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On astheno-Kähler manifolds

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Abstract This survey explores a range of classical findings and recent developments related to our understanding of astheno-Kähler manifolds. Furthermore, we provide various examples of astheno-Kähler manifolds and analyze the challenges associated with their existence.

Keywords complex manifolds · astheno-Kähler metrics · cohomology · strong KT metric · Gauduchon metrics

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1 Introduction

During the last 30 years, astheno-Kähler manifolds have attracted significant interest among geometers worldwide, showing potential relevance in both differential and algebraic contexts. The first four sections present a brief overview of recent advances in the study of astheno-Kähler manifolds, accompanied by examples and counterexamples. The final section delves into specific details regarding the estimates of equations and their solutions within the astheno-Kähler manifolds framework.

2 Preliminaries

An almost complex structure on a real differentiable manifold M of complex dimension m is a tensor field J which at each point $x \in M$ is an endomorphism of the tangent space T_xM such that $J^2 = -I$, where I denotes the identity transformation of T_xM. A manifold with such a structure is called an almost complex manifold. An almost complex structure is called a complex structure if and only if J has no torsion [49].

Definition 2.1 [29] *Let (M,J) be an almost complex manifold, then a vector-valued 2-form $N(J,J)$ associated with J is given by*

$$N(J,J)(X,Y) = -[JX,JY] + J[X,JY] + J[JX,Y] + [X,Y],$$

where X and Y are vector fields. This form is called the torsion tensor, or the Nijenhuis tensor of the almost-complex structure J. J is said to be integrable if $N = 0$, then J is a complex structure and (M,J) is a complex manifold.

An almost complex manifold M [1] together with a compatible Riemannian metric g, that is, $g(JX,JY) = g(X,Y)$ for all vector fields X, Y on M, is called an almost Hermitian manifold and metric g is called almost Hermitian metric. A complex manifold M [1] together with a compatible Riemannian metric g is called a Hermitian manifold and metric g is called a Hermitian metric. The alternating 2-form

$$\Omega(X,Y) = g(JX,Y)$$

is called the associated Kähler form. We can retrieve g from Ω

$$g(X,Y) = \Omega(X,JY).$$

If Ω is closed, then (M,g) is known as a Kähler manifold and g is a Kähler metric.

A quasi-Kähler structure is an almost Hermitian structure whose Kähler form Ω satisfies

$$(d\Omega)^{(1,2)} = \bar{\partial}\Omega = 0.$$

The term astheno-Kähler manifold (astheno is a Greek word, which means weak) was first coined in 1993 by Jost and Yau in the remarkable paper [22, 23].

Definition 2.2 [22,23] *A Hermitian manifold (M,J,g) of complex dimension m is called an astheno-Kähler manifold if it carries a fundamental 2-form (Kähler form) Ω satisfying*

$$\partial\bar{\partial}\Omega^{m-2} = 0,$$

where ∂ and $\bar{\partial}$ are complex exterior differentials and $\Omega^{m-2} = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{m-2 \text{ times}}$.

The above condition is automatically satisfied for $m = 2$. Thus, any Hermitian metric on a complex surface is astheno-Kähler.

Definition 2.3 [32] *Let ∇ be a connection on a smooth manifold M. The torsion tensor T is a $(1,2)$-tensor (alternatively, a vector-valued 2-form) defined as:*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

where, X,Y are vector fields,$\nabla_X Y$ denotes the covariant derivative of Y along X and $[X,Y]$ is the Lie bracket of X and Y.

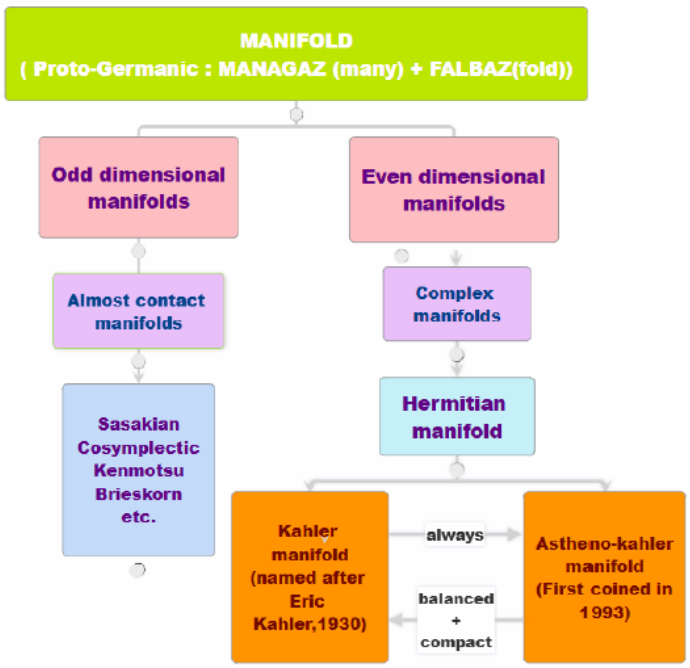

Fig. 1: flow chart

For a connection ∇ with torsion T, a torsion 1-form τ can be defined in terms of certain contraction:

$$\tau(X) = \text{trace}(Y \mapsto T(X, Y))$$

This contraction gives a scalar-valued measure of the torsion in the direction of X.
The torsion 1-form is related to the torsion of a connection on a manifold, which quantifies how the connection deviates from being flat. It characterizes how vectors are transported with respect to that connection.

Definition 2.4 [42] *Let M be an m-dimensional Hermitian manifold. The Hermitian structure (J, g) is called strong Kähler with torsion (strong KT or SKT) and g is called strong KT metric or SKT metric if $\partial\bar{\partial}\Omega = 0$.*

For $m = 3$, astheno-Kähler structure means a strong KT metric.

Definition 2.5 *Let a Hermitian manifold (M, J, g) of complex dimension n carry a fundamental 2-form (Kähler form) Ω defined as $\Omega(X,Y) = g(JX,Y)$. The Lee form, θ, is the unique 1-form defined by:*

$$\theta = -\frac{1}{n-1}\delta\Omega,$$

where δ is the codifferential operator (dual to the exterior derivative d). In this context, the Lee form quantifies the extent to which the almost Hermitian structure deviates from being Hermitian or Kähler.

