ASUPRA EVALUĂRII ERORII LA REZOLVAREA SISTEMELOR LINIARE

Rezumat

Se studiază problema aproximării soluției sistemului de ecuații liniare algebrice (2), cînd în locul acestui sistem se consideră sistemul (4). Evaluările erorii obținute pot fi utile în economie, la calculul prețului de cost.

SOME REMARKS TO THE PAPER PROFESSOR ADOLF HAIMOVICI, SUR QUELQUES INVARIANTS DANS LES ESPACES À CONNEXION AFFINE À TROIS DIMENSIONS

BY

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Dear Professor Haimovici,

In reading your interesting paper „Sur quelques invariants dans les espaces à connexion affine à trois dimensions". (Annales Polonici Mathematici, III, 2, 1957, pp. 300-303), I came across some ideas connected with the topic of your paper which I'd like to bring to your attention in what follows. I am using a little bit different notation than you did, but you will see that this does not confuse the issue.

I.

Denote by A_n a n-dimensional space referred to coordinates x^ν and endowed with a symmetric connection $\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\mu\lambda}^{\nu}$.

Definition (I.1). *Let* $\underset{a}{X}{}^\nu$ $(a,b = 1, \dots, p)$ *be a set of* p *contravariant vectors. Let*

$$f \overset{\text{def}}{=} f(x, \underset{a}{X}) \tag{I.1}$$

represent a scalar function of the coordinates x^ν *and of the vectors* $\underset{a}{X}{}^\nu$, *homogeneous of degree zero in the* $\underset{a}{X}{}^\nu$.

This Definition gives rise to the following

Lemma (I.1). *The* $p \cdot n$ *expressions*

$$\overset{a}{X}_\lambda \overset{\text{def}}{=} \frac{\partial f}{\partial \underset{a}{X}{}^\lambda} \tag{I.2}$$

are components of p *covariant vectors satisfying the conditions*

$$\overset{1}{X}_\lambda \underset{1}{X}{}^\lambda = \overset{2}{X}_\lambda \underset{2}{X}{}^\lambda = \dots = \overset{p}{X}_\lambda \underset{p}{X}{}^\lambda = 0. \tag{I.3}$$

The proof is obvious.

Remark (I.1) If we put

$$X^{\nu}_{\lambda} \overset{\text{def}}{=} \overset{a}{X}_{\lambda} \underset{a}{X}^{\nu} \tag{I.4)a}$$

then we have by virtue of (I.3)

$$X^{\lambda}_{\lambda} = 0. \tag{I.4)b}$$

If ∇_{μ} is the symbol of covariant derivative w. r. to the connection $\Gamma^{\nu}_{\lambda\mu}$ we can write according to (I.4)

$$-X^{\lambda} \nabla_{\mu} \overset{a}{X}_{\lambda} = \overset{a}{X}_{\lambda} \nabla_{\mu} \underset{a}{X}^{\lambda} \tag{I.5}$$

II.

Let us denote by the symbol $f_{,\lambda}$ the derivative of f w. r. to x^{λ} *which is considered as independent of the vectors* $\underset{a}{X}^{\nu}$. It should be stressed that in spite of the fact that f is a scalar, $f_{,\lambda}$ *is not necessarily vector* (gradient). Using this symbol we have obviously

$$\partial f / \partial x^{\mu} = f_{,\mu} + \overset{a}{X}_{\nu}\, \partial \underset{a}{X}^{\nu} / \partial x^{\mu}. \tag{II.1}$$

Theorem (II.1)a. *A necessary and sufficient condition that f be constant if and only if the vectors $\underset{a}{X}^{\nu}$, $\overset{b}{X}_{\lambda}$ are subject to the restriction*

$$\overset{a}{X}_{\lambda} \nabla_{\mu} \underset{a}{X}^{\lambda} (= -X^{\lambda} \nabla_{\mu} \overset{a}{X}_{\lambda}) = 0, \tag{II.2)a}$$

is

$$f_{,\mu} - \Gamma^{\lambda}_{\nu\mu} X^{\nu}_{\lambda} = 0, \tag{II.3}$$

i.e., your equation (3).

Proof. The equation (II.2)a is equivalent to

$$\overset{a}{X}_{\lambda}\, \partial \underset{a}{X}^{\lambda} / \partial x^{\mu} = -\Gamma^{\lambda}_{\nu\mu} X^{\nu}_{\lambda}. \tag{II.2)b}$$

Hence (II.2)b is satisfied if and only if (II.2)a is satisfied. On the other hand a necessary and sufficient condition for the constancy of f is on account of (II.1)

$$f_{,\mu} + \overset{a}{X}_{\lambda}\, \partial \underset{a}{X}^{\lambda} / \partial x^{\mu} = 0. \tag{II.4}$$

The elimination of the terms $\overset{a}{X}_{\lambda}\, \partial \underset{a}{X}^{\lambda} / \partial x^{\mu}$ from (II.2)b and (II.4) leads to (II.3). Assume now that (II.3) is satisfied. Then (II.3) can be converted into (II.4) if and only if (II.2)b (or its equivalent (II.2)a) is satisfied. In other words f satisfying (II.3) is constant if and only if (II.2)a holds.

