcal{N}/\mathcal{P})$ return to hypothesis, we get $2\overline{h(x_0y)} = \overline{0}$ for all $y \in \mathcal{N}$. Therefore, $\overline{h(x_0y)} = \overline{0}$ for all $y \in \mathcal{N}$. Thus, $\overline{h(x_0)h(y)} + \overline{h(x_0)}\bar{y} + \overline{x_0}\overline{h(y)} = \overline{0}$ for all $y \in \mathcal{N}$, replace y by $n \circ m$ to get $\overline{h(x_0)(n \circ m)} = \overline{0}$ for all $n, m \in \mathcal{N}$. That is
$\overline{h(x_0)}\overline{nm} = -\overline{h(x_0)}\overline{mn}$ for all $n, m \in \mathcal{N}$. Put tn in place of n, in the last expression and use it to get

$$\begin{aligned}
\overline{h(x_0)}\overline{tnm} &= -\overline{h(x_0)}\overline{mtn} \\
&= \overline{h(x_0)mt(-n)} \\
&= -\overline{h(x_0)tm(-n)} \\
&= -\overline{h(x_0)t(-m)n} \text{ for all } n, m \in \mathcal{N}.
\end{aligned}$$

Then, $\overline{h(x_0)}\bar{t}(\overline{(-m)n} - \overline{n(-m)}) = \overline{0}$, i.e $\overline{h(x_0)}(\mathcal{N}/\mathcal{P})(\overline{(-m)n} - \overline{n(-m)}) = \{\overline{0}\}$ and we obtain $\overline{h(x_0)} = \overline{0}$ or $\overline{-m} \in Z(\mathcal{N}/\mathcal{P})$ for all $m \in \mathcal{N}$, hence from (3.7), we arrive at $\mathcal{N}/\mathcal{P}$ is a commutative ring, because h is non $\mathcal{P}$-trivial.

Now, returning to our assumption with the fact that $\mathcal{N}/\mathcal{P}$ is a commutative ring and 2-torsion freeness of $\mathcal{N}/\mathcal{P}$, we get $\overline{h(xy)} = \overline{0}$ for all $x, y \in \mathcal{N}$. Substituting xu instead of x in the last equation, we get $\bar{x}(\mathcal{N}/\mathcal{P})\overline{h(y)} = \{\overline{0}\}$ for all $x, y \in \mathcal{N}$, and 3-primality of $\mathcal{N}/\mathcal{P}$ implies that h is $\mathcal{P}$-trivial; a contradiction.

(ii) Suppose that $h(x \circ y) - (x \circ y) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$. So $\overline{h(x \circ y)} = \overline{(x \circ y)}$ for all $x, y \in \mathcal{N}$. Also, $\overline{h(x \circ xy)} = \overline{(x \circ xy)}$ for all $x, y \in \mathcal{N}$. It follows that $\overline{h(x(x \circ y))} = \overline{x(x \circ y)}$ for all $x, y \in \mathcal{N}$. Using the definition of h and the hypothesis, the last expression can be reduced to $2\overline{h(x)(x \circ y)} = \overline{0}$ for all $x, y \in \mathcal{N}$. Via 2-torsion free of $\mathcal{N}/\mathcal{P}$ we get $\overline{h(x)(x \circ y)} = \overline{0}$ for all $x, y \in \mathcal{N}$. For $x = u \circ v$, where $u, v \in \mathcal{N}$, we get $\overline{(u \circ v)((u \circ v) \circ y)} = \overline{0}$ for all $u, v, y \in \mathcal{N}$. Which can be rewritten as

$$\overline{(u \circ v)}\bar{y}\overline{(u \circ v)} = -\overline{(u \circ v)(u \circ v)}\bar{y} \text{ for all } u, v, y \in \mathcal{N} \tag{3.8}$$