Definition 2.6 [26] *Let (M, g, J) be a Hermitian manifold. The Hermitian connection is a unique connection ∇ on M that satisfies $\nabla g = 0$ (metric compatibility) and $\nabla J = 0$ (compatibility with the complex structure). The curvature of the Hermitian connection is encoded in the Hermitian curvature tensor R^{∇}. A Hermitian manifold is said to be Hermitian flat if $R^{\nabla} = 0$, where*

$$R^{\nabla}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Definition 2.7 [18] *Let M be an m-dimensional Hermitian manifold, then the Hermitian structure (J, g) is called standard or Gauduchon and g is called Gauduchon metric if $\partial\bar{\partial}\Omega^{m-1} = 0$. Equivalently, if the Lee form of Ω^{m-1} is co-closed.*

Fino et al. [13] mentioned if $m = 4$, a Hermitian metric that is simultaneously strong KT and astheno-Kähler metric, it must also be Gauduchon.

Fu et al. [16] introduced and investigated the generalization of Gauduchon metrics called the k-th Gauduchon.

Definition 2.8 *Let M be an m-dimensional Hermitian manifold and k be an integer such that $1 \leq k \leq m-1$, then the Hermitian structure (J, g) is called k-th Gauduchon and g is called k-th Gauduchon metric if*

$$\partial\bar{\partial}\Omega^k \wedge \Omega^{m-k-1} = 0.$$

Definition 2.9 [32] *A Hermitian manifold is said to be balanced if the torsion 1-form of its Hermitian connection vanishes everywhere or $d\Omega^{m-1} = 0$. Equivalently, the Hermitian structure is said to be balanced if its Lee form vanishes.*

Now, it is natural to ask, when astheno-Kähler manifold becomes Kähler manifold?
Matsuo and Takahashi [32] gave a significant result and showed that every compact balanced astheno-Kähler manifold is Kähler. In particular, compact Hermitian-flat astheno-Kähler manifolds are Kähler.

Definition 2.10 [24] *An almost Hermitian metric Ω on (M^{2m}, J, ω), where $m \geq 3$ is said to be conformally balanced if it satisfies $d(\|\omega\|_{\Omega}\Omega^{m-1}) = 0$, where ω is a nowhere vanishing holomorphic $(m, 0)$- form admitted by M, which means that ω satisfies $\bar{\partial}\omega = 0$. Then (M, J, ω, Ω) is said to be an almost Calabi-Yau manifold with torsion.*

Definition 2.11 [31] *Let M and N be Riemannian manifolds, a smooth map $f : M \to N$ is harmonic if it is a critical point of the energy functional*

$$\mathbf{E}(f) = \frac{1}{2}\int_M \| \mathrm{d}f\|^2 \, \mathrm{d}v.$$

Equivalently, f solves the corresponding Euler-Lagrange equation $\Delta f = 0$, where the Laplacian Δf (trace of the Hessain $\nabla^2 f$), more commonly denoted by $\tau(f)$ and called the tension field, is a nonlinear operator generalizing the Riemannian Laplacian. We call $\pm$holomorphic a map that is holomorphic or antiholomorphic.

Definition 2.12 [31] *A smooth map $f : M \to N$, where M is a complex manifold, is called pluriharmonic if for every 1-dimensional complex submanifold $C \subseteq M$, the restriction $f|_C : C \to N$ is harmonic.*
Pluriharmonicity can alternatively be defined by :

$$f \text{ pluriharmonic} \quad \Leftrightarrow \quad \mathrm{d}\mathrm{d}^{\mathrm{c}} f = 0.$$

We also have the further characterizations:

$$f \text{ pluriharmonic} \quad \Leftrightarrow \quad \bar{\partial}\mathrm{d}^{1,0} f = 0 \quad \Leftrightarrow \quad \partial \mathrm{d}^{0,1} f = 0 \quad \Leftrightarrow \quad \left(\nabla^2 f\right)^{1,1} = 0.$$

Remark 2.1 We have naturally written $\mathrm{d} = \mathrm{d}^{1,0} + \mathrm{d}^{0,1} = \partial + \bar{\partial}$. To make sense of the Hessian in the last one, choose any compatible Riemannian metric on M. Due to this characterization, pluriharmonic maps have sometimes been called $(1,1)$-geodesic maps.

Definition 2.13 [20] *An n-dimensional vector bundle is a continuous function $p : E \to B$ together with a real vector space structure on $p^{-1}(b)$ for each $b \in B$, such that the following local triviality condition is satisfied: There is a cover of B by open sets U_α for each of which there exists a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ taking $p^{-1}(b)$ to $\{b\} \times \mathbb{R}^n$ by a vector space isomorphism for each $b \in U_\alpha$. This h_α is called a local trivialization of the vector bundle. The space B is called the base space (typically a manifold), E is the total space which is a topological space, and the vector spaces $p^{-1}(b)$ are the fibers.*

2.1 Some known results on astheno-Kähler manifold

Here, we list the results based on the astheno-Kähler metric.
Lemma 2.14 [22, Lemma 6] *Let (M, Ω) be a compact astheno-Kähler manifold. Then, every holomorphic 1-form on M is closed.*

The converse of the above lemma is not true. A counterexample of manifold having no astheno-Kähler metrics with closed holomorphic 1-form is given in [13] which is, let M be a compact quotient $M = \Gamma\backslash G$ of a simply-connected Lie group G by a uniform discrete subgroup Γ, endowed with an invariant

complex structure J and having no invariant astheno-Kähler metrics but any holomorphic 1-form on M is closed.

Jost and Yau gave the following results:

Lemma 2.15 [22, Lemma 7] *Let N be a compact locally Hermitian symmetric space of noncompact type, and assume that the universal cover of N does not have the upper half plane $\mathcal{H}$ as a global factor. Let M be a compact astheno-Kähler manifold of dimension m. Let $f : M \to N$ be Hermitian harmonic. Then f is pluriharmonic. If f has real rank $2\dim_{\mathbb{C}} N$ at some point, then f is $\pm$ holomorphic.*

Theorem 2.16 [22, Theorem 6] *Let N be a compact locally Hermitian symmetric space of noncompact type, without the upper half-plane $\mathcal{H}$ as a global factor of its universal cover, or let N be a compact strongly negatively curved Kähler manifold. Let M be a compact astheno-Kähler manifold. If M is homotopy equivalent to N, then M is $\pm$ biholomorphically equivalent to N.*

Remark 2.2 The above theorem is the extension of Siu's rigidity theorem [41, Theorem 2].

