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Synectic lift metrics on tangent bundles: exploring soliton structures and gradient variants

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Abstract Let TM be the tangent bundle with the synectic lift on the n-dimensional Riemannian manifold M. The objective of this article is to investigate the properties of Ricci soliton and generalized Ricci-Yamabe soliton structures according to the synectic lift metric on the tangent bundle TM. Furthermore, the properties of the gradient Ricci soliton and the generalized gradient Ricci-Yamabe soliton are also investigated in this article.

Keywords synectic lift metric · tangent bundle · Ricci soliton · generalized Ricci-Yamabe soliton · gradient Ricci soliton · generalized gradient Ricci-Yamabe soliton

Mathematics Subject Classification (2010) MSC 53C07 · MSC 53A45

1 Introduction

One of the most fascinating mathematical tools for describing geometric structures in Riemannian geometry in recent years has been the hypothesis of geometric flows. Geometric flows, as a class of important geometric partial differential equations, have been highlighted in many fields of theoretical physics and practical applications, being a very useful tool for understanding the topology of arbitrary Riemannian manifolds and having a profound influence on modern geometric analysis.

Ricci solitons have become a topic of significant importance due to the fact that they correspond to self-similar solutions of the Ricci flow, which is a fundamental concept in Riemannian geometry. At the same time, they are natural generalisations of Einstein metrics, which is another area of great interest in the field. In light of these considerations, it is evident that Ricci soliton and its various generalisations represent a growing area of interest in recent years.

The Yamabe flow was presented by Hamilton [8] at the same time as the Ricci flow. The Ricci soliton and Yamabe soliton represent two special cases

of the Ricci flow and Yamabe flow, respectively. In dimension $n = 2$, these two solitons are equivalent. However, in dimension $n > 2$, they are distinct.

In the context of soliton structures, if the potential vector field can be expressed as the gradient of a function, the soliton structure in question is designated as a gradient soliton (e.g., gradient Ricci soliton or gradient Yamabe soliton).

Several authors studied some soliton structure, for examples, in [1] Abbassi and Amri, interested in natural Ricci soliton structures on tangent and unit tangent bundles of Riemannian manifolds, Chen and Deshmukh [3] introduced the notion of quasi-Yamabe soliton on a Riemannian manifold, Guler and Crasmareanu [7] introduced the notion of Ricci-Yamabe flow on a Riemannian manifold (M, g) by considering a scalar combination of the Ricci flow and Yamabe flow. Here, arbitrary selection of the sign of the coefficients had very useful results differential geometry and the theory of relativity. Dey and Majhi [4] studied, generalized gradient Ricci-Yamabe soliton on a complete Sasakian 3-manifold M with potential function f, then M is compact Einstein and locally isometric to a unit sphere. In [10], Nurowski and Randall introduced generalized Ricci soliton and Kumara et al [9], show that Riemannian concurrent-recurrent manifold (shortly, CR-manifold) is Einstein when its metric is a generalized Ricci-type soliton.

This article aims to explore the characteristics of Ricci soliton and generalized Ricci-Yamabe soliton structures in relation to the synectic lift metric defined on the tangent bundle TM. Additionally, the properties of gradient Ricci soliton and generalized gradient Ricci-Yamabe soliton structures are examined within the scope of this research.

2 The synectic lift metric on tangent bundle

Let $\pi : TM \to M$ denotes the natural projection map, where M is an n-dimensional Riemannian manifold equipped with the Riemannian metric g, and TM is its tangent bundle. Let us consider a local coordinate system $\left(U, x^i\right)$ on M. This induces a local coordinate system $\left(\pi^{-1}\left(U\right), x^i, x^{\overline{i}} = u^i\right)$, $\overline{i} = n+i = n+1, ..., 2n$ on the tangent bundle TM. In this system, the cartesian coordinates $\left(u^i\right)$ refer to the tangent spaces T_PM be a $n-$dimensional Riemannian manifold with Riemannian metric g, TM be its tangent bundle. The local coordinate system $\left(U, x^i\right)$ is induced to the local coordinate system $\left(\pi^{-1}\left(U\right), x^i, x^{\overline{i}} = u^i\right)$, $\overline{i} = n+i = n+1, ..., 2n,$ on TM. Here $\left(u^i\right)$ are the cartesian coordinates in each tangent space T_PM for each point $p \in U$.