Theorem (II.1)b. *Assume that* (I.3) *is satisfied. Then the condition* (II.3) *does not lead necessarily to*

$$\nabla_{\mu} \underset{a}{X}^{\nu} = 0 \tag{II.5}$$

which satisfies (II.2).

Proof. It is sufficient to exhibit one example where (I.3) is satisfied but the condition (II.3) does not lead necessarily to (II.5). Consider the case $p = 1$, denote by a_{μ}, b_{λ} two vectors and put

$$f = \frac{a_{\lambda} X^{\lambda}}{b_{\mu} X^{\mu}}, \quad (b_{\mu} X^{\mu} \neq 0). \tag{II.6)a}$$

Then

$$X_{\lambda} \overset{\text{def}}{=} \frac{\partial f}{\partial X_{\lambda}} = \frac{(a_{\lambda} b_{\mu} - a_{\mu} b_{\lambda})}{(b_{\alpha} X^{\alpha})^2} X^{\mu} \tag{II.6)b}$$

and

(II.6)

$$\text{c)}\ f_{,\mu} = \frac{b_{\alpha}\, \partial a_{\beta} / \partial x^{\mu} - a_{\alpha}\, \partial b_{\beta} / \partial x^{\mu}}{(b_{\gamma} X^{\gamma})^2} X^{\alpha} X^{\beta}$$

$$\text{d)}\ \Gamma^{\lambda}_{\nu\mu} X^{\nu}_{\lambda} = \frac{1}{(b_{\gamma} X^{\gamma})^2} \Gamma^{\lambda}_{\nu\mu} (a_{\lambda} b_{\alpha} - a_{\alpha} b_{\lambda}) X^{\alpha} X^{\nu}.$$

An easy inspection based on (II.6) bcd shows that

$$f_{,\mu} - \Gamma^{\lambda}_{\nu\mu} X^{\nu}_{\lambda} = \frac{X^{\alpha} X^{\beta}}{(b_{\gamma} X^{\gamma})^2} (b_{\alpha} \nabla_{\mu} a_{\beta} - a_{\beta} \nabla_{\mu} b_{\alpha}) \tag{II.7}$$

while by virtue of (I.5) and (II.6)b

$$X^{\lambda} \nabla_{\mu} X_{\lambda} = -X_{\lambda} \nabla_{\mu} X^{\lambda} = \frac{-(a_{\alpha} b_{\beta} - a_{\beta} b_{\alpha}) X^{\beta} \nabla_{\mu} X^{\alpha}}{(b_{\gamma} X^{\gamma})^2}. \tag{II.8}$$

Comparing (II.7) and (II.8) one sees at once that the condition (II.3) does not necessarily lead to (II.5) in this case.

Remark (II.1). Taking the covariant derivative of the identity

$$X^{\alpha} X^{\beta} (a_{\alpha} b_{\beta} - a_{\beta} b_{\alpha}) = 0$$

one finds easily that in the case (II.6)a the equations (II.2)a and (II.3) are compatibile and lead to a constant f.

III.

In this Section we shall deal with your integrability conditions of (II.3). If $R_{\omega\mu\lambda}{}^{\nu}$ is the curvature tensor of the connection then your integrability conditions (4_1), (4_2) ... $(4'_1)$ ... are equivalent to

$$
\text{(III.1)}\qquad
\begin{aligned}
&\text{a)}\ R_{\omega\mu\lambda}{}^{\nu}\underset{\nu}{X}^{\lambda}\\
&\text{b)}\ (\nabla_{\xi_s}\cdots\nabla_{\xi_1} R_{\omega\mu\lambda}{}^{\nu})\underset{\nu}{X}^{\lambda}=0 \quad s=1,2,3,\ldots\\
&\text{c)}\ (R_{\omega\mu\lambda}{}^{\alpha}R_{\rho\sigma\alpha}{}^{\nu}-R_{\rho\sigma\lambda}{}^{\alpha}R_{\omega\mu\alpha}{}^{\nu})\underset{\nu}{X}^{\lambda}=0\\
&\text{d)}\ [\nabla_{\xi_s}\cdots\nabla_{\xi_1}(R_{\omega\mu\lambda}{}^{\alpha}R_{\rho\sigma\alpha}{}^{\nu}-R_{\rho\sigma\lambda}{}^{\alpha}R_{\omega\mu\alpha}{}^{\nu})]\underset{\nu}{X}^{\lambda}=0.
\end{aligned}
$$