Fino and Tomassini also find the condition when astheno-Kähler becomes Kähler.

Theorem 2.17 [13, Theorem 4.1] *A conformally balanced astheno-Kähler structure (J, g) on a compact manifold of complex dimension $m \geq 3$ whose Bismut connection has (restricted) holonomy contained in $SU(m)$ is necessarily Kähler and therefore it is a Calabi-Yau structure.*

Latorre and Ugarte [27] proved the following results:

Theorem 2.18 [27, Corollary 2.3] *Let M be a homogeneous compact complex manifold of complex dimension $m \geq 3$ and let g be an invariant Hermitian metric on M. If g is SKT or astheno-Kähler, then g is k-th Gauduchon for any $1 \leq k \leq m-1$.*

Theorem 2.19 [27, Theorem 2.4] *For each $m \geq 4$, there is a non-Kähler compact complex manifold M of complex dimension m admitting a balanced metric $\tilde{g}$ and an astheno-Kähler metric g, which is additionally k-th Gauduchon for any $1 \leq k \leq m-1$.*

In [12] Fino and Grantcherov proved that the twistor spaces of compact hyper-Kähler and negative quaternionic-Kähler manifolds do not admit astheno-Kähler metrics. They proved that compact semisimple Lie group of even dimension ($m > 6$) and rank two endowed with its Samelson's complex structure admits an astheno-Kähler non-SKT metric. In contrast, its canonical SKT metric is not astheno-Kähler. In particular, the Lie groups $SU(3)$ and G_2 admit SKT and astheno-Kähler metrics, which are different.

Before listing further results, let us recall the following definitions from [5].

Definition 2.20 *The Bott-Chern cohomology of a complex manifold X is the bi-graded algebra.*

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\ker\partial\cap\ker\bar{\partial}}{\mathrm{im}\,\partial\bar{\partial}}.$$

Unlike in the case of the Dolbeault cohomology groups, for every $(p,q)\in\mathbb{N}^2$, the conjugation induces an isomorphism

$$H_{BC}^{p,q}(X)\xrightarrow{\simeq} H_{BC}^{q,p}(X).$$

Definition 2.21 *The Aeppli cohomology of a complex manifold X is the bi-graded $H_{BC}^{\bullet,\bullet}(X)$-module*

$$H_{A}^{\bullet,\bullet}(X) := \frac{\ker\partial\bar{\partial}}{\mathrm{im}\,\partial+\mathrm{im}\,\bar{\partial}}.$$

As for the Bott-Chern cohomology, for every $(p,q)\in\mathbb{N}^2$, the conjugation induces the isomorphism

$$H_{A}^{p,q}(X)\xrightarrow{\simeq} H_{A}^{q,p}(X).$$

Notation: For every $(p,q)\in\mathbb{N}^2$, for every $k\in\mathbb{N}$, and for $\sharp\in\{\bar{\partial},\partial,BC,A\}$, we will denote

$$h_{\sharp}^{p,q}:=\dim_{\mathbb{C}}H_{\sharp}^{p,q}(X)\in\mathbb{N}\quad\text{ and }\quad h_{\sharp}^{k}:=\sum_{p+q=k}h_{\sharp}^{p,q}\in\mathbb{N}.$$

Chiose and Rasdeaconu gave the following significant result on cohomologies on astheno-Kähler manifold:

Theorem 2.22 [9, Theorem 1.1] *On a compact astheno-Kähler manifold (M,Ω), the following inequalities hold:*

$$h_{BC}^{0,1}(M)\leq h_{A}^{0,1}(M)\leq h_{BC}^{0,1}(M)+1.$$

Theorem 2.23 [9, Remark 4.2] *On an astheno-Kähler manifold (M,Ω), at least one of the morphisms*

$$H_{\bar{\partial}}^{0,1}(M)\to H_{A}^{0,1}(M)\quad\textit{and}\quad H_{BC}^{0,1}(M)\to H_{\bar{\partial}}^{0,1}(M)$$

is an isomorphism.

Theorem 2.24 [9, Remark 4.4] *On an astheno-Kähler manifold (M,Ω) of complex dimension m one has $h_{BC}^{0,1}=h_{A}^{0,1}$ if and only if $\partial(e^{(m-1)f}\Omega^{m-1})$ is $\partial\bar{\partial}$- exact, where f is a C^∞ real function on M such that $e^{(m-1)f}\Omega^{m-1}$ is $i\partial\bar{\partial}-$ closed.*

Theorem 2.25 [9, Proposition 4.2] *Let M be a compact complex astheno-Kähler manifold of dimension m, and let*

$$i_{p,q} : H^{p,q}_{BC}(M) \to H^{p,q}_{A}(M)$$

be the map induced by the identity. Then $i_{m-1,m-1}$ is surjective if and only if $i_{0,1}$ is injective.

Theorem 2.26 [9, Proposition 5.4] *If M_1 and M_2 be two astheno-Kähler manifolds saturating the upper bound in $h^{0,1}_{BC}(M_1 \times M_2) \leq h^{0,1}_{A}(M_1 \times M_2) \leq h^{0,1}_{BC}(M_1 \times M_2) + 1$, then $M_1 \times M_2$ does not admit astheno-Kähler metrics.*

Definition 2.27 [48] *A locally conformally Kähler (LCK) manifold (M, Ω) is Vaisman if $\nabla\theta = 0$, where ∇ denotes the Levi-Civita connection associated to Ω and θ is the Lee form.*

Theorem 2.28 [9, Theorem 1.6] *Any compact Vaisman manifold of dimension at least three satisfies $h^{0,1}_{A}(M) = h^{0,1}_{BC}(M) + 1$ and carries no astheno-Kähler metric and so being a Vaisman there exists no astheno-Kähler metrics on similarity Hopf manifolds of dimension at least three, diagonal Hopf manifold and standard Hopf manifold.*

Theorem 2.29 [9, Corollary 6.4] *The class of astheno-Kähler manifolds is not invariant under modifications.*

In support of this, we can consider an example: let M be the 3-dimensional manifold constructed by Hironaka [21], which is a proper modification of the projective space $\mathbb{CP}^3$ and which unlike $\mathbb{CP}^3$, cannot carry an astheno-Kähler metric.

