A GENERALIZATION OF THE MÖBIUS FUNCTION

Summary

In the present Note the author defines the arithmetical function $\mu_k(n)$ as follows:

(1) $$\mu_k(n) = \begin{cases} 0 & \text{if } p^k/n \\ (-1)^{\nu(n)} & \text{otherwise} \end{cases}$$

($\nu(n)$ denotes the number of prime divisors of n) and shows some properties of $\mu_k(n)$; when $k = 2$, this function coincides with the well-known Möbius function $\mu(n)$.

COVARIANCE ANALYSIS IN UNBALANCED SCHEMES

BY

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In the analysis of variance, covariance and regression, the observation data obtained as a result of the investigation performed on a statistic characteristic are being interpreted as independent random variables subject to one and the same distribution law and admitting the representation

(I) $$y_i = \sum_{j=1}^{p} x_{ji}\,\beta_j + e_i, \quad (i = 1, 2, \ldots, n)$$

where

$\{y_i\}$ — values obtained through observing the characteristic to be investigated Y.

$\{e_i\}$ — observation errors which we suppose to be independent random variables subject to the law $N(0, \sigma^2)$, i. e.

(II) $$E(e_i) = 0, \quad (i = 1, 2, \ldots, n), \quad E(e_i\, e_j) = \sigma^2\,\delta_{ij}$$

where σ^2 is an unknown constant and δ_{ij} is Kronecker's symbol.

$\{\beta_j\}$ — unknown parameters designing the action of various constant or random factors which act upon the Y-characteristic during observation time. The β_j parameters may be either unknown constants or random variables whose distribution law depends upon other unknown parameters.

$\{x_{ji}\}$ — characteristic variables which can take the values zero or one depending on the absence or presence of the action of the factor designed by β_j.

The analysis of variance can be obtained when all $\{x_{ji}\}$ are characteristic variables. When $\{x_{ji}\}$ are not characteristic variables, but they describe continuous sets of values [1], the $\{x_{ji}\}$ variables are called independent variables; the observation data y_i are the values of the dependent variable Y and we deal with analysis of regression. It may happen that among the $\{x_{ji}\}$ there exist characteristic variables as well as independent ones. Then we obtain covariance analysis.

In analysis of this type the investigation is being performed with the help of certain models:

α) The constant factor model is that one in which all $\{\beta_j\}$ are unknown constants. It often happens that one of the factors designed by β_j acts constantly, i.e. $x_{ji}=1$, for $i=1,2,\ldots,n$. Such a β_j may be called an additive constant. The additive constant of a model may be interpreted to a certain extent as „the general average".

β) The model with random factors is being obtained when all $\{\beta_j\}$ are random variables, except for a β_j which is the additive constant.

γ) The mixed model represents the case when at least one β_j parameter is a random one and at least one of them is non-random, and the non-random parameter is different from the additive constant.

We shall consider the covariance analysis unbalanced on a model of (α) type with two factors A and B in two cases:

Case a): the absence of interaction between factors;

Case b): the presence of interaction between factors.

The schema after which we shall perform the analysis being unbalanced, the number of observation on each factor combination varies from combination to combination.

Let Y be a statistic characteristic depending on a non-random amount X and on the action of two constant factors A and B, each of them having a certain number of variants. The equations of the linear models corresponding to the (a) and (b) cases are given in [1], the problem of performing the unbalanced covariance analysis being open. We shall consider the two cases in turn.

Case a). The equation of the model is

$$Y_{ijh}=\mu+\alpha_i+\beta_j+\gamma X_{ijh}+e_{ijh}, \tag{1}$$

$$(i=1,2,\ldots,v;\ j=1,2,\ldots,r;\ h=1,2,\ldots,n_{ij}),$$

where

μ — the average effect (the additive constant);

α_i — the effect of the „i" variant of the A factor;

β_j — the effect of the „j" variant of the B factor;

γ — the average coefficient of the linear regression of Y over X which we suppose to be the same in all observations;

Y_{ijh} — the value of the random variable dependent on Y, obtained in the h^{th} observation, the simultaneous action of the A_i and B_j variants being present. The number of performed observations for variants combination $A_i B_j$ is n_{ij}. We suppose $n_{ij}\neq 0$ for any i and j;

X_{ijh} — the value of the non-random independent variable X obtained in the h^{th} observation, the action of the $A_i B_j$ combination being present.

e_{ijh} — the observation error of the Y_{ijh} value.

We shall accept some simplifying hypotheses:

i_1) the independent variable X is a non-random amount observed without errors; to each observed value of X there corresponds only one value of Y;

i_2) the observed values Y_{ijh} of the independent random variable Y represent independent random variables with distributions $N(\mu+\alpha_i+\beta_j+\gamma X_{ijh},\sigma^2)$;

i_3) the errors e_{ijh} represent independent random variables subject to the distribution law $N(0,\sigma^2)$.

Notations:

$\sum_i\sum_j n_{ij}=n_{..}=$ the total number of observations;

$$\sum_i n_{ij}=n_{.j},\quad \sum_j n_{ij}=n_{i.},\quad \sum_h Y_{ijh}=Y_{ij.},\quad \sum\sum Y_{ijh}=Y_{.j.}$$

$\sum_j\sum_h Y_{ijh}=Y_{i..}$, $\sum_i\sum_j\sum_h Y_{ijh}=Y_{...}$ and the same for the variable X.