In what follows we always assume

$$\text{(III.1)e}\qquad R_{\omega\mu\lambda}{}^{\nu}\neq 0\,.$$

In order to derive some consequences from (III.1) we need the following

T h e o r e m (III.1) *The first integrability conditions of* (II.2)a *are*

$$\text{(III.2)}\qquad 2(\nabla_{[\omega}\underset{a}{X}^{\lambda})\nabla_{\mu]}\overset{a}{X}_{\lambda}+R_{\omega\mu\lambda}{}^{\nu}\underset{\nu}{X}^{\lambda}=0.$$

Proof. Applying to the second equation of (II.2)a the operator ∇_ω and alternating it with ∇_μ one obtains

$$(\nabla_\omega\underset{a}{X}^{\lambda})\nabla_\mu\overset{a}{X}_{\lambda}-(\nabla_\mu\underset{a}{X}^{\lambda})\nabla_\omega\overset{a}{X}_{\lambda}+\underset{a}{X}^{\lambda}(\nabla_\omega\nabla_\mu-\nabla_\mu\nabla_\omega)\overset{a}{X}_{\lambda}=0.$$

On the other hand we have the identity

$$2\underset{a}{X}^{\lambda}\nabla_{[\omega}\nabla_{\mu]}\overset{a}{X}_{\lambda}=\underset{a}{X}^{\lambda}(\nabla_\omega\nabla_\mu-\nabla_\mu\nabla_\omega)\overset{a}{X}_{\lambda}=R_{\omega\mu\lambda}{}^{\nu}\underset{\nu}{X}^{\lambda}\,.$$

The last two equations lead at once to (III.2).

In the next corollary we shall use the words ,,weak condition (II.2)a" which means that this condition is not a consequence of a stronger condition, like (II.5).

Corollary (III.1). *In your paper you do not consider the weak condition* (II.2)a.

Proof. The equation (II.3) ia a consequence of (II.2)a and of

$$\text{(III.3)}\qquad \partial f/\partial x^{\mu}=0\,.$$

Hence the first integrability condition of (II.3) must be a consequence of the first integrability condition (III.2) of (II.2)a and the first integrability condition (III.1) of (III.3). Comparing these two sets of integrability conditions we see that they are compatibile if and only if *the following new condition for f*

$$\text{(III.4)}\qquad (\nabla_\omega\underset{a}{X}^{\lambda})\nabla_\mu\overset{a}{X}_{\lambda}-(\nabla_\mu\underset{a}{X}^{\lambda})\nabla_\omega\overset{a}{X}_{\lambda}=0^{*})$$

is satisfied. In your paper you do not use this condition and therefore you do not consider the weak condition (II.2)a.

*) The integrability conditions of (III.4) impose a new set of conditions on f.

Remark (III.1). Observe that (III.4) is identically satisfied by (II.5) as it was to be expected.

It is known*) that if the holonomy group G_r of A_n if of r parameters then there is a set of r tensors $\underset{A}{\mathfrak{L}}{}^{\nu}_{\lambda}$ with the following properties. They

a) are linearly independent,

b) are defined up to transformations

$$\underset{A'}{\mathfrak{L}}{}^{\nu}_{\lambda}=T^{G}_{A'}\,\underset{G}{\mathfrak{L}}{}^{\nu}_{\lambda}\,,$$

where the coefficients $T^{G}_{A'}$ are subject only to one condition, namely

$$\operatorname{Det}\,((T^{G}_{A'}))\neq 0,$$

c) describe the $(^{\nu}_{\lambda})$-domain of all tensors

$$\text{(III.5)}\qquad R_{\omega\mu\lambda}{}^{\nu},\quad \nabla_{\xi_s}\cdots\nabla_{\xi_1}R_{\omega\mu\lambda}{}^{\nu},\quad s=1,\,2,\,\ldots$$

so that

$$\text{(III.6)}\qquad
\begin{gathered}
R_{\omega\mu\lambda}{}^{\nu}=\overset{A}{M}_{\omega\mu}\,\underset{A}{\mathfrak{L}}{}^{\nu}_{\lambda}\\
\nabla_{\xi_s}\cdots\nabla_{\xi_1}R_{\omega\mu\lambda}{}^{\nu}=\overset{A}{M}_{\xi_s\cdots\xi_1\omega\mu}\underset{A}{\mathfrak{L}}{}^{\nu}_{\lambda}\\
s=1,2,\,\ldots
\end{gathered}
$$