In [3], Biswas in his two-page paper gave a result that if G/Γ, where Γ is a lattice in a connected complex Lie group G such that G/Γ is compact, admits an astheno-Kähler metric, then G is abelian and G/Γ is a compact complex torus . The author used a lemma due to Jost and Yau [22] that "any holomorphic 1-form on a compact astheno-Kähler manifold is closed" as a key element in his proof.

In [38], Podesta showed that for a suitable invariant complex structure there exists an invariant Hermitian metric, which is Chern–Einstein and astheno-Kähler.

The main result of the paper reads as follows:

Theorem 2.30 [38, Theorem 3] *Let $M = G/L$ be a manifold in the class* C *(special subclass of compact simply connected complex homogeneous manifolds that includes the Calabi–Eckmann manifolds) endowed with an invariant complex structure $J_{a,b}(a, b \in \mathbb{R}, b \neq 0)$ and complex dimension m, where G is a compact connected Lie group and L is some compact subgroup. Then M is simply connected, non-Kähler and there exist $a_o, b_o \in \mathbb{R}$ and an invariant metric, Hermitian concerning J_{a_o,b_o}, which is Chern-Einstein and astheno-Kähler.*

In [7], Chen et al., in the second part of the paper, derived a characterization of k-Gauduchon metrics in terms of the Chern scalar curvature and a Riemannian type scalar curvature of Ω concerning the Chern connection, where in particular, a pluriclosed metric (i.e., $\partial\bar{\partial}\Omega = 0$) is 1-Gauduchon. In contrast, an astheno-Kähler metric (i.e., $\partial\bar{\partial}\Omega^{m-2} = 0$) is $(m-2)$-Gauduchon.

In [24], Kawamura showed a relation between almost Gauduchon, astheno-Kähler and quasi-Kähler manifold. The author proved the following result:

Theorem 2.31 [24, Theorem 1.1,1.2] *Suppose $(M^{2m}, J, \omega, \Omega)$ is a compact almost-Calabi-Yau manifold with torsion for $m \geq 3$. Suppose Ω is either almost pluri-closed (Gauduchon) or almost-astheno-Kähler. Then Ω is quasi-Kähler metric, where ω is a nowhere vanishing holomorphic $(m,0)$-form admitted by M. Here holomorphic means that ω satisfies $\bar{\partial}\omega = 0$.*

Calamai [6] provided a list of possible projectively flat metrics studied in [30] and showed that projectively flat astheno-Kähler metrics are globally conformally flat Kähler metrics.

In [45], Sferruzza and Tomassini provided families of compact astheno-Kähler nilmanifolds endowed with left-invariant complex structure and studied the behaviour of blowup of compact complex manifolds endowed with astheno-Kähler metrics, satisfying certain extra differential conditions (i.e., $dd^c\Omega^{m-3} = 0$) and showed that in this case the existence of astheno-Kähler metric is not preserved by blowup.

We will now present several findings related to nilmanifolds.

Remark 2.3 [11] Starting from late 1990, a "complex nilmanifold" means a quotient of a real nilpotent Lie group equipped with a left-invariant complex structure by the left action of a discrete, co-compact subgroup. This definition is much more general; indeed, left-invariant complex structures are found on many even-dimensional nilpotent Lie groups that are not complex. The complex structure on a Kodaira surface is one such example.

Complex structures on a nilmanifold have impeccable algebraic characterization. Let G be a real nilpotent Lie group and $\mathfrak{g}$ be a Lie algebra. By Newlander-Nirenberg theorem [34], a complex structure on G is the same as a sub-bundle $T^{1,0}G \subset TG \otimes_{\mathbb{R}} \mathbb{C}$ such that $\left[T^{1,0}G, T^{1,0}G\right] \subset T^{1,0}G$ and $T^{1,0}G \oplus \overline{T^{1,0}G} = TG \otimes_{\mathbb{R}} \mathbb{C}$. The left-invariant sub-bundles in $T^{1,0}G$ are the same as subspaces $W \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and the condition $\left[T^{1,0}G, T^{1,0}G\right] \subset T^{1,0}G$ is equivalent to $[W, W] \subset W$. Therefore, left-invariant complex structures on G are the same as complex sub-algebras $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

Sferruzza [44] proved necessary cohomological conditions for the existence of curves of astheno-Kähler metrics along curves of deformations starting from an initial compact complex manifold endowed with an astheno-Kähler metric. Furthermore, he applied his results, providing obstructions to the

existence of curves of astheno-Kähler metrics on two different families of real 8-dimensional nilmanifolds endowed with invariant nilpotent complex structures.

Theorem 2.32 [9, Corollary 1.5] *Let M be non-Kähler nilmanifold equipped with a nilpotent complex structure. If M admits an astheno-Kähler metric, then $h^{0,1}_A(M) = h^{0,1}_{BC}(M) + 1$.*

3 Some known results on bundles on astheno-Kähler manifold

Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial, where a_i are nonnegative integers and E be a holomorphic vector bundle on a compact complex manifold X. Let $f(E) = \bigoplus_{i=0}^{n} \left(E^{\otimes i}\right)^{\oplus a_i}$ be the vector bundle on X. An holomorphic vector bundle E on a compact complex manifold X is called finite [4] if there are two distinct polynomials f, g with nonnegative integral coefficients, such that the vector bundle $f(E)$ is isomorphic to $g(E)$. It is known that a vector bundle E is finite if and only if there is a finite collection of vector bundles $\{F_j\}_{j=1}^m$ and nonnegative integers $\{a_{i,j}\}_{j=1}^m$ such that

$$E^{\otimes i} = \bigoplus_{i=0}^{n} (F_j)^{\oplus a_{i,j}}$$

for all $i \geq 1$ [36, p. 35, Lemma 3.1].

Now, we give some results based on bundles.

In [4], Biswas and Pingali characterized finite vector bundles on a compact complex manifold admitting a Gauduchon and astheno-Kähler metric. The main result of the paper is as follows:

Theorem 3.1 [4, Theorem 1.1] *Let M be a compact complex manifold that admits a Hermitian metric that is both Gauduchon and astheno-Kähler. Then a holomorphic vector bundle E over M is finite if and only if it corresponds to a representation of a finite quotient of the fundamental group of M, or equivalently, if and only if E admits a flat connection, compatible with its holomorphic structure, that has a finite monodromy group.*

Before stating the next result, we need to state the following definition:

Definition 3.2 [28] *Let (X, Ω) be a m-dimensional compact Hermitian manifold. A line bundle L over (X, Ω) is said to be numerically effective if for every $\epsilon > 0$, there exists a smooth metric h_ϵ on L such that the curvature $\sqrt{-1}\Theta\,(L, h_\epsilon) = \sqrt{-1}\bar{\partial}\partial \log h_\epsilon \geq -\epsilon\Omega$.*

A Hermitian flat-line bundle is a numerically effective. A vector bundle, E of rank $r \geq 2$, is said to be numerically effective flat if the anti-tautological line bundle $\mathcal{O}_E(1)$ on the projective bundle PE is numerically effective flat. A vector bundle, E, is said to be numerically flat if both E and its dual E^* are numerically effective flat.