We define ∇ as the Levi-Civita connection of the Riemannian metric g. In the horizontal distribution determined by the Levi-Civita connection ∇ and the vertical distribution defined by $\ker \pi_*$, the local frames are given respectively as:

$$E_i = \frac{\partial}{\partial x^i} - \Gamma^h_{i0}\frac{\partial}{\partial u^h};\ i = 1, ..., n$$

and

$$E_{\bar{i}} = \frac{\partial}{\partial u^i};\ \bar{i} = n+1, ..., 2n.$$

In this context, the Christoffel symbols for ∇, represented by $\Gamma^h_{i0} = u^j\ \Gamma^h_{ij}$, are defined.

In the tangent bundle, the local frame $\{E_\beta\} = (E_i, E_{\bar{i}})$ is referred to as the adapted frame, and for any vector field $X = X^i \frac{\partial}{\partial x^i}$, the horizontal, complete and vertical lifts of X on the adapted frame can be represented as follows [12]:

$$^HX = X^iE_i,\ ^CX = X^iE_i + u^s\nabla_s X^i E_{\bar{i}} \text{ and } ^VX = X^iE_{\bar{i}}.$$

In the tangent bundle TM, the local $1-$form system $\left(dx^i, \delta u^i\right)$ is the dual frame of the adapted frame $\{E_\beta\}$. Here

$$\delta u^i = \ ^H\left(dx^i\right) = du^i + \Gamma^i_{h0}dx^h.$$

As previously stated, a number of Riemannian or pseudo-Riemannian metrics have been defined using the natural lifts of the Riemannian metric g. One such metric is the synectic lift metric on the tangent bundle TM.

Definition 2.1 *[11] Let g represent a Riemannian metric with components g_{ij}. It can be demonstrated that the following expression can be regarded as a pseudo-Riemannian metric on TM:*

$$^{Syn}g = a_{ij}dx^idx^j + 2g_{ij}dx^i\delta y^j.$$

This metric is known as the synectic lift metric. The synectic lift metric ^{Syn}g is defined as $^{Syn}g =^C g+\ ^Va$, where $a = (a_{ij})$ is a symmetric tensor field of the type $(0, 2)$ on M. In this context, the symbols ^Cg and ^Va represent the complete lift and vertical lift of g and a to TM, respectively. Its components are

$$^{Syn}g = \begin{pmatrix} a_{ij} + \partial g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

with respect to induced coordinates.

In the adapted frame $\{E_\beta\}$, the synectic lift metric ^{Syn}g and its inverse are respectively given by:

$$^{Syn}g = \begin{pmatrix} a_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix} \tag{2.1}$$

and

$$^{Syn}g^{-1} = \begin{pmatrix} 0 & g^{jk} \\ g^{jk} & -a^{jk} \end{pmatrix}. \tag{2.2}$$

For the Levi-Civita connection $^{Syn}\nabla$ of the synectic lift metric, we provide the following proposition:

Proposition 2.2 *[2] The Levi-Civita connection ${}^{Syn}\nabla$ of the synectic lift metric ${}^{Syn}g$ on tangent bundle is given as follows:*

$$\begin{aligned} {}^{Syn}\nabla_{E_i}E_j &= \Gamma_{ij}^k E_k + M_{ij}^k + y^s R_{sij}{}^k, \\ {}^{Syn}\nabla_{E_{\bar{i}}}E_j &= 0, \\ {}^{Syn}\nabla_{E_i}E_{\bar{j}} &= \Gamma_{ij}^k E_{\bar{k}}, \\ {}^{Syn}\nabla_{E_{\bar{i}}}E_{\bar{j}} &= 0, \end{aligned} \tag{2.3}$$

where $M_{ij}^k = \frac{1}{2}g^{kh}\left(\nabla_i a_{hj} + \nabla_j a_{hi} - \nabla_h a_{ij}\right)$ is a tensor of type $(1,2)$ and $R_{ijk}{}^h$ are components of the Riemannian curvature tensor field R of ∇.