The first problem to be solved is to find estimations for the parameters ocurring in (1). We shall obtain these estimations by means of the least squares method. We may show that the estimations obtained through this method are such, that their mean values are equal to the values of the parameters to be estimated, while their variances are minimum; i.e. the estimations we obtain by the least squares method are efficient estimations.

According to the above-mentioned method we wish to choose such values of parameters that the residual sum of squares

$$R=\sum_i\sum_j\sum_h(Y_{ijh}-\mu-\alpha_i-\beta_j-\gamma X_{ijh})^2 \tag{2}$$

be minimum. By partial derivation of (2) with respect to the parameters we obtain an equation system called normal equations. Any solution of normal equations is the estimation obtained by means of the least squares method and vice-versa. The partial derivatives of (2) with respect to the parameters are the following:

$$\begin{aligned}
&\frac{\partial R}{\partial\mu}=-2\sum_i\sum_j\sum_h(Y_{ijh}-\mu-\alpha_i-\beta_j-\gamma X_{ijh}),\\
&\frac{\partial R}{\partial\alpha_i}=-2\sum_j\sum_h(Y_{ijh}-\mu-\alpha_i-\beta_j-\gamma X_{ijh}),\quad (i=1,2,\ldots,v),\\
&\frac{\partial R}{\partial\beta_j}=-2\sum_i\sum_h(Y_{ijh}-\mu-\alpha_i-\beta_j-\gamma X_{ijh}),\quad (j=1,2,\ldots,r),\\
&\frac{\partial R}{\gamma\,\partial}=-2\sum_i\sum_j\sum_h X_{ijh}(Y_{ijh}-\mu-\alpha_i-\beta_j-\gamma X_{ijh}).
\end{aligned} \tag{3}$$

We shall denote the estimations thus obtained by: $\widehat{\mu}$, $\widehat{\alpha}_i$, $\widehat{\beta}_j$, $\widehat{\gamma}$. For the uniqueness of estimations it is necessary and sufficient that the rank of the matrix of the coefficients of unknown parameters in (I) be equal to the number of independent parameters, which require some supplementary conditions namely:

$$(4) \qquad \sum_i \hat{\alpha}_i = 0, \quad (5) \ \sum_j \hat{\beta}_j = 0.$$

Making the partial derivates (3) equal to zero, we obtain the following equations:

$$(6) \qquad n_{..}\hat{\mu} + \sum_i n_{i.}\hat{\alpha}_i + \sum_j n_{.j}\hat{\beta}_j + \hat{\gamma}X_{...} = Y_{...},$$

$$(7) \qquad n_{i.}(\hat{\mu} + \hat{\alpha}_i) + \sum_j n_{ij}\hat{\beta}_j + \hat{\gamma}X_{i..} = Y_{i..}, \ (i = 1, 2, \ldots, v),$$

$$(8) \qquad n_{.j}(\hat{\mu} + \hat{\beta}_j) + \sum_i n_{ij}\hat{\alpha}_i + \hat{\gamma}X_{.j.} = Y_{.j.}, \ (j = 1, 2, \ldots, r),$$

$$(9) \qquad X_{...}\hat{\mu} + \sum_i X_{i..}\hat{\alpha}_i + \sum_j X_{.j.}\hat{\beta}_j + \hat{\gamma}\sum_i\sum_j\sum_h X_{ijh}^2 = \sum_i\sum_j\sum_h X_{ijh}Y_{ijh}.$$

These estimations may be obtained from the equations (6) — (9) together with the conditions (4) and (5). From the „j" equation of (8) we have

$$(10) \qquad \hat{\beta}_j = \bar{y}_{.j.} - \hat{\mu} - \frac{1}{n_{.j}}\sum_i n_{ij}\hat{\alpha}_i - \hat{\gamma}\bar{x}_{.j.}$$

where

$$\bar{y}_{.j.} = \frac{Y_{.j.}}{n_{.j}} \ \text{and} \ \bar{x}_{.j.} = \frac{X_{.j.}}{n_{.j}}.$$

The equations (7) and the condition (5) lead to

$$(11) \qquad \hat{\mu} = \frac{1}{r}\left(\sum_j \bar{y}_{.j.} - \sum_j \frac{1}{n_{.j}}\sum_i n_{ij}\hat{\alpha}_i - \hat{\gamma}\sum_j \bar{x}_{.j.}\right).$$

Substituting $\hat{\mu}$ from (11) in (10) we get:

$$(12) \qquad \hat{\beta}_j = \bar{y}_{.j.} - \frac{1}{r}\left(\sum_j \bar{y}_{.j.} - \sum_j \frac{1}{n_{.j}}\sum_i n_{ij}\hat{\alpha}_i - \hat{\gamma}\bar{x}_{.j.}\right) - \frac{1}{n_{.j}}\sum_i n_{ij}\hat{\alpha}_i - \hat{\gamma}\bar{x}_{.j.}$$

Introducing $\hat{\mu}$ from (11) and $\hat{\beta}_j$ from (12) into (9) we have

$$(13) \qquad \hat{\gamma} = \frac{\sum_i\sum_j\sum_h X_{ijh}Y_{ijh} - \sum_j X_{.j.}\bar{y}_{.j.} - \sum_i X_{i..}\hat{\alpha}_i + \sum_j \bar{x}_{.j.}\sum_i n_{ij}\hat{\alpha}_i}{\sum_i\sum_j\sum_h X_{ijh}^2 - \sum_j X_{.j.}\bar{x}_{.j.}}$$