Chen [8] stated and proved the following result:

Theorem 3.3 *Let E be a pseudo-effective vector bundle on the Gauduchon and astheno-Kähler manifold (M, Ω) with vanishing first Chern number. Then, E is numerically flat.*

In [28], Li et al. consider numerically flat vector bundles over compact non-Kähler manifolds, mainly astheno-Kähler manifolds by introducing the notion of approximate Hermitian flatness (a holomorphic vector bundle E over a compact complex manifold X is approximate Hermitian flat if there exists a Hermitian metric h_ϵ such that the curvature satisfies $\sup_X |\Theta(L, h_\epsilon)| < \epsilon$, where L is line bundle over a compact complex manifold X and $\Theta(L, h_\epsilon)$ is the Chern form of an arbitrary Hermitian metric h_ϵ on L.).

Theorem 3.4 [28] *Let $(X,\widehat{\Omega})$ be an m-dimensional compact astheno-Kähler manifold, Ω a Gauduchon metric conformal to $\widehat{\Omega}$, and E a holomorphic vector bundle over X. Then the following statements on E are equivalent:*
(1) E is numerically flat.
(2) E is numerically effective flat with $\mathrm{ch}_1(E) \cdot \left[\Omega^{m-1}\right] = 0$.
(3) E is Ω-semistable with $\mathrm{ch}_1(E) \cdot \left[\Omega^{m-1}\right] = \mathrm{ch}_2(E) \cdot \left[\widehat{\Omega}^{m-2}\right] = 0$.
(4) E is approximate Hermitian flat.
(5) There exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l = E$$

by subbundles whose quotients are Hermitian flat.

Biswas with Loftin [2] contributed to the field of astheno-Kähler manifold. They proved the existence and uniqueness of a Harder-Narasimhan filtration for flat vector bundles over M, as well as a Bogomolov-type inequality for semistable flat vector bundles over M under the assumption that the Gauduchon metric g is astheno-Kähler.

Shen [40] studied the problem of Hitchin-Kobayashi correspondence for twisted holomorphic vector bundles over compact Gauduchon manifolds. The main result proven is that a twisted holomorphic vector bundle over a compact Gauduchon manifold is semi-stable if and only if it admits an approximate Hermitian-Einstein structure. The author derived a Bogomolov-type inequality for a semi-stable twisted holomorphic vector bundle over a compact astheno-Kähler manifold.

4 Examples

Now, we give some examples of astheno-Kähler manifolds.

Example 4.1 [32] Any product manifold of curves and surfaces is astheno-Kähler.

Consider the product manifold $M = M_1 \times M_2$, where $M_i (i = 1, 2)$ be a $(2m_i + 1)$-dimensional compact normal almost contact metric manifold with the structure tensor fields (ϕ_i, ξ_i, η_i), we consider an almost complex structure J defined by

$$J = \phi_1 - \eta_2 \otimes \xi_1 + \phi_2 + \eta_1 \otimes \xi_2 \tag{4.1}$$

Here M endowed with J is a compact complex manifold of complex dimension $m = m_1 + m_2 + 1$. Moreover, if g_i is the compatible Riemannian metric on M_i for each $i = 1, 2$, then the Riemannian product metric $g = g_1 + g_2$ on M is compatible with J, that is, g is a Hermitian metric on M. Then its Kähler form Ω is given by

$$\Omega = \Phi_1 + \Phi_2 - 2\eta_1 \wedge \eta_2 \tag{4.2}$$

where Φ_i denotes the fundamental 2-form on M_i for each $i = 1, 2$.

Example 4.2 **Product of two Sasakian manifolds**
[32] Let (M_i, g_i) be a 3-dimensional compact Sasakian manifold for each $i = 1, 2$. Then the product manifold $M = M_1 \times M_2$ with the Hermitian structure (4.1) and Kähler form (4.2) is astheno-Kähler. Since M_1 and M_2 are both Sasakian, we have

$$d\Omega = -2 \left(\Phi_1 \wedge \eta_2 - \Phi_2 \wedge \eta_1 \right)$$

and

$$d^{\mathrm{c}}\Omega = Jd\Omega = -2 \left(J\Phi_1 \wedge J\eta_2 - J\Phi_2 \wedge J\eta_1 \right) = 2 \left(\Phi_1 \wedge \eta_1 + \Phi_2 \wedge \eta_2 \right).$$

Using the fact that Ω and Φ_i are J-invariant, and $J\eta_1 = \eta_2, J\eta_2 = -\eta_1$, we get

$$dd^{\mathrm{c}}\Omega = 2 \left(\Phi_1^2 + \Phi_2^2 \right)$$

Since $\dim_{\mathbb{C}} M = m = 3$, i.e., $m_i = 1$ for each $i = 1, 2$, $\Phi_i^2 = 0$ on M_i, and hence

$$dd^{\mathrm{c}}\Omega^{m-2} = dd^{\mathrm{c}}\Omega = 2 \left(\Phi_1^2 + \Phi_2^2 \right) = 0 \quad \text{on } M.$$

Therefore, the Hermitian structure (4.1) on M is astheno-Kähler.

Note: If $\dim_{\mathbb{C}} M = m > 3$ in above example, the Hermitian structure (4.1) is not astheno-Kähler.