Proposition 2.3 *Let TM be the tangent bundle equipped with the synectic lift metric ${}^{Syn}g$ on a Riemannian manifold M. In terms of the adapted frame $\{E_\beta\}$, the Riemannian curvature tensor ${}^{Syn}R$ on the tangent bundle has the following properties:*

$$\begin{aligned} {}^{Syn}R_{mij}{}^k &= R_{mij}{}^k, \\ {}^{Syn}R_{mij}{}^{\bar{k}} &= \nabla_m M_{ij}^k - \nabla_i M_{mj}^k + y^s\left(\nabla_m R_{sij}{}^k - \nabla_i R_{smj}{}^k\right), \\ {}^{Syn}R_{mi\bar{j}}{}^{\bar{k}} &= R_{mij}{}^k \end{aligned} \tag{2.4}$$

and other components are zero. Here $M_{ij}^k = \frac{1}{2}g^{kh}\left(\nabla_i a_{hj} + \nabla_j a_{hi} - \nabla_h a_{ij}\right)$ is a tensor of type $(1,2)$. Also, Γ_{ij}^h and $R_{ijk}{}^h$ respectively represent the component of the Levi-Civita connection ∇ of g and its curvature tensor field on M (see, also [6]).

Proposition 2.4 *Let TM be the tangent bundle equipped with the synectic lift metric ${}^{Syn}g$ on a Riemannian manifold M. According to the adapted frame $\{E_\beta\}$, the Ricci curvature tensor of ${}^{Syn}\nabla$ on tangent bundle TM has the following properties:*

$$\begin{aligned} {}^{Syn}R_{ij} &= R_{ij}, \\ {}^{Syn}R_{\bar{i}j} &= {}^{Syn}R_{i\bar{j}} = {}^{Syn}R_{\bar{i}\bar{j}} = 0, \end{aligned} \tag{2.5}$$

where $R_{ij} = R_{kij}{}^k$.

Proposition 2.5 *Let TM be the tangent bundle equipped with the synectic lift metric ${}^{Syn}g$ on a Riemannian manifold M. The scalar curvature tensor ${}^{Syn}r$ of ${}^{Syn}\nabla$ on tangent bundle is zero.*

Let TM be the tangent bundle over M and let Φ be a transformation of TM. If the transformation Φ preserves the fibers, it is referred to as a fibre-preserving transformation. Consider a vector field $\tilde{X}$ on TM and the local one-parameter group $\{\Phi_t\}$ of local transformations of TM generated by $\tilde{X}$. The vector field $\tilde{X}$ is termed an infinitesimal fibre-preserving transformation if each Φ_t is a local fibre-preserving transformation of TM.

Lemma 2.6 *[5] The Lie derivative of ${}^{Syn}g$ with respect to the fibre- preserving vector field $\widetilde{X}$ is given as follows:*

$$L_{\widetilde{X}}\ {}^{Syn}g = \left\{L_V a_{ij} - 2g_{im}\left(y^b v^c R_{jcb}{}^m - v^{\bar{b}}\Gamma^m_{bj} - E_j v^{\bar{m}}\right)\right\} dx^i dx^j \quad (2.6)$$
$$+2\left\{L_V g_{ij} - g_{im}\left(\nabla_j v^m - E_{\bar{j}} v^{\overline{m}}\right)\right\} dx^i \delta y^j,$$

where $L_V g_{ij}$ and $L_V a_{ij}$ denote the components of the Lie derivative $L_V g$ and $L_V a$ respectively. Also, $\nabla_i v^m$ denotes the components of the covariant derivative of V.