The values of $\hat{\mu}$ in (11), $\hat{\beta}_j$ in (12) and $\hat{\gamma}$ in (13) introduced in equations (7) lead to the system of equations

$$(14) \qquad n_{l.}\hat{\alpha}_e - \sum_j \frac{n_{lj}}{n_{.j}}\sum_i n_{ij}\hat{\alpha}_i + \left(X_{l..} - \sum_j n_{lj}\bar{x}_{.j.}\right)\frac{\sum_j \bar{x}_{.j.}\sum_i n_{ij}\hat{\alpha}_i - \sum_i X_{i..}\hat{\alpha}_i}{\sum_i\sum_j\sum_h X_{ijh}^2 - \sum_j X_{.j.}\bar{x}_{.j.}} =$$

$$= Y_{l..} - \sum_j n_{lj}\bar{y}_{.j.} - \left(X_{l..} - \sum_j n_{lj}\bar{x}_{.j.}\right)\frac{\sum_i\sum_j\sum_h X_{ijh}Y_{ijh} - \sum_j X_{.j.}\bar{y}_{.j.}}{\sum_i\sum_j\sum_h X_{ijh}^2 - \sum_j X_{.j.}\bar{x}_{.j.}}$$

$$(l = 1, 2, \ldots, v)$$

which implies the estimations $\hat{\alpha}_i$, only.

The estimations $\hat{\alpha}_i$ of the α_i parameters which satisfy the system (14) as well as the supplementary condition (4) are obtained by means of the least squares method of the effects of the A factor variants adjusted by linear regression.

If in the equation (1) we consider $\gamma = 0$, i.e. in the equations (14) we omit the terms containing values of the variable X, we obtain the following system of equations

$$\alpha_1'\left(n_{1.} - \sum_j \frac{n_{1j}^2}{n_{.j}}\right) - \sum_{i=2}^{v}\alpha_i'\sum_j\frac{n_{1j}n_{ij}}{n_{.j}} = Y_{1..} - \sum_j n_{1j}\bar{y}_{.j.}$$

$$-\alpha_1'\sum_j\frac{n_{1j}n_{2j}}{n_{.j}} + \alpha_2'\left(n_{2.} - \sum_j\frac{n_{2j}^2}{n_{.j}}\right) - \sum_{i=3}^{v}\alpha_i'\sum_j\frac{n_{2j}n_{ij}}{n_{.j}} = Y_{2..} - \sum_j n_{2j}\bar{y}_{.j.}$$

(15) . . . .

$$-\sum_{i=1}^{v-1}\alpha_i'\sum_j\frac{n_{vj}n_{ij}}{n_{.j}} + \alpha_v'\left(n_{v.} - \sum_j\frac{n_{vj}^2}{n_{.j}}\right) = Y_{v..} - \sum_j n_{vj}\bar{y}_{.j.}$$

The system of equations (15) together with the supplementary condition

$$(16) \qquad \sum_i \alpha_i' = 0$$

lead to unique solutions for the estimations of the A factor variants, but this time unadjusted by linear regression.

On the other hand, the equation (7) as well as the condition (5) lead to

$$(17) \qquad \hat{\mu} = \frac{1}{v}\left(\sum_i \bar{y}_{i..} - \sum_i\frac{1}{n_{i.}}\sum_j n_{ij}\hat{\beta}_j - \hat{\gamma}\sum_i \bar{x}_{i..}\right).$$

From the equation „i" of (7) and the relation (17) we have

$$(18)\quad \hat{\alpha}_i = \bar{y}_{i..} - \frac{1}{v}\left(\sum_i \bar{y}_{i..} - \sum_i \frac{1}{n_{i.}} \sum_j n_{ij}\hat{\beta}_j - \hat{\gamma}\sum_i \bar{x}_{i..}\right) - \frac{1}{n_{i.}}\sum_j n_{ij}\hat{\beta}_j - \hat{\gamma}\bar{x}_{i..}$$

The equation (9) gives

$$(19)\qquad \hat{\gamma} = \frac{\sum_i\sum_j\sum_h X_{ijh}Y_{ijh} - X_{...}\hat{\mu} - \sum_j X_{.j.}\hat{\beta}_j - \sum_i X_{i..}\hat{\alpha}_i}{\sum_i\sum_j\sum_h X^2_{ijh}}.$$

Substituting $\hat{\gamma}$ from (17) and $\hat{\alpha}_i$ from (18) into (19) we obtain

$$(20)\qquad \hat{\gamma} = \frac{\sum_i\sum_j\sum_h X_{ijh}Y_{ijh} - \sum_j X_{.j.}\hat{\beta}_j - \sum_i X_{i..}\bar{y}_{i..} + \sum_i \frac{X_{i..}}{n_{i.}}\sum_j n_{ij}\hat{\beta}_j}{\sum_i\sum_j\sum_h X^2_{ijh} - \sum_i X_{i..}\bar{x}_{i..}}.$$