Replace y by ty in (3.8) and using it again to get

$$\begin{aligned}\overline{(u\circ v)}\bar{t}\bar{y}\overline{(u\circ v)} &= -\overline{(u\circ v)(u\circ v)}\bar{t}\bar{y}\\ &= \overline{(u\circ v)(u\circ v)\bar{t}(-y)}\\ &= (-\overline{(u\circ v)}\overline{\bar{t}(u\circ v)(-y)}\\ &= \overline{(u\circ v)\bar{t}(-(u\circ v))(-y)}\\ &= -\overline{(u\circ v)}\overline{\bar{t}(-(u\circ v))}\bar{y} \text{ for all } u,v,y,t\in\mathcal{N}.\end{aligned}$$

Therefore, $\overline{(u\circ v)}\bar{t}[\overline{(-(u\circ v))},\bar{y}]=\overline{0}$. That is $\overline{(u\circ v)}(\mathcal{N}/\mathcal{P})[\overline{(-(u\circ v))},\bar{y}]=\{\overline{0}\}$ for all $u,v,y\in\mathcal{N}$, 3-primeness of $(\mathcal{N}/\mathcal{P})$ forces $-\overline{(u\circ v)}\in Z(\mathcal{N}/\mathcal{P})$ for all $u,v\in\mathcal{N}$. It means that $-\overline{(u\circ uv)}=\bar{u}(\overline{(u\circ v)})\in Z(\mathcal{N}/\mathcal{P})$ for all $u,v\in\mathcal{N}$. By Lemma 2.2(ii), we arrive at

$$\overline{(u\circ v)}=\overline{0} \text{ or } \bar{u}\in Z(\mathcal{N}/\mathcal{P}) \text{ for all } u,v\in\mathcal{N}. \tag{3.9}$$

If there is an element $u_0\in\mathcal{N}$ such that $\overline{u_0\circ v}=\overline{0}$ for all $v\in\mathcal{N}$, which means that $\bar{v}\overline{u_0}=-\overline{u_0}\bar{v}$ for all $v\in\mathcal{N}$, for $v=tv$, where $t\in\mathcal{N}$, we get

$$\begin{aligned}\overline{tv}\bar{u}_0 &= -\overline{u_0}\overline{tv}\\ &= \overline{u_0}\bar{t}(-\bar{v})\\ &= \left(-\overline{tu_0}\right)(-\bar{v})\\ &= -\bar{t}\overline{(-u_0)}\bar{v} \text{ for all } v,t\in\mathcal{N}.\end{aligned}$$

Hence, $\bar{t}\overline{[(-u_0)},\bar{v}\Big]=\overline{0}$ for all $v,t\in\mathcal{N}$, we find that $-\overline{u_0}\in Z(\mathcal{N}/\mathcal{P})$, return to (3.9), we conclude that $\mathcal{N}/\mathcal{P}$ is a commutative ring. The last result with the fact $\overline{h(x\circ y)}=\overline{(x\circ y)}$ for all $x,y\in\mathcal{N}$, implies that $\overline{h(xy)}=\overline{xy}$ for all $x,y\in\mathcal{N}$. Putting xu instead of x in latest equation we obtain $2\bar{x}\bar{u}\overline{h(y)}=\overline{0}$ for all $x,y,u\in\mathcal{N}$, by 2-torsion freeness of $\mathcal{N}/\mathcal{P}$, we get $\bar{x}(\mathcal{N}/\mathcal{P})\overline{h(y)}=\{\overline{0}\}$ for all $x,y\in\mathcal{N}$, 3-primeness of $\mathcal{N}/\mathcal{P}$ implies that h is a $\mathcal{P}$-trivial; a contradiction.

(*iii*) Suppose that $h([x,y])-(x\circ y)\in\mathcal{P}$ for all $x,y\in\mathcal{N}$. For $y=x$ we find that $\overline{x^2}=\overline{0}$ for all $x\in\mathcal{N}$. Hence, $\overline{x(x+y)^2}=\overline{0}$ for all $x,y\in\mathcal{N}$. This implies $\overline{x}(\mathcal{N}/\mathcal{P})\overline{x}=\{\overline{0}\}$ for all $x\in\mathcal{N}$ and using 3-primeness of $\mathcal{N}/\mathcal{P}$ leads to a contradiction.