The tensors M are uniquely defined by the set (III.5) and the tensors $\underset{A}{\mathfrak{L}}{}^{\nu}_{\lambda}$. If $c_{AG}^{\ \ H}$ are the structural ,,constants" of G_r then

$$\text{(III.7)}\qquad \underset{A}{\mathfrak{L}}{}^{\alpha}_{\nu}\,\underset{G}{\mathfrak{L}}{}^{\lambda}_{\alpha}-\underset{G}{\mathfrak{L}}{}^{\alpha}_{\nu}\,\underset{A}{\mathfrak{L}}{}^{\lambda}_{\alpha}=c_{AG}^{\ \ H}\,\underset{H}{\mathfrak{L}}{}^{\lambda}_{\nu}\,,$$

We shall use these results for simplifying your equations (III.1). The following three Theorems will be proved simultaneously.

T h e o r e m (III.2)a. *The infinite set of equations* (III.1)ab *is equivalent to r equations*

$$\text{(III.8)}\qquad \underset{A}{\mathfrak{L}}{}^{\lambda}_{\nu}\underset{\lambda}{X}^{\nu}=0.$$

T h e o r e m (III.2)b. *The infinite set of equations* (III.1)c)d) *is a consequence of* (III.8).

T h e o r e m (III.2)c. *A necessary and sufficient condition that the set* (III.1) cd *be satisfied even if the set* (III.1)ab *is not satisfied is*: *The holonomy group G_r is Abelian.*

Proof. The statement about (III.8) follows from the fact that every

*) See for more detail: H l a v a t ý, *The holonomy group* I. ... , VI, Journal of Math. & Mechanics, Vol. 8, 285—308, 597—622; Vol. 9, 89—122, 453—496; Vol. 10, 317—348; Vol. 11, 35—60.—The German indices run from 1 to r.

tensor in the infinite set (III.5) which is not zero has the property that its $\binom{\nu}{\lambda}$-domain is either identical with or contained in the $\{\binom{\nu}{\lambda}\}$-domain spanned by the tensors $\underset{A}{\mathfrak{L}}{}^{\lambda}_{\nu}$ (Cfr. (III.6).) Using the first equations of (III.6) we may rewrite (III.1)c in the form

$$\overset{A}{M}_{\omega\mu}\overset{G}{M}_{\rho\sigma}(\underset{A}{\mathfrak{L}}{}^{\lambda}_{\alpha}\underset{G}{\mathfrak{L}}{}^{\alpha}_{\nu} - \underset{G}{\mathfrak{L}}{}^{\alpha}_{\lambda}\underset{A}{\mathfrak{L}}{}^{\nu}_{\alpha})X^{\lambda}_{\nu} = 0,$$

or according to (III.7)

$$\overset{A}{M}_{\omega\mu}\overset{G}{M}_{\rho\sigma}c_{AG}^{\ \ H}\underset{H}{\mathfrak{L}}{}^{\nu}_{\lambda}X^{\lambda}_{\nu} = 0.$$

Hence (III.1)c is a consequence of (III.8), i.e. of (III.1)ab. The proof concerning (III.1)d follows the same pattern. All the equations (III.1)cd reduce then to $r(r-1)/2$ equations

$$c_{AG}^{\ \ H}\underset{H}{L}{}^{\gamma}_{\lambda}X^{\lambda}_{\gamma} = 0. \tag{III.9}$$

Hence if the group is Abelian

$$c_{AG}^{\ \ H} = 0,$$

then (III.9) and consequently also (III.1)cd are always satisfied even if (III.1)ab are not.

According to the previous Theorems it is sufficient to consider only the finite set (III.8) instead of the double infinite set (III.1).

T h e o r e m (III.3). *A necessary and sufficient condition for the system* (III.8) (*equivalent to* (III.1)) *to admit a non-trivial solution* X^{ν}_{λ} *is: The* $n^2 \times r$ *matrix of the array*

$$\begin{matrix} \underset{1}{\mathfrak{L}}{}^{1}_{1} & \underset{1}{\mathfrak{L}}{}^{2}_{1} & \ldots & \underset{1}{\mathfrak{L}}{}^{n}_{n} \\ \vdots & \vdots & & \\ \underset{r}{\mathfrak{L}}{}^{1}_{1} & \underset{r}{\mathfrak{L}}{}^{2}_{1} & \ldots & \underset{r}{\mathfrak{L}}{}^{n}_{n} \end{matrix} \tag{III.10}$$

is of rank $\rho < n^2$. *If this is the case then there are exactly* $m = n^2 - \rho$ *linearly independent solutions*

$$\underset{(1)}{X}{}^{\lambda}_{\nu}, \ldots, \underset{(m)}{X}{}^{\lambda}_{\nu}$$

each defined up to an arbitrary factor.