The values of $\hat{\gamma}$ in (20), $\hat{\mu}$ in (17) and $\hat{\alpha}_i$ in (18) substituted in the equations (8) lead to the system of equations

$$(21)\quad n_{.k}\hat{\beta}_k - \sum_i \frac{n_{ik}}{n_{i.}}\sum_j n_{ij}\hat{\beta}_j + \left(X_{.k.} - \sum_i n_{ik}\bar{x}_{i..}\right)\frac{\sum_i \bar{x}_{i..}\sum_j n_{ij}\hat{\beta}_j - \sum_j X_{.j.}\hat{\beta}_j}{\sum_i\sum_j\sum_h X^2_{ijh} - \sum_i X_{i..}\bar{x}_{i..}} =$$

$$= Y_{.k.} - \sum_i n_{ik}\bar{y}_{i..} - \left(X_{.k.} - \sum_i n_{ik}\bar{x}_{i..}\right)\frac{\sum_i\sum_j\sum_h X_{ijh}Y_{ijh} - \sum_i X_{i..}\bar{y}_{i..}}{\sum_i\sum_j\sum_h X^2_{ijh} - \sum_i X_{i..}\bar{x}_{i..}}$$

$$(k = 1, 2, \ldots, r),$$

which implies the estimations $\hat{\beta}_j$ only. The system of equations (21) together with the supplementary condition (5) lead to unique solutions for the estimations $\hat{\beta}_j$ of the effects of the B factor variants, adjusted by linear regression.

If in the equation (1) we consider $\gamma = 0$, i.e. in the equations (21) we omit the terms containing values of the variable X, we obtain the following system of equations

$$(22)\quad \begin{aligned} &\beta'_1\left(n_{.1} - \sum_i \frac{n^2_{i1}}{n_{i.}}\right) + \sum_{j=2}^{r}\beta'_j\sum_i\frac{n_{i1}n_{ij}}{n_{i.}} = Y_{.1.} - \sum_i n_{i1}\bar{y}_{i..},\\ &-\beta'_1\sum_i\frac{n_{i1}n_{i2}}{n_{i.}} + \beta'_2\left(n_{.2} - \sum_i\frac{n^2_{i2}}{n_{i.}}\right) - \sum_{j=3}^{r}\beta'_j\sum_i\frac{n_{i2}n_{ij}}{n_{i.}} = Y_{.2.} - \sum_i n_{i2}\bar{y}_{i..},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &-\sum_{j=1}^{r-1}\beta'_j\sum_i\frac{n_{ir}n_{ij}}{n_{i.}} + \beta'_r\left(n_{.r} - \sum_i\frac{n^2_{ir}}{n_{i.}}\right) = Y_{.r.} - \sum_i n_{ir}\bar{y}_{i..}\end{aligned}$$

From the system of equations (22) and the supplementary condition

$$(23)\qquad \sum_j \beta'_j = 0$$

we obtain unique estimations for the effects of B factor variants, unadjusted by linear regression.

Remark. If in the equation (1) we consider $\gamma = 0$, we obtain the equation of a model with constant factors used in variance analysis; therefore, variance analysis is a particular case of covariance analysis. The estimations given by the systems of equations (15) and (22) together with the supplementary conditions (16) and (23) respectively, are estimations in the case of unbalanced variance analysis for a model with two constant factors.

So far we have made a quantitative analysis of the action of factors acting over the random variable Y, connected to the non random variable X by means of the equation (1). Thus, the estimation $\hat{\gamma}$ given by (20) shows how much the resultative characteristic Y varies when the factorial characteristic X varies with an unity.

The qualitative analysis of the factors action, or, to be more precise, to appreciate on the observation data whether the action of the factors A and B over the characteristic Y is significant or random is important for practical applications. In order to be able to answer to this problem the hypothesis (i_2) is absolutely necessary. This hypothesis gives the opportunity of constructing some verifying tests of the hypothesis concerning the significance (or its absence) of the action of the A and B factors over Y-characteristic.

Aiming to this we shall compute certain sums of squares and certain products.

The general sums

$$T_{yy} = \sum_i\sum_j\sum_h\left(Y_{ijh} - \frac{Y_{...}}{n_{..}}\right)^2 = \sum_i\sum_j\sum_h Y^2_{ijh} - \frac{Y^2_{...}}{n_{..}},$$

$$(24)\quad T_{xy} = \sum_i\sum_j\sum_h\left(X_{ijh} - \frac{X_{...}}{n_{..}}\right)\left(Y_{ijh} - \frac{Y_{...}}{n_{..}}\right) = \sum_i\sum_j\sum_h X_{ijh}Y_{ijh} - \frac{X_{...}Y_{...}}{n_{..}},$$

$$T_{xx} = \sum_i\sum_j\sum_h\left(X_{ijh} - \frac{X_{...}}{n_{..}}\right)^2 = \sum_i\sum_j\sum_h X^2_{ijh} - \frac{X^2_{...}}{n_{..}}$$

have one and the same number degrees of freedom equal to $n_{..} - 1$.

The sums corresponding to the influence of the A factor, omitting the influence of the B factor

$$V_{yy} = \sum_i \sum_j \sum_h \left(\frac{Y_{i..}}{n_{i.}} - \frac{Y_{...}}{n_{..}}\right)^2 = \sum_{i=1}^{v} \frac{Y_{i..}^2}{n_{i.}} - \frac{Y_{...}^2}{n_{..}},$$

$$V_{xy} = \sum_i \sum_j \sum_h \left(\frac{X_{i..}}{n_{i.}} - \frac{X_{...}}{n_{..}}\right)\left(\frac{Y_{i..}}{n_{i.}} - \frac{Y_{...}}{n_{..}}\right) =$$

(24 A)
$$= \sum_{i=1}^{v} \frac{X_{i..}\, Y_{i..}}{n_{i.}} - \frac{X_{...}\, Y_{...}}{n_{..}},$$

$$V_{xx} = \sum_i \sum_j \sum_h \left(\frac{X_{i..}}{n_{i.}} - \frac{X_{...}}{n_{..}}\right)^2 = \sum_{i=1}^{v} \frac{X_{i..}}{n_{i.}} - \frac{X_{...}^2}{n_{..}}$$

have the same number of degrees of freedom equal to $v - 1$.