Example 4.3 **Product of Sasakian manifold and Cosymplectic manifold**
[32] Let (M_1, g_1) be a 3-dimensional compact Sasakian manifold, and (M_2, g_2) a compact cosymplectic manifold of dimension ≥ 3. Then, the product manifold $M = M_1 \times M_2$ with the Hermitian structure (4.1) is astheno-Kähler. Since M_1 is Sasakian and M_2 is cosymplectic, we have

$$d\Omega = -2\Phi_1 \wedge \eta_2 \quad \text{and} \quad d^c\Omega = 2\Phi_1 \wedge \eta_1.$$

By simple calculation, we have

$$dd^c\Omega = 2\Phi_1^2$$

Since $m_1 = 1, \Phi_1^2 = 0$ on M_1, that is, $dd^c\Omega = 0$ and $d\Omega \wedge d^c\Omega = 0$ on M, and hence we obtain

$$dd^c\Omega \wedge \Omega + (m-3)d\Omega \wedge d^c\Omega = 0$$

Therefore, the Hermitian structure (4.1) on M is astheno-Kähler.

Note: Let (M_1, g_1) be a Sasakian manifold of dimension greater than 3, and (M_2, g_2) a cosymplectic manifold. Then the Hermitian structure (4.1) is not astheno-Kähler.

Example 4.4 [33] There exist astheno- Kähler structure (non-Kähler) on the Calabi- Eckmann manifold $M = S^{2m_1+1} \times S^{2m_2+1}$ with an almost complex structure J given by Tsukada defined by

$$J = \phi_1 - \left(\frac{a}{b}\eta_1 + \frac{a^2+b^2}{b}\eta_2\right) \otimes \xi_1 + \phi_2 + \left(\frac{1}{b}\eta_1 + \frac{a}{b}\eta_2\right) \otimes \xi_2, \qquad (4.3)$$

where $a, b \in \mathbb{R}$ and $b \neq 0$. The Hermitian metric g on the complex manifold (M, J) is given by

$$g = g_1 + g_2 + a\left(\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1\right) + \left(a^2 + b^2 - 1\right)\eta_2 \otimes \eta_2.$$

Then the Kähler form Ω on the Hermitian manifold (M, J, g) is given by

$$\Omega = \Phi_1 + \Phi_2 - 2b\eta_1 \wedge \eta_2$$

where Φ_i denotes the fundamental 2-form on M_i for each $i = 1, 2$.

Note: When $a = 0$ and $b = 1$, the almost complex structure (4.3) coincides with A. Morimoto's complex structure (4.1).

Example 4.5 [9, Theorem 1.3] A cartesian product of two compact complex surfaces admits an astheno-Kähler metric if and only if at least one of the surfaces admits a Kähler metric.

Example 4.6 [9, Corollary 5.1] The complex parallelizable Nakamura manifold does not admit astheno-Kähler metrics.

Example 4.7 [9, Corollary 5.3] The Oeljeklaus-Toma manifolds of type (s,t) with $s \geq 2$ and $s = 1$ do not admit astheno-Kähler metrics.

Example 4.8 **By blow-ups and resolutions**
[13] Let (M, J, g) be an astheno-Kähler manifold of complex dimension m such that its fundamental 2-form Ω satisfies

$$\partial\bar{\partial}\Omega = 0, \partial\bar{\partial}\Omega^2 = 0.$$

Then both the blow-up $\tilde{M}_p$ of M at a point $p \in M$ and the blow-up $\tilde{M}_Y$ of M along a compact complex submanifold Y admit an astheno-Kähler metric satisfying $\partial\bar{\partial}\Omega = 0, \partial\bar{\partial}\Omega^2 = 0$. Thus, it is possible to construct new examples of astheno Kähler manifolds by blowing up a given astheno-Kähler manifold M (satisfying $\partial\bar{\partial}\Omega = 0,\ \partial\bar{\partial}\Omega^2 = 0$) at one or more points or along a compact complex submanifold.

Example 4.9 **By twist construction**
[13] Applying the twist construction [43, Prop. 4.5] to 8-dimensional astheno-Kähler manifolds with torus action, one can get new simply connected astheno-Kähler manifolds. Let $\left(N^6, J, \right)$ be a 6-dimensional simply connected compact complex manifold with a Hermitian structure g, which is strong KT and standard. Consider the product $M^8 = N^6 \times \mathbb{T}^2$, where $\mathbb{T}^2$ is a 2 -torus with an invariant Kähler structure. Then M^8 is astheno-Kähler and strong KT with torsion c supported on N^6.

Example 4.10 8**-dimensional nilmanifolds**
[13, Example 2.3]
Let $\left\{\eta^1, \ldots, \eta^4\right\}$ be the set of complex forms of type $(1,0)$ such that

$$\begin{cases} d\eta^j = 0, \quad j = 1,2,3 \\ d\eta^4 = a_1\eta^1 \wedge \eta^2 + a_2\eta^1 \wedge \eta^3 + a_3\eta^1 \wedge \bar{\eta}^1 + a_4\eta^1 \wedge \bar{\eta}^2 + a_5\eta^1 \wedge \bar{\eta}^3 \\ \qquad + a_6\eta^2 \wedge \eta^3 + a_7\eta^2 \wedge \bar{\eta}^1 + a_8\eta^2 \wedge \bar{\eta}^2 + a_9\eta^2 \wedge \bar{\eta}^3 + a_{10}\eta^3 \wedge \bar{\eta}^1 \\ \qquad + a_{11}\eta^3 \wedge \bar{\eta}^2 + a_{12}\eta^3 \wedge \bar{\eta}^3 \end{cases}$$

span the dual of a 2-step nilpotent Lie algebra $\mathfrak{n}$, depending on the complex parameters $a_1, \ldots, a_{12}$ and define an integrable almost complex structure J on $\mathfrak{n}$. Let $a_1, \ldots, a_{12} \in \mathbb{Q}[i]$ and $(M = \Gamma\backslash N, J)$ be the corresponding compact complex nilmanifold of real dimension 8, where N is connected Lie group with Lie algebra $\mathfrak{n}$ and Γ is uniform discrete subgroup, then the Hermitian metric

$$g = \frac{1}{2}\sum_{j=1}^{4} \eta^j \otimes \bar{\eta}^j + \bar{\eta}^j \otimes \eta^j$$

is astheno-Kähler if and only if

$$|a_1|^2 + |a_2|^2 + |a_4|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2 + |a_9|^2 + |a_{10}|^2 + |a_{11}|^2 = 2\mathfrak{Re}\left(a_3\bar{a}_8 + a_3\bar{a}_{12} + a_8\bar{a}_{12}\right).$$

If, in addition $a_8 = 0$ and $|a_4|^2 + |a_{11}|^2 \neq 0$, then the astheno-Kähler metric g is not SKT. Moreover, if $a_8 = 0$, the astheno-Kähler metric g is SKT if and only if $a_1 = a_4 = a_6 = a_7 = a_9 = a_{11} = 0$.