3 Some soliton structures with respect to synectic lift

3.1 Hessian operator

To elucidate our main topic, we will provide the requisite details regarding the Hessian operator (with respect to the synectic lift metric) of any smooth function f on M, given by:

$$\left(Hess_{{}^{Syn}g} f\right)(X,Y) = XYf - (\nabla_X Y) f$$

or equivalently,

$$\left({}^{Syn}\nabla^2 f\right)_{ij} = \partial_i\partial_j f - {}^{Syn}\Gamma^k_{ij}\partial_k f, \quad (3.1)$$

where X, Y are vector fields on M.

Lemma 3.1 *If f is a smooth function on a Riemannian manifold (M, g), then the Hessian (with respect to the synectic lift metric) of its vertical lift is expressed as follows:*

$$\begin{aligned}
&\left({}^{Syn}\nabla^2\ {}^V f\right)_{ij} = \partial_i\partial_j f - {}^{Syn}\Gamma^k_{ij}\partial_k f = \left(\nabla^2 f\right)_{ij},\\
&\left({}^{Syn}\nabla^2\ {}^V f\right)_{i\bar{j}} = 0, \quad (3.2)\\
&\left({}^{Syn}\nabla^2\ {}^V f\right)_{\bar{i}j} = 0,\\
&\left({}^{Syn}\nabla^2\ {}^V f\right)_{\bar{i}\bar{j}} = 0.
\end{aligned}$$

Here, the vertical lift ${}^V f$ of a smooth function f on M is defined as ${}^V f = f \circ \pi$.

3.2 Ricci soliton and gradient Ricci solitons

Let us define a Ricci soliton on a smooth manifold M of dimension $n \geq 2$ as a triple (g, X, λ), where g is a pseudo-Riemannian metric on M, Ric is

the corresponding Ricci tensor, X is a vector field, and λ is a real constant. This triple must satisfy the equation:

$$Ric + \frac{1}{2} L_X g = \lambda g. \tag{3.3}$$

where L_X represents the Lie derivative along X.

A gradient Ricci soliton is characterized by having the vector field X as the gradient of a smooth function f on M. In this scenario, the equation can be expressed as:

$$\left(\nabla^2 f\right) + Ric = \lambda g, \tag{3.4}$$

here ∇^2 being the Hessian operator.

Theorem 3.2 *In the context of a Riemannian manifold (M, g) of dimension $n \geq 2$, where ${}^{Syn}g = {}^{C}g + {}^{V}a$ represents a synectic lift metric on the tangent bundle TM with a symmetric tensor field $a = (a_{ij})$ of type $(0, 2)$ on M, the quartet $(TM, {}^{Syn}g, \widetilde{X}, \lambda)$ is a Ricci soliton if and only if the following conditions are satisfied:*

$$\begin{aligned} &i) \quad \widetilde{X} = v^a E_a + v^{\bar{a}} E_{\bar{a}} = v^a E_a + (y^s A^a_s + B^a) E_{\bar{a}}, \\ &ii) \quad \lambda = \tfrac{1}{n} \left(g^{ij} L_V g_{ij} + A^j_j - \nabla_j v^j \right), \\ &ii) \quad v^c R_{cs} - \nabla_j A^j_s = 0. \end{aligned}$$

Here, λ is a function on TM, and $B = (B^h)$ and $A = (A^h_s)$ represent $(1, 0)$ and $(1, 1)$ tensor fields on M, respectively.