The sums corresponding to the influence of the B factor, omitting the influence of the A factor

$$R_{yy} = \sum_i \sum_j \sum_h \left(\frac{Y_{.j.}}{n_{.j}} - \frac{Y_{...}}{n_{..}}\right)^2 = \sum_{j=1}^{r} \frac{Y_{.j.}^2}{n_{.j}} - \frac{Y_{...}^2}{n_{..}},$$

$$R_{xy} = \sum_i \sum_j \sum_h \left(\frac{X_{.j.}}{n_{.j}} - \frac{X_{...}}{n_{..}}\right)\left(\frac{Y_{.j.}}{n_{.j}} - \frac{Y_{...}}{n_{..}}\right) =$$

(24 B)
$$\sum_{j=1}^{r} \frac{X_{.j.}\, Y_{.j.}}{n_{.j}} - \frac{X_{...}\, Y_{...}}{n_{..}},$$

$$R_{xx} = \sum_i \sum_j \sum_h \left(\frac{X_{.j.}}{n_{.j}} - \frac{X_{...}}{n_{..}}\right)^2 = \sum_{j=1}^{r} \frac{X_{.j.}}{n_{.j}} - \frac{X_{...}^2}{n_{..}}$$

have the same number of degrees of freedom equal to $r - 1$.

The residual sums corresponding to random factors

$$D_{yy} = \sum_i \sum_j \sum_h \left(Y_{ijh} - \frac{Y_{i..}}{n_{i.}} - \frac{Y_{.j.}}{n_{.j}} + \frac{Y_{...}}{n_{..}}\right)^2 =$$

$$= \sum_i \sum_j \sum_h Y_{ijh}^2 - \left(\sum_i \frac{Y_{i..}^2}{n_{i.}} + \sum_j \frac{Y_{.j.}^2}{n_{.j}} - \frac{Y_{...}^2}{n_{..}}\right),$$

$$D_{xy} = \sum_i \sum_j \sum_h \left(X_{ijh} - \frac{X_{i..}}{n_{i.}} - \frac{X_{.j.}}{n_{.j}} + \frac{X_{...}}{n_{..}}\right)\left(Y_{ijh} - \frac{Y_{i..}}{n_{i.}} - \right.$$

$$\left. - \frac{Y_{.j.}}{n_{.j}} + \frac{Y_{...}}{n_{..}}\right) =$$

$$= \sum_i \sum_j \sum_h X_{ijh}\, Y_{ijh} - \left(\sum_i \frac{X_{i..}}{n_{i.}} Y_{i..} + \sum_j \frac{X_{.j.}}{n_{.j}} Y_{.j.} - \right.$$

(24 R)
$$\left. - \frac{X_{...}\, Y_{...}}{n_{..}}\right),$$

$$D_{xx} = \sum_i \sum_j \sum_h \left(X_{ijh} - \frac{X_{i..}}{n_{i.}} - \frac{X_{.j.}}{n_{.j}} + \frac{X_{...}}{n_{..}}\right)^2 =$$

$$= \sum_i \sum_j \sum_h X_{ijh}^2 - \left(\sum_i \frac{X_{i..}^2}{n_{i.}} + \sum_j \frac{X_{.j.}^2}{n_{.j}} - \frac{X_{...}^2}{n_{..}}\right).$$

The equalities considered from (24) to (24 R) are justified by the existence of decompositions:

(25)
$$\sum_i \sum_j \sum_h \left(Y_{ijh} - \frac{Y_{...}}{n_{..}}\right)^2 = \sum_i \sum_j \sum_h \left(\frac{Y_{i..}}{n_{i.}} - \frac{Y_{...}}{n_{..}}\right)^2 +$$

$$+ \sum_i \sum_j \sum_h \left(\frac{Y_{.j.}}{n_{.j}} - \frac{Y_{...}}{n_{..}}\right)^2 +$$

$$+ \sum_i \sum_j \sum_h \left(Y_{ijh} - \frac{Y_{i..}}{n_{i.}} - \frac{Y_{.j.}}{n_{.j}} + \frac{Y_{...}}{n_{..}}\right)$$

which are true under the condition

(25′)
$$\sum_i \sum_j \sum_h \frac{Y_{i..}}{n_{i.}} \frac{Y_{.j.}}{n_{.j}} = \frac{Y_{...}^2}{n_{..}}$$

and

(26)
$$\sum_i \sum_j \sum_h \left(X_{ijh} - \frac{X_{...}}{n_{..}}\right)\left(Y_{ijh} - \frac{Y_{...}}{n_{..}}\right) =$$

$$= \sum_i \sum_j \sum_h \left(\frac{Y_{i..}}{n_{i.}} - \frac{Y_{...}}{n_{..}}\right)\left(\frac{X_{i..}}{n_{i.}} - \frac{X_{...}}{n_{..}}\right) +$$

$$+ \sum_i \sum_j \sum_h \left(\frac{X_{.j.}}{n_{.j}} - \frac{X_{...}}{n_{..}}\right)\left(\frac{Y_{.j.}}{n_{.j}} - \frac{Y_{...}}{n_{..}}\right) +$$