The proof is elementary.

Remark (III.2). Let us stess again that the system (III.8) *does not represent integrability conditions of the weak equation* (II.2) unless special conditions are prescribed. (Cf. Theorems (III.1) and (II.1)b).)

IV.

Theorem (III.1) shows that the integrability conditions of (II.2)a *are in general different from your equations* (III.1). The question arises: *What condition should one prescribe in order that both sets* (II.2)a *and* (II.3) *have the same integrability condition*? This question is of paramount importance because the integrability conditions (III.1) of (II.3) are not necessarily integrability conditions of the weak condition (II.2)a so that the solution X^{λ}_{ν} of (III.8) *does not necessarily satisfy the weak condition* (II.2)a.

You answered this question choosing one special condition reducible to (II.5). Indeed you have written at the beginning of your interesting paper „*il s'agit de trouver les invariants au transport paralèlle d'un système de plusieurs directions...*" *You used the equation* (1), *i. e.*

$$dX^{\nu} + \Gamma^{\ \nu}_{\lambda\mu}dx^{\mu}X^{\lambda} = 0, \tag{IV.1}$$

which — according to your following text — *should hold for any direction* and this is satisfied if and only if the strong condition (III.5) is satisfied. In order to consider this situation we shall need the following

L e m m a (IV.1). *The integrability conditions of*

$$\nabla_{\mu}Y^{\nu} = Y^{\nu}\partial_{\mu}\ln\rho \tag{IV.2}$$

are

$$\begin{aligned} &\text{a)}\ R_{\omega\mu\lambda}^{\ \ \ \ \nu}Y^{\lambda} = 0, \\ &\text{b)}\ (\nabla_{\xi_s}\cdots\nabla_{\xi_1}R_{\omega\mu\lambda}^{\ \ \ \ \nu})Y^{\lambda} = 0 \quad s = 1, 2, 3, \ldots \end{aligned} \tag{IV.3}$$

If they are satisfied, then for every solution Y^{ν} *of* (IV.3) *there is an appropriate scalar* ρ *such that* (IV.2) *is satisfied. Moreover, the vector*

$$X^{\nu} \overset{\text{def}}{=} Y^{\nu}\rho^{-1}\ \text{const} \tag{IV.4}$$

satisfies (II.5).

Proof. Applying to (IV.2) the operator ∇_{ω} and alternating it with ∇_{μ} one obtains

$$(\nabla_{\omega}\nabla_{\mu} - \nabla_{\mu}\nabla_{\omega})Y^{\nu} = 0$$

so that the known identity

$$(\nabla_{\omega}\nabla_{\mu} - \nabla_{\mu}\nabla_{\omega})Y^{\nu} = -R_{\omega\mu\lambda}^{\ \ \ \ \nu}Y^{\lambda}$$

leads to the first of (IV.3). The next equation is obtained by a very well known procedure. Applying the operator ∇_{ξ} to (IV.3)a one gets

$$(\nabla_{\xi}R_{\omega\mu\lambda}^{\ \ \ \ \nu})Y^{\nu} + R_{\omega\mu\lambda}^{\ \ \ \ \nu}\nabla_{\xi}Y^{\lambda} = 0.$$

But

$$R_{\omega\mu\lambda}^{\ \ \ \ \nu}\nabla_{\xi}Y^{\lambda} = R_{\omega\mu\lambda}^{\ \ \ \ \nu}(\partial_{\xi}\ln\rho)Y^{\lambda} = 0.$$

The remaining equations are obtained in the same way. The system (IV.3) is homogeneous in Y and therefore if a solution Y exists, it is known up to a factor of proportionality. Therefore, to every such solution there is an appropriate scalar ρ such that (IV.2) is satisfied. On the other hand we obtain from (IV.4) and (IV.2)

$$\frac{1}{\text{const}}\nabla_\mu X^\nu = -\rho^{-2}(\partial\rho/\partial x^\mu)Y^\nu + \rho^{-1}\nabla_\mu Y^\nu = \rho^{-2}Y^\nu(-\partial\rho/\partial x^\mu + \partial\rho/\partial x^\mu) = 0.$$

Remark (IV.1). It is clear that (IV.2) and (II.5) have the same integrability conditions. Therefore, we may limit ourselves to one of these two equations, say (II.5). Moreover, we proved (Theorem (III.2)b) that the integrability conditions (III.1)cd of (II.3) are satisfied, whenever (III.1)ab are satisfied. Hence we could limit ourselves to (III.1)ab.