Proof. The presence of the scalar λ can be established by utilizing the $L_{\widetilde{X}}\, {}^{Syn}g$ expression from Lemma 2.6 in equation (3.3):

$$R_{ij} + \frac{1}{2} \left\{ L_V a_{ij} - 2 g_{im} \left(y^b v^c R_{jcb}{}^m - v^{\bar{b}} \Gamma^m_{bj} - E_j v^{\bar{m}} \right) \right\} = \lambda a_{ij} \tag{3.5}$$

and

$$L_V g_{ij} - g_{im} \left(\nabla_j v^m - E_{\bar{j}} v^{\overline{m}} \right) = \lambda g_{ij}. \tag{3.6}$$

Applying $E_{\bar{k}}$ to both sides of equation (3.6) gives:

$$\begin{aligned} E_{\bar{k}} E_{\bar{j}} v^{\bar{m}} &= 0 \\ v^{\bar{m}} &= y^s A^m_s + B^m. \end{aligned} \tag{3.7}$$

Substituting equation (3.7) into equation (3.5) yields:

$$R_{ij} + \frac{1}{2} L_V a_{ij} - g_{im} \left[y^s \left(v^c R_{jcs}{}^m - \nabla_j A^m_s \right) - \nabla_j B^m \right] = \lambda a_{ij}.$$

From the last equation, we infer:

$$g_{im} \left(v^c R_{jcs}{}^m - \nabla_j A^m_s \right) = 0 \tag{3.8}$$

and

$$\frac{1}{2}L_V a_{ij} + R_{ij} + g_{im}\nabla_j B^m = \lambda a_{ij}. \tag{3.9}$$

Contracting with g^{ij} in equation (3.8) gives:

$$v^c R_{cs} - \nabla_j A^j_s = 0. \tag{3.10}$$

Inserting equation (3.7) into equation (3.6) results in:

$$L_V g_{ij} - g_{im}\left(\nabla_j v^m - A^m_j\right) = \lambda g_{ij}. \tag{3.11}$$

Contracting with g^{ij} in equation (3.11) leads to:

$$\lambda = \frac{1}{n}\left(g^{ij}L_V g_{ij} + A^j_j - \nabla_j v^j\right).$$

Conversely, a straightforward calculation confirms the conditions i and ii of the theorem in all cases.

□

Theorem 3.3 *On a Riemannian manifold (M, g) of dimension $n \geq 2$, with ${}^{Syn}g = {}^{C}g + {}^{V}a$ representing a synectic lift metric on the tangent bundle TM where $a = (a_{ij})$ is a symmetric tensor field of type $(0, 2)$ on M, the quartet $(TM, {}^{Syn}g, \widetilde{X}, \lambda)$ forms a gradient Ricci soliton if and only if the following conditions are met:*

$$\begin{array}{ll} i) & \lambda = 0, \\ ii) & \left(\nabla^2 f\right)_{ij} = -R_{ij}. \end{array}$$

Proof. By utilizing Lemma 3.1 and the Ricci curvature tensor of TM with respect to the synectic lift metric in equation (3.4), we can derive the following expressions. Here, ${}^{Syn}\nabla$ and ${}^{Syn}Ric$ represent the Levi-Civita connection and the Ricci curvature tensor of TM with respect to the synectic lift metric g, respectively. Starting from:

$$\left({}^{Syn}\nabla^2 f\right)_{ij} + {}^{Syn}R_{ij} = \lambda\, {}^{Syn}g_{ij},$$

we can deduce:

$$\left(\nabla^2 f\right)_{ij} = \lambda a_{ij} - R_{ij}. \tag{3.12}$$

Given:

$$\left({}^{Syn}\nabla^2 f\right)_{\bar{i}\bar{j}} + {}^{Syn}R_{\bar{i}\bar{j}} = \lambda\, {}^{Syn}g_{\bar{i}\bar{j}},$$

we find that $\lambda = 0$. By substituting this result in equation (3.12), we obtain:

$$\left(\nabla^2 f\right)_{ij} = -R_{ij}.$$

Conversely, a routine calculation confirms the validity of conditions i and ii of the theorem in all scenarios.