$$+ \sum_i \sum_j \sum_h \left(X_{ijh} - \frac{X_{i..}}{n_{i.}} - \frac{X_{.j.}}{n_{.j}} + \frac{X_{...}}{n_{..}}\right)\left(Y_{ijh} - \frac{Y_{i..}}{n_{i.}} - \frac{Y_{.j.}}{n_{.j}} + \frac{Y_{...}}{n_{..}}\right)$$

which are true under the condition

(26′)
$$\sum_i \sum_j \sum_h \frac{X_{i..}}{n_{i.}} \frac{Y_{.j.}}{n_{.j}} = \sum_i \sum_j \sum_h \frac{Y_{i..}}{n_{i.}} \frac{X_{.j.}}{n_{.j}} = \frac{X_{...}\, Y_{...}}{n_{..}}$$

and

$$(27)\qquad \sum_i\sum_j\sum_h\left(X_{ijh}-\frac{X_{...}}{n_{..}}\right)^2=\sum_i\sum_j\sum_h\left(\frac{X_{i..}}{n_{i.}}-\frac{X_{...}}{n_{..}}\right)^2+$$

$$+\sum_j\sum_h\sum\left(\frac{X_{.j.}}{n_{.j}}-\frac{X_{...}}{n_{..}}\right)^2+\sum_i\sum_j\sum_h\left(X_{ijh}-\frac{X_{i..}}{n_{i.}}-\frac{X_{.j.}}{n_{.j}}+\frac{X_{...}}{n_{..}}\right)^2$$

which are true under the condition

$$(27')\qquad \sum_i\sum_j\sum_h\frac{X_{i..}}{n_{i.}}\frac{X_{.j.}}{n_{.j}}=\frac{X^2_{...}}{n_{..}}.$$

Under the conditions (25′), (26′) and (27′), the following relations occur:

$$(28)\qquad \begin{aligned} T_{yy}&=V_{yy}+R_{yy}+D_{yy},\\ T_{xy}&=V_{xy}+R_{xy}+D_{xy},\\ T_{xx}&=V_{xx}+R_{xx}+D_{xx}.\end{aligned}$$

Among the numbers of degrees of freedom of the left and right terms in (28) the following relations takes place

$$(29)\qquad n_{..}-1=(r-1)+(v-1)+(n_{..}-r-v+1).$$

According to the hypothesis (i_2) the sums T_{yy}, T_{xy}, T_{xx} have $\sigma^2\chi^2$-distribution with $n_{..}-1$ degrees of freedom. Using (i_2) as well as the relation (29), according to Cochran's theorem, we may state that the sums V_{yy} and V_{xx}, R_{yy} and R_{xx}, D_{yy} and D_{xx} have $\sigma^2\chi^2$ independent distributions with $v-1$, $r-1$, $n_{..}-r-v+1$ degrees of freedom, respectively. The obtained results may be arranged in the following tabular form

Variation source \ Sums	Y^2	XY	X^2	Degrees of freedom
A factor	V_{yy}	V_{xy}	V_{xx}	$v-1$
B factor	R_{yy}	R_{xy}	R_{xx}	$r-1$
Random factors	D_{yy}	D_{xy}	D_{xx}	$n_{..}-r-v+1$
Total	T_{yy}	T_{xy}	T_{xx}	$n_{..}-1$

We now introduce the sampling variances for the random variable Y

$$(30)\qquad s_Y^2=\frac{T_{yy}}{n_{..}-1},\; s^2_{y_A}=\frac{V_{yy}}{v-1},\; s^2_{y_B}=\frac{R_{yy}}{r-1},\; s^2_{y_R}=\frac{D_{yy}}{n_{..}-r-v+1}$$

and the sums of squares adjusted by linear regression

$$(31)\qquad T'_{yy}=T_{yy}-\frac{T^2_{xy}}{T_{xx}},\; V'_{yy}=V_{yy}-\frac{V^2_{xy}}{V_{xx}},\; R'_{yy}=R_{yy}-\frac{R^2_{xy}}{R_{xx}},$$

$$D'_{yy}=D_{yy}-\frac{D^2_{xy}}{D_{xx}}$$

with $n_{..}-2$, $v-2$, $r-2$, $n_{..}-r-v$ degrees of freedom, respectively.

The sampling variances, adjusted by linear regression

$$(32)\qquad s'^2_y=\frac{T'_{yy}}{n_{..}-2},\; s'^2_{y_A}=\frac{V'_{yy}}{v-2},\; s'^2_{y_B}=\frac{R'_{yy}}{r-2},\; s'^2_{y_R}=\frac{D'_{yy}}{n_{..}-r-v},$$

$$s'^2_{y_{ABR}}=\frac{T'_{yy}-V'_{yy}-R'_{yy}-D'_{yy}}{1}$$

are unbiased estimations of the general variance σ^2. According to the hypothesis (i_2), we may state that the quantities $\frac{T'_{yy}}{\sigma^2}$, $\frac{V'_{yy}}{\sigma^2}$, $\frac{R'_{yy}}{\sigma^2}$, $\frac{D'_{yy}}{\sigma^2}$, $\frac{T'_{yy}-V'_{yy}-R'_{yy}-D'_{yy}}{\sigma^2}$ have χ^2-distribution with $n_{..}-2$, $v-2$, $r-2$, $n_{..}-r-v$, 1 degrees of freedom, respectively.