T h e o r e m (IV.1). *The integrability conditions of* (II.5) *constitute a final set of* $r\ n$ *equations*

$$\text{(IV.5)a} \qquad \underset{A}{\pounds}_\lambda^\nu X^\lambda = 0.$$

The integrability conditions (III.8) *of* (II.3) *are always satisfied, whenever* (IV.5)a *are satisfied and therefore do not yield any new information concerning* f.

Proof. The integrability conditions of (II.5) are (IV.3) where X has to be substituted for Y. Using the method of the proof of Theorem (III.2)a one finds easily that the above mentioned conditions are equivalent to (IV.5)a. Assume that the system (IV.5)a admits exactly m linearly independent solutions*). If $m \geqslant p$ we take just p of these solutions $\underset{a}{X}^\nu$ which satisfy (IV.5)a, i.e.

$$\text{(IV.5)b} \qquad \underset{A}{\pounds}_\lambda^\nu \underset{a}{X}{}^\lambda_\nu = 0.$$

Whatever are the vectors $\overset{a}{X}_\nu$ *satisfying* (I.4) we get at once (III.8) as a consequence of (IV.5)b and (I.4)a. Therefore, these equations do not yield any new information concerning the vectors $\overset{a}{X}_\nu = \partial f/\partial x^\nu$ and therefore also they do not yield any new information on f. If $m < p$, we take the m linearly independent solution and complete this set by $p - m$ linear combinations of these solutions with constant coefficients. In this way we get again (IV.5)b and the statement about f holds even here.

In the next Theorem we shall also make use of the consequences of Theorems (III.1), (III.2), and (IV.1).

*) Necessary and sufficient conditions for the existence of at least one non-trivial solution are exhibited in Theorem (IV.3)a.

T h e o r e m (IV.2). *Integrability conditions* (III.8) *equivalent to* (III.1) a—d) *pertaining to the problem defined by* (II.2) *and* (II.3) *are necessary but in general*) not sufficient to the existence of a solution* f *of the problem. — Integrability conditions* (IV.5)a *are necessary and sufficient for the existence of a solution* f *of the problem defined by* (II.3) *and* (II.5). *If they are satisfied, then* (II.3) *and* (II.5) *are completely integrable. However, this equation* (II.3) *does not yield necessarily any pertinent information on the structure of the solution* f.

Proof. The part of Theorems (IV.2) dealing with the problem defined by (II.2) and (II.3) has been already proved. (Cf. Theorems (II.1) b, (III.1), (III.2), and Remark (III.2)).—It has been proved in Theorem (IV.1) that integrability conditions of (II.3) are the consequence of the integrability conditions (IV.5) of (II.5). Hence if (IV.5) are satisfied, then the system (II.3) is completely integrable. Therefore the only part of Theorem (IV.2) to be proved is the last part of it. To this effect it is sufficient to prove that at least in one example the last part of Theorem (IV.2) holds: Consider the case $n = 4$ and the connection $\Gamma_{\lambda\mu}^{\ \nu} = \Gamma_{\mu\lambda}^{\ \nu}$ given by

$$\text{(IV.6) a} \quad \begin{cases} \Gamma_{\omega^*2}^{\ \ 2} = \partial \ln \lambda/\partial x^2, \quad \Gamma_{22}^{4} = -\dfrac{1}{2}\,\partial \ln \lambda^2/\partial x^3 \\ \text{the remaining } \Gamma_{\lambda\mu}^{\ \nu} = 0, \quad \omega^* = 2, 3, \end{cases}$$

where

$$\text{(IV.6) b} \qquad \lambda = \lambda(x^2, x^3).$$

It could be very easily established that the connection (IV.6) admits *only* the following parallel vector fields

$$w^\nu = A\delta_1^\nu + B\delta_4^\nu \text{ **)} \quad A, B = \text{const.}$$

Hence if we assume $p = 3$ we could define three fields of parallel vectors

$$\text{(IV.7) a} \qquad \underset{1}{X}^\nu = \delta_1^\nu, \quad \underset{2}{X}^\nu = \delta_4^\nu, \quad \underset{3}{X}^\nu = \delta_1^\nu - \delta_4^\nu$$

so that according to (I.3) we must have

$$\text{(IV.7) b} \qquad \overset{1}{X}_\nu = (O\overset{1}{X}_2\overset{1}{X}_3\overset{1}{X}_4), \quad \overset{2}{X}_\nu = (\overset{2}{X}_1\overset{2}{X}_2\overset{2}{X}_3 O), \quad \overset{3}{X}_\nu = (\xi\overset{3}{X}_2\overset{3}{X}_3\xi),$$

*) i.e., in the case when the weak condition (II.2) is not a consequence of a stronger condition like f. i. the condition (II.5).