(*iv*) we have $\overline{h(x\circ y)}=\overline{[x,y]}$ for all $x,y\in\mathcal{N}$

So, $\overline{h(x\circ y)}=\overline{[yx,y]}$ for all $x,y\in\mathcal{N}$ and then $\overline{h(y(x\circ y))}=\overline{y([x,y])}$ for all $x,y\in\mathcal{N}$ and developing last equation implies

$$\overline{h(y)(x\circ y)}+\overline{h(y)[x,y]}=\overline{0} \text{ for all } x,y\in\mathcal{N} \tag{3.10}$$

For $x=y$ in (3.10) and use 2-torsion freeness of $\mathcal{N}/\mathcal{P}$ to get $\overline{h(y)y^2}=\overline{0}$ for all $y\in\mathcal{N}$, now if we replace x by yx in (3.10) and develop it with the

fact $\overline{h(y)y^2} = \overline{0}$ for all $y \in \mathcal{N}$ and 2-torsion freeness of $\mathcal{N}/\mathcal{P}$ we will get $\overline{h(y)yxy} = \overline{0}$ for all $x, y \in \mathcal{N}$, that is $\overline{h(y)y}(\mathcal{N}/\mathcal{P})\bar{y} = \{\overline{0}\}$ for all $y \in \mathcal{N}$, thus $\overline{h(y)y} = \overline{0}$ for all $y \in \mathcal{N}$ because 3-primeness of $\mathcal{N}/\mathcal{P}$. Applying the last result on (3.10) together with 2-torsion freeness of $\mathcal{N}/\mathcal{P}$ implies last $\overline{h(y)}(\mathcal{N}/\mathcal{P})\bar{y} = \{\overline{0}\}$ for all $y \in \mathcal{N}$ and using 3-primeness of $\mathcal{N}/\mathcal{P}$ leads to a contradiction because h is non $\mathcal{P}$-trivial.

□

Corollary 3.4 [6, Theorem 3.8] *Let $\mathcal{N}$ be 2-torsion 3-prime near-ring. Then there is no nonzero homoderivation h that satisfies any one of the following claims:*

(i) $h(x \circ y) = 0$ *for all* $x, y \in \mathcal{N}$,
(ii) $h(x \circ y) = (x \circ y)$ *for all* $x, y \in \mathcal{N}$,
(iii) $h([x, y]) = (x \circ y)$ *for all* $x, y \in \mathcal{N}$.

Corollary 3.5 *Let $\mathcal{N}$ be 2-torsion 3-prime near-ring. Then there is no nonzero homoderivation h that satisfies $h(x \circ y) = [x, y]$ for all $x, y \in \mathcal{N}$.*

4 $\mathcal{P}$-homoderivation with right Jordan ideal

Theorem 4.1 *Let $\mathcal{P}$ be 3-prime ideal of a near-ring $\mathcal{N}$ and $\mathcal{J}$ be a Jordan right ideal of $\mathcal{N}$ with $\mathcal{J} \nsubseteq \mathcal{P}$. If h is a non $\mathcal{P}$-trivial $\mathcal{P}$-homoderivation of $\mathcal{N}$ s.t $h(\mathcal{J}) \subseteq \mathcal{P}$, then $\bar{u}\bar{v} = \bar{v}\bar{u}$ and $\bar{u} + \bar{v} = \bar{v} + \bar{u}$ for all $u, v \in \mathcal{J}$.*

Proof. We have $\overline{h(u)} = \overline{0}$ for all $u \in \mathcal{J}$, then $\overline{h(u \circ n)} = \overline{0}$ for all $u \in \mathcal{J}$, $n \in N$. Hence, $\bar{u}\overline{h(n)} + \overline{h(n)}\bar{u} = \overline{0}$ for all $u \in \mathcal{J}$, $n \in \mathcal{N}$ and this leads to the following equation

$$\overline{h(n)}\bar{u} = -\bar{u}\overline{h(n)} \text{for all} u \in \mathcal{J}, n \in \mathcal{N}. \tag{4.1}$$