We introduce the sampling quantities.

$$F_A=\frac{s'^2_{y_A}}{s'^2_{y_R}} \text{ has } F\text{-distribution with } k_1=v-2,\; k_2=n_{..}-r-v \text{ degrees of freedom}$$

$$(33)\qquad F_B=\frac{s'^2_{y_B}}{s'^2_{y_R}} \text{ has } F\text{-distribution with } k_1=r-2,\; k_2=n_{..}-r-v \text{ degrees of freedom.}$$

$$F_{ABR}=\frac{s'^2_{y_{ABR}}}{s'^2_{y_R}} \text{ has } F\text{-distribution with } k_1=1,\; k_2=n_{..}-r-v \text{ degrees of freedom.}$$

The quantities F_A and F_B given in (33) for a significance level given before, serve to appreciate whether the divergence between the variances adjusted for the A and B factors respectively and the residual one is either significant or random, or to be more precise, to verify the hypothesis of the absence of the A or B factor influence, respectively, over the Y variable, taking into account the linear relation between X and Y given by the equation (1). The quantity F_{ABR} gives the opportunity to appreciate the divergence between the variance due to the simultaneous action of the

A, B and random factors on the one hand and the variance due to random factors only, on the other hand.

Case b) The equation of the model is

$$Y_{ijh} = \mu + \alpha_i + \beta_j + \delta_{ij} + \gamma X_{ijh} + e_{ijh} \tag{34}$$

where δ_{ij} represents a characteristic effect of the interaction of the A_i and B_j variants, while the other amounts are defined in just the same way as in (1).

We try to find estimations making the residual sum

$$R = \sum_i \sum_j \sum_h (Y_{ijh} - \mu - \alpha_i - \beta_j - \delta_{ij} - \gamma X_{ijh})^2 \tag{35}$$

to be minimum, for the parameters of the equations (34). The partial derivatives of (35) with respect to various parameters are

$$\frac{\partial R}{\partial \mu} = -2 \sum_i \sum_j \sum_h (Y_{ijh} - \mu - \alpha_i - \beta_j - \delta_{ij} - \gamma X_{ijh}), \tag{36}$$

$$\frac{\partial R}{\partial \alpha_i} = -2 \sum_j \sum_h (Y_{ijh} - \mu - \alpha_i - \beta_j - \delta_{ij} - \gamma X_{ijh}), \quad (i = 1, 2, \ldots v) \tag{37}$$

$$\frac{\partial R}{\partial \beta_j} = -2 \sum_i \sum_h (Y_{ijh} - \mu - \alpha_i - \beta_j - \delta_{ij} - \gamma X_{ijh}), \quad (j = 1, 2, \ldots, r) \tag{38}$$

$$\frac{\partial R}{\partial \delta_{ij}} = -2 \sum_h (Y_{ijh} - \mu - \alpha_i - \beta_j - \delta_{ij} - \gamma X_{ijh}), \quad \begin{pmatrix} i = 1, 2, \ldots, v \\ j = 1, 2, \ldots, r \end{pmatrix} \tag{39}$$

$$\frac{\partial R}{\partial \gamma} = -2 \sum_i \sum_j \sum_h X_{ijh} (Y_{ijh} - \mu - \alpha_i - \beta_j - \delta_{ij} - \gamma X_{ijh}). \tag{40}$$

By taking the above-mentioned partial derivatives equal to zero we obtain equations giving the estimations

$$n_{..} \hat{\mu} + \sum_i n_{i.} \hat{\alpha}_i + \sum_j n_{.j} \hat{\beta}_j + \sum_i \sum_j n_{ij} \hat{\delta}_{ij} + \hat{\gamma} X_{...} = Y_{...} \tag{41}$$

$$n_{k.} (\hat{\mu} + \hat{\alpha}_k) + \sum_j n_{kj} (\hat{\beta}_j + \hat{\delta}_{kj}) + \hat{\gamma} X_{k..} = Y_{k..}, \quad (k = 1, 2, \ldots, v) \tag{42}$$

$$n_{.l} (\hat{\mu} + \hat{\beta}_l) + \sum_i n_i (\hat{\alpha}_i + \hat{\delta}_i) + \hat{\gamma} X_{.j.} = Y_{.j.}, \quad (j = 1, 2, \ldots, r) \tag{43}$$

$$n_{ij} (\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\delta}_{ij}) + \hat{\gamma} X_{ij.} = Y_{ij.}, \quad \begin{matrix} (i = 1, 2, \ldots, v) \\ (j = 1, 2, \ldots, r) \end{matrix} \tag{44}$$

$$X_{...} \hat{\mu} + \sum_i X_{i..} \hat{\alpha}_i + \sum_j X_{.j.} \hat{\beta}_j + \sum_i \sum_j X_{ij.} \hat{\delta}_{ij} + \hat{\gamma} \sum_i \sum_j \sum_h X^2_{ijh} = \sum_i \sum_j \sum_h X_{ijh} Y_{ijh}. \tag{45}$$

In order to obtain unique estimations from the above-mentioned equations some supplementary conditions are being imposed

$$\sum_i \hat{\alpha}_i = 0, \quad (47) \ \sum_j \hat{\beta}_j = 0, \quad (48) \ \sum_i \hat{\delta}_{ij} = 0, \quad (j = 1, 2, \ldots, r). \tag{46}$$

$$\sum_j \hat{\delta}_{ij} = 0, \quad (i = 1, 2, \ldots, v). \tag{49}$$

The equations (44) together with the conditions (47) and (49) lead to

$$\hat{\mu} + \hat{\alpha}_i = \frac{1}{r} \sum_j (\bar{y}_{ij.} - \hat{\gamma} \bar{x}_{ij.}), \quad (i = 1, 2, \ldots, v). \tag{50}$$

The equations (50) give the estimations of the A factor effects, adjusted by linear regression, in the hypothesis of the presence of the interaction between A and B factors.