□

3.3 Generalized Ricci-Yamabe soliton and generalized gradient Ricci-Yamabe soliton

Guler and Crasmareanu [7] originally introduced the concept of Ricci - Yamabe flow on a Riemannian manifold (M, g) by formulating a scalar combination of the Ricci flow and Yamabe flow represented as:

$$\frac{\partial g}{\partial t}(t) + 2\alpha R(t) + \beta r(t) g(t) = 0. \tag{3.13}$$

In this equation, g signifies the Riemannian metric, R denotes the Ricci tensor, r represents the scalar curvature tensor, and α, β are real numbers.

A Riemannian manifold $(M, g), n > 2$ is recognized to possess a generalized Ricci-Yamabe soliton $(g, V, \lambda, \alpha, \beta, \gamma)$ if it satisfies the equation:

$$L_V g + 2\alpha R = (2\lambda - \beta r)\, g + 2\gamma V^{\#} \otimes V^{\#}. \tag{3.14}$$

Here, λ, α, β and γ are real numbers, and $V^{\#}$ is the 1-form dual to V.

When V is the gradient of a smooth function f on M, the notion transforms into a generalized gradient Ricci-Yamabe soliton, and equation (3.14) simplifies to:

$$\nabla^2 f + \alpha R = \left(\lambda - \frac{1}{2}\beta r\right) g + \gamma df \otimes df. \tag{3.15}$$

The generalized (gradient) Ricci-Yamabe soliton is said to be expanding, steady or shrinking according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$ respectively.

Theorem 3.4 *In the context of a Riemannian manifold (M, g) with dimension $n \geq 2$, let ${}^{Syn}g = {}^{C}g + {}^{V}a$ denote a synectic lift metric on the tangent bundle TM, where $a = (a_{ij})$ is a symmetric tensor field of type $(0, 2)$ on M. $\left({}^{Syn}g, {}^{C}V, \lambda, \alpha, \beta, \gamma\right)$ is categorized as a generalized Ricci-Yamabe soliton if and only if the following conditions are met:*

$$\begin{aligned}
&i)\ \ \lambda = \tfrac{1}{n}\left[2\nabla_i v^i - \gamma v_i (a^i_m v^m + y^s \nabla_s v^i)\right], \\
&ii)\ L_V a_{ij} - \partial_t a_{ij} - 2\lambda a_{ij} - 2\gamma a_{mi} a_{nj} v^m v^n = 0, \\
&iii) g_{im}(v^c R_{jcs}{}^{m} - \nabla_s \nabla_j v^m) + \gamma(a_{mi} v^m \nabla_s v^j + a_{nj} v^n \nabla_s v_i) = 0, \\
&iv)\ \nabla_s v_i \nabla_p v_j = 0.
\end{aligned}$$

Here, the potential vector field is considered the complete lift ${}^{C}V$ of the vector field V on M.

Proof. To establish the presence of the scalar λ, we initiate by noting the scalar curvature tensor under the synectic lift metric is zero. From (3.13) we get

$$2\alpha R_{ij} = -\partial_t a_{ij}. \tag{3.16}$$

Utilizing the information from Lemma 2.6 and equation (3.16 in equation (3.14), we derive:

$$L_V a_{ij} - 2g_{im}\left(y^b v^c R_{jcb}{}^{m} - v^{\bar{b}} \Gamma^{m}_{bj} - E_j v^{\bar{m}}\right) - \partial_t a_{ij} = 2\lambda a_{ij} + 2\gamma V^{\#} \otimes V^{\#} \tag{3.17}$$

and

$$2\left\{L_V g_{ij} - g_{im}\left(\nabla_j v^m - E_{\bar{j}} v^{\overline{m}}\right)\right\} = 2\lambda g_{ij} + 2\gamma V^{\#} \otimes V^{\#}. \tag{3.18}$$