Replace n by nv, where $v \in \mathcal{J}$, in (4.1) and using it again to get

$$\begin{aligned}\overline{h(n)}\bar{v}\bar{u} &= -\bar{u}\overline{h(n)}\bar{v} \\ &= \bar{u}\overline{h(n)}(-\bar{v}) \\ &= \overline{h(n)}(-\bar{u})(-\bar{v}) \text{ for all } u, v \in \mathcal{J}, n \in \mathcal{N},\end{aligned}$$

hence we get

$$\overline{h(n)}[(-\bar{u}), \bar{v}] = \overline{0} \text{ for all } u, v \in \mathcal{J}, n \in \mathcal{N}.$$

Since $\overline{(h([-u, v]))} = \overline{0}$for all $u, v \in \mathcal{J}$, hence we can use Lemma 2.4, when we replace n by nm in the last expression and use the definition of h to get $\overline{h(n)}\bar{m}[(-\bar{u}), \bar{v}] = \overline{0}$ for all $u, v \in \mathcal{J}$, $n, m \in \mathcal{N}$, it follows $\overline{h(n)}(\mathcal{N}/\mathcal{P})[(-\bar{u}), \bar{v}] = \{\overline{0}\}$ for all $u, v \in \mathcal{J}$, $n \in \mathcal{N}$, since h is non $\mathcal{P}$-trivial

and $\mathcal{N}/\mathcal{P}$ is 3-prime we conclude that $[(-\bar{u}), \bar{v}] = \overline{0}$ for all $u, v \in \mathcal{J}$, putting $-u$ instead of u in the last equation we get $[\bar{u}, \bar{v}] = \overline{0}$. Then

$$\begin{aligned}(\bar{u}+\bar{v})(\bar{k}+\bar{k}) &= (\bar{u}+\bar{v})\bar{k} + (\bar{u}+\bar{v})\bar{k} \\ &= \bar{u}\bar{k} + v\bar{k} + \bar{u}\bar{k} + \bar{v}\bar{k} \text{ for all } i, j, k \in \mathcal{J}.\end{aligned}$$

Also,

$$\begin{aligned}(\bar{k}+\bar{k})(\bar{u}+\bar{v}) &= (\bar{k}+\bar{k})\bar{u} + (\bar{k}+\bar{k})\bar{v} \\ &= \bar{k}\bar{u} + \bar{k}\bar{u} + \bar{k}\bar{v} + \bar{k}\bar{v} \\ &= \bar{u}\bar{k} + \bar{u}\bar{k} + \bar{v}\bar{k} + \bar{v}\bar{k} \text{ for all } i, j, k \in \mathcal{J}.\end{aligned}$$

Equalizing the last equations, we get $((\bar{u}+\bar{v})-(\bar{v}+\bar{u}))\bar{k} = \overline{0}$ for all $u, v, k \in \mathcal{J}$. Replacing k by $k \circ n$ in the last expression we get $((\bar{u}+\bar{v})-(\bar{v}+\bar{u}))\bar{n}\bar{k} = \overline{0}$ for all $u, v, k \in J, n \in \mathcal{N}$. Using the fact $\mathcal{J} \nsubseteq \mathcal{P}$ and the 3-primeness of $\mathcal{N}/\mathcal{P}$ implies that $\bar{u} + \bar{v} = \bar{v} + \bar{u}$ for all $u, v \in \mathcal{J}$.

□

Corollary 4.2 [6, Theorem 4.9] *Let $\mathcal{J}$ be a nonzero Jordan right ideal of 3-prime near-ring $\mathcal{N}$. If h is a nonzero homoderivation of $\mathcal{N}$ such that $h(\mathcal{J}) = \{0\}$, then $\mathcal{J}$ is commutative.*

Theorem 4.3 *Let $\mathcal{P}$ be 3-prime ideal of a near-ring $\mathcal{N}$ and $\mathcal{J}$ be a Jordan right ideal of $\mathcal{N}$ with $\mathcal{J} \nsubseteq \mathcal{P}$, and $\mathcal{N}/\mathcal{P}$ is 2-torsion free. Then there is no non $\mathcal{P}$-trivial P-homoderivation h satisfying any one of the following claims:*

(i) $\overline{h(u)} = \bar{u}$ *for all* $u \in \mathcal{J}$,
(ii) $\overline{h(uv)} = \overline{uv}$ *for all* $u, v \in \mathcal{J}$.