**) See V. H l a v a t ý, *Weierstrass Characteristic and Covariant Constant Vector Fields* (Journal of Math. & Mech., Vol. 14, 737–768), p. 765.

where the components in (IV.7) b are *unknown*. We obtain from (IV.7) by virtue of (I.4) a

(IV.7) c
$$X^{\nu}_{\lambda} = \delta^{\nu}_{1} M_{\lambda} + \delta^{\nu}_{4} N_{\lambda}$$

where M_{λ} and N_{λ} are functions of the components involved in (IV.7) b. An easy inspection shows that

$$\overset{a}{X}_{\lambda} \partial \underset{a}{X}^{\lambda} / \partial x^{\mu} = M_{\lambda} \partial \delta^{\nu}_{1} / \partial x^{\mu} + N_{\lambda} \partial \delta^{\nu}_{4} / \partial x^{\mu}$$

and therefore according to (II.5)*), (II.4), and (II.3) we must have also

(IV.8)
$$\text{a)} \quad \Gamma_{\lambda\mu}^{\ \ \nu} X^{\lambda}_{\nu} = 0, \qquad \text{b)} \quad f_{,\mu} = 0.$$

The first equation reduces to identity $0 = 0$ (See (IV.6) and (4.7) c) and the second equation does not give us any pertinent information about f. One of the reasons is in particular the fact that in the coordinate system in which (IV.6) holds the vectors $\underset{a}{X}^{\nu}$ have *constant components*. Therefore, if a_{λ}, b_{λ}, c_{λ} are vectors which *in the actual coordinate system* have constant components then f could be f.i. a function of three expressions

$$\frac{a_{\lambda} b_{\mu} \underset{1}{X}^{\lambda} \underset{2}{X}^{\mu}}{a_{\alpha} b_{\beta} \underset{2}{X}^{\alpha} \underset{1}{X}^{\beta}}, \quad \frac{b_{\lambda} c_{\mu} \underset{2}{X}^{\lambda} \underset{3}{X}^{\mu}}{b_{\alpha} c_{\beta} \underset{3}{X}^{\alpha} \underset{2}{X}^{\beta}}, \quad \frac{c_{\lambda} a_{\mu} \underset{3}{X}^{\lambda} \underset{1}{X}^{\mu}}{c_{\alpha} a_{\beta} \underset{3}{X}^{\beta} \underset{1}{X}^{\alpha}}$$

(provided that neither one of the vectors $\underset{1}{X}^{\nu}, \underset{2}{X}^{\nu}, \underset{3}{X}^{\nu}$ is incident with either one of a_{λ}, b_{λ}, c_{λ}, which can always be achieved. Then (IV.7) b is satisfied but it does not give us any pertinent information about the structure of f, for (IV.7)b *could be satisfied by other functions of* $\underset{1}{X}^{\nu}, \underset{2}{X}^{\nu}, \underset{3}{X}^{\nu}$.

In the next Theorems we denote by $\underset{A}{((\mathfrak{L}))}$ the $n \times n$ matrix of the array

$$\begin{matrix} \underset{A}{\mathfrak{L}}{}^{1}_{1}, & \underset{A}{\mathfrak{L}}{}^{1}_{2}, & \dots, & \underset{A}{\mathfrak{L}}{}^{1}_{n} \\ \vdots & & & \\ \underset{A}{\mathfrak{L}}{}^{n}_{1}, & \underset{A}{\mathfrak{L}}{}^{n}_{2}, & \dots, & \underset{A}{\mathfrak{L}}{}^{n}_{n} \end{matrix}$$

*) If (II,5) is satisfied, so is (2,2)b,

and by $\mathfrak{L}$ the $nr \times n$ matrix

(IV.9)
$$\mathfrak{L} \overset{\text{def}}{=} \left\| \begin{matrix} \underset{1}{((\mathfrak{L}))} \\ \cdot \\ \cdot \\ \underset{r}{((\mathfrak{L}))} \end{matrix} \right\|.$$

The following two Theorems will be proved simultaneously.