The potential vector field ${}^{C}V$ and the $1-$form dual to ${}^{C}V$ are described as:

$${}^{C}V = \begin{pmatrix} v^m \\ v^{\bar{m}} \end{pmatrix} = \begin{pmatrix} v^m \\ y^s \nabla_s v^m \end{pmatrix} \tag{3.19}$$

and

$$V_j^{\#} = v^I \tilde{g}_{Ij} = v^i \tilde{g}_{ij} + v^{\bar{i}} \tilde{g}_{\bar{i}j} = a_{ij} v^i + g_{ij} v^{\bar{i}} = a_{ij} v^i + y^s \nabla_s v_j,$$
$$V_{\bar{j}}^{\#} = v^I \tilde{g}_{I\bar{j}} = v^i \tilde{g}_{i\bar{j}} + v^{\bar{i}} \tilde{g}_{\bar{i}\bar{j}} = v^i g_{ij} = v_j,$$

from which

$$(V^{\#}) = \begin{pmatrix} a_{ij} v^i + y^s \nabla_s v_j \\ v_j \end{pmatrix}. \tag{3.20}$$

Upon substituting equations (3.19) and (3.20) into equation (3.18), we arrive at:

$$\nabla_i v^j + \nabla_j v^i - \gamma v_i (a_{mj} v^m + y^s \nabla_s v_j) = \lambda g_{ij}.$$

Contracting both sides of the above equation with g^{ij}, we determine the scalar λ as:

$$\lambda = \frac{1}{n}\left[2\nabla_i v^i - \gamma v_i (a_m^i v^m + y^s \nabla_s v^i)\right].$$

Substitution of equations (3.19) and (3.20) into equation (3.17) yields:

$$L_V a_{ij} - \partial_t a_{ij} - 2\lambda a_{ij} - 2\gamma a_{mi} a_{nj} v^m v^n = 0,$$

$$g_{im}(v^c R_{jcs}{}^m - \nabla_s \nabla_j v^m) + \gamma(a_{mi} v^m \nabla_s v^j + a_{nj} v^n \nabla_s v_i) = 0$$

and

$$\nabla_s v_i \nabla_p v_j = 0.$$

Through standard calculations, the fulfillment of conditions $i - iv$ of the theorem can be verified.

□

Theorem 3.5 *In a Riemannian manifold (M, g) of dimension $n \geq 2$, where the metric g is a synectic lift metric on the tangent bundle TM given by ${}^{Syn}g = {}^{C}g + {}^{V}a$, with $a = (a_{ij})$ being a symmetric tensor field of type $(0,2)$ on M, $({}^{Syn}g, f, \lambda, \alpha, \beta, \gamma)$ is classified as a generalized gradient Ricci-Yamabe soliton if and only if the following condition is satisfied:*

$$\lambda = \frac{1}{Tr_g a}\left[a^{ij}(\nabla^2 f)_{ij} + \alpha a^{ij} R_{ij} - \gamma a^{ij} \partial_i f \partial_j f\right].$$

Proof. Using the expression from Lemma 3.1 as denoted in 3.15, we find:

$$(\nabla^2 f)_{ij} + \alpha R_{ij} = \lambda a_{ij} + \gamma \partial_i f \partial_j f.$$

By taking the contraction with a^{ij} on both sides of the above equation, we derive:

$$a^{ij}(\nabla^2 f)_{ij} + \alpha a^{ij} R_{ij} - \gamma a^{ij} \partial_i f \partial_j f = \lambda Tr_g a.$$

This leads to the conclusion:

$$\lambda = \frac{1}{Tr_g a} \left[a^{ij}(\nabla^2 f)_{ij} + \alpha a^{ij} R_{ij} - \gamma a^{ij} \partial_i f \partial_j f \right],$$

where $Tr_g a = g^{ij} a_{ij}$. Hence, the proof is now fully established.

□

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