Proof. *(i)* We have $\overline{h(u \circ un)} = \overline{u \circ un}$ for all $u \in J$, $n \in \mathcal{N}$, it follows that $\overline{h(u(u \circ n))} = \bar{u}\overline{(u \circ n)}$ for all $u \in \mathcal{J}$, $n \in \mathcal{N}$ and using the definition of h leads to $2\bar{u}\overline{(u \circ n)} = \overline{0}$ for all $u \in J$, $n \in \mathcal{N}$, which implies $\bar{u}\overline{(u \circ n)} = \overline{0}$ for all $u \in \mathcal{J}$, $n \in \mathcal{N}$, since $\mathcal{N}/\mathcal{P}$ is 2-torsion free. Therefore,

$$\bar{u}\bar{n}\bar{u} = -\overline{u^2}\bar{n} \text{ for all } u \in \mathcal{J}, n \in \mathcal{N}. \tag{4.2}$$

Replace n by nm in (4.2) and using it to get

$$\begin{aligned}\bar{u}\bar{n}\bar{m}\bar{u} &= -\overline{u^2}\bar{n}\bar{m} \\ &= \overline{u^2}\bar{n}(-\bar{m}) \\ &= \bar{u}\bar{n}\overline{(-u)(-m)} \text{ for all } u \in \mathcal{J}, n, m \in \mathcal{N}.\end{aligned}$$

This means that $-\bar{u}\bar{n}\bar{m}\overline{(-u)} = -\bar{u}\bar{n}\overline{(-u)}\bar{m}$ for all $u \in \mathcal{J}$, $n, m \in \mathcal{N}$, which can be written as $\bar{u}\bar{n}\bar{m}\overline{(-u)} = \bar{u}\bar{n}\overline{(-u)}\bar{m}$ for all $u \in \mathcal{J}$, $n, m \in \mathcal{N}$, substituting $-u$ instead of u in the last equation implies $\overline{(-u)}(\mathcal{N}/\mathcal{P})[\bar{u}, \bar{m}] = \{\overline{0}\}$ for all $u \in \mathcal{J}$, $m \in \mathcal{N}$, by 3-primeness of $\mathcal{N}/\mathcal{P}$ we get $\bar{u} \in Z(\mathcal{N}/\mathcal{P})$ for all $u \in \mathcal{J}$. Returning to (4.2) and using the fact $\bar{u} \in Z(\mathcal{N}/\mathcal{P})$ for all $u \in \mathcal{J}$

and 2-torsion freeness of $\mathcal{N}/\mathcal{P}$, we get $\bar{u}\bar{n}\bar{u} = \overline{0}$ for all $u \in \mathcal{J}$, $n \in \mathcal{N}$, that is $\bar{u}(\mathcal{N}/\mathcal{P})\bar{u} = \{\overline{0}\}$ for all $u \in \mathcal{J}$. The 3-primeness of $\mathcal{N}/\mathcal{P}$ leads to a contradiction, because $\bar{J} \neq \{\overline{0}\}$.

(ii) Assume that $\overline{h(uv)} = \overline{uv}$ for all $u, v \in \mathcal{J}$. It follows that

$$\overline{h(u)h(v)} + \overline{h(u)}\bar{v} + \bar{u}\overline{h(v)} = \bar{u}\bar{v} \text{ for all } u, v \in \mathcal{J}. \tag{4.3}$$

Substitute v by $2v^2$ in (4.3) then use it once again beside 2-torsion freeness of $\mathcal{N}/\mathcal{P}$ to get

$$\overline{h(u)v^2} = \overline{0} \text{ for all } u, v \in \mathcal{J}. \tag{4.4}$$