T h e o r e m (IV.3) b. *A necessary and sufficient condition for the existence of a non-trivial solution of the problem defined by* (II.3) *and* (II.5) *is: The matrix* $\mathfrak{L}$ *defined by* (IV.9) *has the rank* $\rho < n$. *If this is the case then there are exactly* $m = n - \rho$ *linearly independent solutions of* (IV.5) *which satisfy* (IV.5) b.*)

T h e o r e m (IV.3) b. *If* $m = n$ *the space* A_n *is flat, i.e. the condition* (III.1)e *is not satisfied.*

Proof. We know Theorem (IV.2) that (IV.5)a are integrability conditions for the existence of a solution of the problems defined by (II.3), (II.5). In order to construct (II. 3) we need first the vectors $\underset{a}{X}^{\nu}$ satisfying (II.5). By virtue of Theorem (IV.1) the construction of $\underset{a}{X}^{\nu}$ is obvious and it is described in its main features by Theorem (IV.3)a which is an application of linear algebra to (IV.5)a. — If $m = n$ we have $\rho = 0$ and consequently $\underset{A}{\mathfrak{L}}{}^{\nu}_{\lambda} = 0$ so that according to (III.6) we have

$$R_{\omega\mu\lambda}^{\ \ \ \nu} = 0.$$

Remark (IV.2). It is interesting that in the Riemann space ρ must satisfy the condition $\rho < n - 1$ if (II.1)e should be satisfied.

Corollary (IV.1). *The maximal possible number of linearly independent solutions of* (II.5) *in a non flat space* A_n *is* $m = n - 1$.

Remark (IV.3). In view of this Corollary your invariants I and J introduced by you in (I), p. 301

(IV.10)
$$I = \frac{(XUV)}{(YUV)} : \frac{(XZV)}{(YZV)}, \quad J = \frac{(XYZ)}{(VYZ)} : \frac{(XUZ)}{(VUZ)}$$

are undefined for (III.1) e. Indeed for $n = 3$ we have in a non flat space only two linearly independent covariant constant vector fields, say U, V. The remaining covariant constant vector fields X, Y, Z must be linear combinations of U, V with constant coefficients. Hence all determinants in (IV.10) are equal to zero. Similar remark holds also for your invariants J, I, defined by III, p. 302.

*) Cf. Remark (IV.1).

UNELE OBSERVAŢII ASUPRA ARTICOLULUI „SUR QUELQUES INVARIANTS DANS LES ESPACES A CONNEXION AFFINE A TROIS DIMENSIONS" (Annales Polonici Mathematici, III, 2, 1957, 300–303) DE PROF. ADOLF HAIMOVICI

Rezumat

Pornind de la articolul citat în titlu, autorul demonstrează între altele că o condiţie necesară şi suficientă pentru existenţa unei soluţii nebanale a problemei definite prin (II.3) şi (II.5) este ca matricea $\mathfrak{L}$ definită de (IV.9) să aibă rangul $\rho < n$. Dacă această condiţie este satisfăcută, atunci există $m = n - \rho$ soluţii liniar independente pentru (IV.5), care satisfac (IV.5) b; dacă $m = n$, spaţiul e plan; aşadar, numărul maxim de soluţii liniar independente pentru (II.5) într-un spaţiu cu curbură diferită de zero este $m = n - 1$.

LA THÉORIE DES VARIÉTÉS D'UN ESPACE À CONNEXION PROJECTIVE SANS TORSION (I)

PAR

GH. MURĂRESCU

1. *Introduction*

L'étude des espaces à connexion projective, à n dimensions, peut être faite à l'aide des espaces à connexion affine à $n+1$ dimensions ([3], [4]).

Une connexion projective sur une variété V_n peut être identifiée à une certaine connexion affine sur une variété différentiable $V_{n+1} = V_n \times R$, où R est la droite réelle sur laquelle les seuls changements admis de coordonnées sont les translations. Si U est un voisinage de coordonnées sur V_n, alors un voisinage de coordonnées sur V_{n+1} sera de la forme $U \times R$. Une carte locale sur $U \times R$ sera de la forme

$$\{x^i\} = \{x^a,\ x^{n+1}\},\ (i = 1, \ldots, n+1\ ;\ a = 1, \ldots, n),$$

où $\{x^a\}$ est une carte locale sur U, et $\{x^{n+1}\}$ une carte sur R.

Par sa définition, la variété V_{n+1} admet la congruence de courbes

$$(+) \qquad x^a = \text{const. et } x^{n+1} = \text{variable}$$

qui s'appellent des rayons de V_{n+1}.

L'ensemble de ces rayons forme une variété qu'on peut identifier avec V_n. Cette congruence détermine sur V_{n+1} un champ de vecteurs ξ tangents en chaque point à un rayon, qui dans la carte donnée ont les composantes $\xi^i = \delta^i_{n+1}$.

On peut utiliser des cartes locales par rapport auxquelles les équations des rayons sont de la forme $(+)$.

Un changement de coordonnées entre deux cartes locales de ce type sera de la forme :

$$(1) \qquad x'^a = x'^a(x^1, \ldots, x^n),\ x'^{n+1} = x^{n+1} + \varphi(x^1, \ldots, x^n).$$

Nous appellerons *l'atlas préférentiel* l'ensemble des cartes locales liées entre eux par les relations (1).