In their study, Latorre and Ugarte [27] presented instances of non-Kähler compact complex manifolds supporting balanced and astheno-Kähler metrics. They provided constructions of compact complex non-Kähler manifolds with complex dimension $m \geq 4$, which possess a balanced metric and an astheno-Kähler metric that is k-Gauduchon for any k within the range $\{1, \ldots, m-1\}$. Additionally, they offered examples of nilmanifolds characterized by a Lie algebra that is a product of a $(2n+1)$-dimensional Heisenberg Lie algebra and a real line featuring an invariant abelian complex structure. Furthermore, they also explored examples on nilmanifolds with an invariant non-abelian complex structure specifically in complex dimension 4.

Example 4.11 [12, Example 4.2] The total space of a principal T^2-bundle over a 6-dimensional torus admits a balanced and astheno-Kähler metric. Let $\pi : M \to T^6$ be the principal T^2 bundle over T^6 with characteristic classes

$$a_1 = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 - 2dz_3 \wedge d\bar{z}_3, \quad a_2 = dz_2 \wedge d\bar{z}_2 - dz_3 \wedge d\bar{z}_3,$$

where (z_1, z_2, z_3) are complex coordinates on T^6. Consider on T^6 the standard complex structure and let

$$F_1 = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3, \quad F_2 = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + 5dz_3 \wedge d\bar{z}_3,$$

then the a_i 's are traceless with respect to F_1 and $\left(a_1^2 + a_2^2\right) \wedge F_2 = 0$. Let θ^j be connection 1-forms such that $d\theta^j = \pi^* a_j$ and define

$$\Omega_1 = \pi^* F_1 + \theta^1 \wedge \theta^2, \quad \Omega_2 = \pi^* F_2 + \theta^1 \wedge \theta^2.$$

The 2-forms Ω_1 and Ω_2 define a balanced metric g_1 and an astheno Kähler metric g_2 on M, respectively, compatible with the integrable complex structure so that the projection map π is holomorphic. M can be alternatively described as the 2-step nilmanifold G/Γ, where G is the 2-step nilpotent Lie group with structure equations

$$\begin{cases} de^j = 0, \quad j = 1, \ldots, 6 \\ de^7 = e^1 \wedge e^2 + e^3 \wedge e^4 - 2e^5 \wedge e^6, \\ de^8 = e^3 \wedge e^4 - e^5 \wedge e^6 \end{cases}$$

and Γ is a co-compact discrete subgroup, endowed with the invariant complex structure I such that $Ie_1 = e_2, Ie_3 = e_4, Ie_5 = e_6, Ie_7 = e_8$. In this setting the 2-form $e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 + e^7 \wedge e^8$ defines a balanced metric and $e^1 \wedge e^2 + e^3 \wedge e^4 + 5e^5 \wedge e^6 + e^7 \wedge e^8$ gives an astheno-Kähler metric.

Example 4.12 [12, Proposition 5.7] The homogeneous space $SU(5)/T^2$ (flag manifold) for appropriate action of T^2 is simply connected. It has an invariant complex structure which admits both balanced and astheno-Kähler metrics but does not admit any SKT metric.

Example 4.13 In [33] Matsuo showed the existence of astheno-Kähler metrics on Calabi-Eckmann manifolds. Since Calabi-Eckmann manifolds are principal T^2-bundles over $\mathbb{CP}^n \times \mathbb{CP}^m$, the construction of astheno-Kähler structures on torus bundles over Kähler manifolds.

In [12], Fino et al. generalised Matsuo's result to principal torus fibrations over compact Kähler manifolds. Using this idea Fino et al. gave a new example, which is

Example 4.14 Let $\pi : P \to M$ be an n-dimensional principal torus bundle over a Kähler manifold (M, J, Ω) equipped with 2 connections 1-forms θ_1, θ_2 whose curvatures Q_1, Q_2 are of type $(1,1)$ and are pull-backs from forms α_1 and α_2 on M. Let I be the complex structure on P defined as the pull-back of J to the horizontal subspaces and as $I(\theta_1) = \theta_2$ along vertical directions. Let $\Phi = \pi^*(\Omega) + \theta_1 \wedge \theta_2$; then $dd^c\Phi^k = k(Q_1^2 + Q_2^2) \wedge (\pi^*(\Omega^{k-1}))$, where $1 \leq k \leq n-2$. In particular, if $\alpha_1^2 + \alpha_2^2 \wedge \Omega^{n-3} = 0$, then Φ is astheno-Kähler and if a torus bundle with 2-dimensional fiber over a Kähler base admits an SKT metric, then it is astheno-Kähler.

Example 4.15 [9, Proposition 5.6] Let S be a compact complex surface, M a compact Kähler manifold. If N denotes the blow-up of $M \times S$ along a smooth submanifold, then N is an astheno-Kähler manifold.

5 Equations and estimates of their solutions on astheno-Kähler manifold

This section explores the equation on compact Hermitian manifolds and demonstrates the existence of solutions for the equation on astheno-Kähler manifolds.

5.1 The Fu-Yau equation on compact astheno-Kähler manifolds

Let (M, Ω) be an m-dimensional compact Kähler manifold. As a reduced generalized Strominger system in higher dimensions, Fu and Yau introduced the following fully nonlinear equation for φ, which is usually called Fu-Yau equation [37,17]

$$\sqrt{-1}\partial\bar{\partial}\big(e^{\varphi}\Omega - \alpha e^{-\varphi}\rho\big)\wedge\Omega^{m-2} + m\alpha\sqrt{-1}\partial\bar{\partial}\varphi\wedge\sqrt{-1}\partial\bar{\partial}\varphi\wedge\Omega^{m-2} + \mu\frac{\Omega^m}{m!} = 0, \quad (5.1)$$

where α is a non-zero constant called the slope parameter, ρ is a real smooth $(1,1)$ form, μ is a smooth function. For φ, we can impose the elliptic condition

$$\tilde{\Omega} = e^{\varphi}\Omega + \alpha e^{-\varphi}\rho + 2m\alpha\sqrt{-1}\partial\bar{\partial}\varphi \in \Gamma_2(M)$$

and the normalization condition

$$\left\|e^{-\varphi}\right\|_{L^1} = A \tag{5.2}$$

where

$$\Gamma_2(M) = \left\{\alpha \in A^{1,1}(M) \,\middle|\, \frac{\alpha^1 \wedge \Omega^{m-1}}{\Omega^m} > 0, \frac{\alpha^2 \wedge \Omega^{m-2}}{\Omega^m} > 0\right\}$$

and $A^{1,1}(M)$ is the space of smooth real $(1,1)$ forms on M. When $m = 2$, (5.1) is equivalent to the Strominger system on a toric fibration over a $K3$ surface constructed by Goldstein and Prokushki [19], which was solved by Fu and Yau for $\alpha > 0$ and $\alpha < 0$ in [14,15]. In case of $\alpha < 0$, Phong et al. [37] proved the existence of solutions of (5.1) such that the condition (5.2) is replaced by

$$\|e^{\varphi}\|_{L^1} = \frac{1}{A} \gg 1.$$

Chu et al. [10] proved the following result:

Theorem 5.1 *Let (M, Ω) be an m-dimensional compact astheno-Kähler manifold. Then there exist constants A_0, C_0, δ_0, M_0 and D_0 depending only on α, ρ, μ and (M, Ω) such that for any $A \leqslant A_0$, there exists a unique solution φ of the Fu-Yau equation and satisfying the elliptic condition and the normal condition*

$$e^{-\varphi} \leqslant \delta_0, \quad |\partial\bar{\partial}\varphi|_g \leqslant D, \quad D_0 \leqslant D \quad \text{and } A \leqslant \frac{1}{C_0 M_0 D}.$$

Remark 5.1 They used the Moser iteration to do C^0-estimate for solutions φ of the Fu-Yau equation and also gave a sketch of C^1,C^2 estimates of φ.

Remark 5.2 Furthermore, they improved a result for the case $\alpha \leq 0$ and provided another proof to derive a prior $\mathrm{C}^1, \mathrm{C}^2$ estimate for φ of the Fu-Yau equation in the case of $\alpha \leq 0$, but without the restriction condition.

Kawamura[25] investigated the Fu-Yau equation on compact almost astheno-Kähler manifolds and showed an a priori C^0-estimate for a smooth solution of the equation.

5.2 Monge-Ampère equation on compact Astheno-Kähler manifolds

The complex Monge-Ampère equation

$$(\Omega + \sqrt{-1}\partial\bar{\partial}u)^m = e^F \Omega^m, \quad \Omega + \sqrt{-1}\partial\bar{\partial}u > 0,$$

on a compact Kähler manifold (M, Ω) of dimension m was solved by Yau in the 1978 [50] and has consistently been influenced by its pervasive role in Kähler geometry since its inception. Here F is a given smooth function on M, normalized so that $\int_M e^F \Omega^m = \int_M \Omega^m$. Yau's Theorem, conjectured in the 1950s by Calabi, is that the Monge-Ampère equation has a unique solution u with $\sup_M u = 0$.

For a general Hermitian metric Ω, the complex Monge-Ampère equation was solved in full generality by the authors in [46]. In this case, there exists a unique pair (u, b) with u a smooth function satisfying $\sup_M u = 0$ and b a constant such that

$$(\Omega + \sqrt{-1}\partial\bar{\partial}u)^m = e^{F+b} \Omega^m, \quad \Omega + \sqrt{-1}\partial\bar{\partial}u > 0.$$

This equation applies to studying cohomology classes and notions of positivity on complex manifolds.

Tosatti and Weincove [47] showed unique smooth solutions to the Monge-Ampère equation for $(m-1)$-plurisubharmonic functions on Hermitian manifolds. They obtained Calabi-Yau theorems for Gauduchon and strongly Gauduchon metrics on a class of non-Kähler manifolds those satisfying the astheno-Kähler condition. They discussed another Monge-Ampère equation (introduced by Popovici [39]) and showed that the full Gauduchon conjecture can be reduced to a second order estimate of Hou-Ma-Wu type.

The main result of the authors is given below:

Theorem 5.2 *Let M be a compact complex manifold equipped with an astheno-Kähler metric Ω. Let Ω_0 be a Gauduchon (resp. strongly Gauduchon) metric on M and F' a smooth function on M. Then there exists a unique constant b' and a unique Gauduchon (resp. strongly Gauduchon) metric, which we write as Ω_u, with*

$$\Omega_u^{m-1} = \Omega_0^{m-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \Omega^{m-2}$$

for some smooth function u.

5.3 Hermitian-Yang-Mills flow

Let $\left(E, \bar{\partial}_E\right)$ be a rank r holomorphic vector bundle over a m-dimensional compact complex manifold M with Hermitian metric g and associated $(1,1)$-form Ω. Let H_0 be the initial Hermitian metric on E and $\mathcal{A}_{H_0}$ be the space

of connections of E compatible with H_0. Let F_A denote the curvature of the connection $A \in \mathcal{A}_{H_0}$ and Λ_Ω denote the adjoint of the operation $\eta \mapsto \eta \wedge \Omega$.

In [35], Nie and Zhang studied the limiting behaviour of solutions of the Hermitian-Yang-Mills flow on $\mathcal{A}_{H_0}$,

$$\begin{aligned} \frac{\partial A}{\partial t} &= i\left(\bar{\partial}_A - \partial_A\right)\Lambda_\Omega F_A, \\ A(0) &= A_0. \end{aligned} \tag{5.3}$$

When (M, g) is Kähler, the flow equations (5.3) reduce to the Yang-Mills flow,

$$\frac{\partial A}{\partial t} = -d_A^* F_A. \tag{5.4}$$

In [35], the authors proved that

Theorem 5.3 *Let (M, g) satisfy the Gauduchon condition $\partial\bar{\partial}\Omega^{m-1} = 0$ and the astheno-Kähler condition $\partial\bar{\partial}\Omega^{m-2} = 0$ and let $A(t)$ be a solution of (5.3). Then for every sequence $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}$ with $\lim\limits_{k\to\infty} t_k = \infty$, there is a subsequence t_{k_j} and a closed set $\Sigma \subset M$ of Hausdorff codimension at least four such that $A(t_{k_j})$ converges modulo gauge transformations to a connection A_∞ on a Hermitian vector bundle (E_∞, H_∞) in C^∞_{loc} topology on $M\backslash\Sigma$, where A_∞ satisfies*

$$d_{A_\infty}\Lambda_\Omega F_{A_\infty} = 0.$$

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