Replacing v by $v \circ vn = v(v \circ n)$ in (4.3) and using the hypothesis, we get $2\overline{h(u)}\bar{v}\overline{(v \circ n)} = \overline{0}$ for all $u, v \in \mathcal{J}$, $n \in \mathcal{N}$, so $\overline{h(u)}\bar{v}\overline{(v \circ n)} = \overline{0}$ for all $u, v \in \mathcal{J}$, $n \in \mathcal{N}$, because $\mathcal{N}/\mathcal{P}$ is 2-torsion free. So

$$\overline{h(u)}\bar{v}\bar{n}\bar{v} = -\overline{h(u)v^2}\bar{n} \text{ for all } u, v \in \mathcal{J}, n \in \mathcal{N} \tag{4.5}$$

Using (4.4) in (4.5) we get $\overline{h(u)}\bar{v}(\mathcal{N}/\mathcal{P})\bar{v} = \{\overline{0}\}$ for all $u, v \in \mathcal{J}$. Since $\mathcal{N}/\mathcal{P}$ is 3-prime, we arrive at $\overline{h(u)}\bar{v} = \overline{0}$ for all $u, v \in \mathcal{J}$. Replacing v by $v \circ n$, where, $n \in \mathcal{N}$, in last equation, ensure that $\overline{h(u)}(\mathcal{N}/\mathcal{P})\bar{v} = \{\overline{0}\}$ for all $u, v \in \mathcal{J}$, again 3-primeness of $\mathcal{N}/\mathcal{P}$ implies $\overline{h(u)} = \overline{0}$ for all $u \in J$ and using the last result with hypothesis follow $\bar{u}\bar{v} = \overline{0}$ for all $u, v \in \mathcal{J}$. i.e. $\overline{0} = \bar{u}\overline{(v \circ n)} = \bar{u}\bar{n}\bar{v}$ for all $u, v \in \mathcal{J}$, $n \in \mathcal{N}$, but $\mathcal{N}/\mathcal{P}$ is 3-prime which forces a contradiction because $\bar{J} \neq \{\overline{0}\}$.

□

Corollary 4.4 [6, Theorem 4.10] *Let $\mathcal{J}$ be a nonzero Jordan right ideal of a 2-torsion free 3-prime near-ring, then There is no nonzero homoderivation h that satisfies any one of the following claim:*

(i) $h(J) = J$.
(ii) $h(uv) = uv$ *for all* $u, v \in J$.

The following example proves that the 3-primeness of $\mathcal{P}$ in all Theorems can not be omitted.

Example 4.1 Let $\mathcal{K}$ be a 2-torsion free near-ring. Define $\mathcal{N}$, $\mathcal{P}$, $\mathcal{J}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & u & v \\ (0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid u, v, 0 \in \mathcal{K} \right\}, \mathcal{P} = \left\{ \begin{pmatrix} 0 & 0 & s \\ (0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid s, 0 \in \mathcal{K} \right\},$$

$$\mathcal{J} = \left\{ \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid t, 0 \in \mathcal{K} \right\}$$

It is easily to show that $\mathcal{N}$ is a near-ring, $\mathcal{P}$ is a symmetric ideal of $\mathcal{N}$ not 3-prime and $\mathcal{J}$ is Jordan ideal of $\mathcal{N}$ with $\mathcal{J} \nsubseteq \mathcal{P}$. $h : \mathcal{N} \to \mathcal{N}$ defined by:

$$h\begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly h is a non $\mathcal{P}$-trivial $\mathcal{P}$-homoderivation on $\mathcal{N}$ and satisfies all the conditions below:

(i) $\overline{h(\mathcal{N})} \subseteq Z(\mathcal{N}/\mathcal{P})$,
(ii) $h([A, B]) \in \mathcal{P}$,
(iii) $h([A, B]) - [A, B] \in \mathcal{P}$ for all $A, B \in \mathcal{N}$. However $\mathcal{N}/\mathcal{P}$ is not commutative ring.

Also, h satisfies

(iv) $h(A \circ B) - (A \circ B) \in \mathcal{P}$,
(v) $h([A, B]) - (A \circ B) \in \mathcal{P}$, Where h is a non $\mathcal{P}$-trivial.

Moreover, h is a non P-trivial that satisfies: $\overline{h(A)} = \bar{A}$ for all $A \in \mathcal{J}$ and $\overline{h(AB)} = \overline{AB}$ for all $A, B \in \mathcal{J}$.

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