Summing up the equations (50) and taking into account the condition (46) we obtain

$$\hat{\mu} = \frac{1}{rv} \sum_i \sum_j (\bar{y}_{ij.} - \hat{\gamma} \bar{x}_{ij.}). \tag{51}$$

From the equations (44) together with the conditions (46) and (48) we obtain

$$\hat{\mu} + \hat{\beta}_j = \frac{1}{v} \sum_i (\bar{y}_{ij.} - \hat{\gamma} \bar{x}_{ij.}). \quad (j = 1, 2, \ldots, r) \tag{52}$$

from where we can get the estimations of the B factors effects, adjusted by linear regression.

From one of the equations (44) we obtain

$$\hat{\delta}_{ij} = \bar{y}_{ij.} - \hat{\gamma} \bar{x}_{ij.} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j, \quad \begin{pmatrix} i = 1, 2, \ldots, v \\ j = 1, 2, \ldots, r \end{pmatrix} \tag{53}$$

Replacing $\hat{\delta}_{ij}$ from (53) into (45) we have the estimation for $\hat{\gamma}$

$$\hat{\gamma} = \frac{\sum_i \sum_j \left(\sum_h X_{ijh} Y_{ijh} - X_{ij.} Y_{ij.} / n_{ij} \right)}{\sum_i \sum_j \left(\sum_h X^2_{ijh} - X^2_{ij.} / n_{ij} \right)} \tag{54}$$

The $\hat{\gamma}$ estimation of the linear regression coefficient in significant when all $n_{ij} \neq 0$, $(i = 1, 2, \dots, v,\ j = 1,2, \dots, r)$.

Similarly to case (a), we introduce some sums of squares and products corresponding to the variation due to different sources.

The general sums

$$
(55)\qquad
\begin{aligned}
T_{yy} &= \sum_i \sum_j \sum_h \left(Y_{ijh} - \frac{Y_{...}}{n_{..}}\right)^2 = \sum_i \sum_j \sum_h Y_{ijh}^2 - \frac{Y_{...}^2}{n_{..}} \\
T_{xy} &= \sum_i \sum_j \sum_h \left(X_{ijh} - \frac{X_{...}}{n_{..}}\right)\left(Y_{ijh} - \frac{Y_{...}}{n_{..}}\right) = \\
&= \sum_i \sum_j \sum_h X_{ijh} Y_{ijh} - \frac{X_{...} Y_{...}}{n_{..}} \\
T_{xx} &= \sum_i \sum_j \sum_h \left(X_{ijh} - \frac{X_{...}}{n_{..}}\right)^2 = \sum_i \sum_j \sum_h X_{ijh}^2 - \frac{X_{...}^2}{n_{..}}
\end{aligned}
$$

have $n_{..} - 1$ degrees of freedom each.

The sums corresponding to the A factor action, omitting the B factor action

$$
(55A)\qquad
\begin{aligned}
V_{yy} &= \sum_i \sum_j \sum_h \left(\frac{Y_{i..}}{n_{i.}} - \frac{Y_{...}}{n_{..}}\right)^2 = \sum_{i=1}^{v} \frac{Y_{i..}^2}{n_{i.}} - \frac{Y_{...}^2}{n_{..}} \\
V_{xy} &= \sum_i \sum_j \sum_h \left(\frac{X_{i..}}{n_{i.}} - \frac{X_{...}}{n_{..}}\right)\left(\frac{Y_{i..}}{n_{i.}} - \frac{Y_{...}}{n_{i.}}\right) = \\
&= \sum_{i=1}^{v} \frac{X_{i..} Y_{i..}}{n_{i.}} - \frac{X_{...} Y_{...}}{n_{.}} \\
V_{xx} &= \sum_i \sum_j \sum_h \left(\frac{X_{i..}}{n_{i.}} - \frac{X_{...}}{n_{..}}\right)^2 = \sum_{i=1}^{v} \frac{X_{i..}^2}{n_{i..}} - \frac{X_{...}^2}{n_{..}}
\end{aligned}
$$

have $v - 1$ degrees of freedom each.

The sums corresponding to the B factor action, omitting the A factor action

$$
(55B)\qquad
\begin{aligned}
R_{yy} &= \sum_i \sum_j \sum_h \left(\frac{Y_{.j.}}{n_{.j}} - \frac{Y_{...}}{n_{..}}\right)^2 = \sum_{j=1}^{r} \frac{Y_{.j.}^2}{n_{.j.}} - \frac{Y_{...}^2}{n_{..}} \\
R_{xy} &= \sum_i \sum_j \sum_h \left(\frac{X_{.j.}}{n_{.j}} - \frac{X_{...}}{n_{..}}\right)\left(\frac{Y_{.j.}}{n_{.j}} - \frac{Y_{...}}{n_{..}}\right) = \sum_{j=1}^{r} \frac{X_{.j.} Y_{.j.}}{n_{.j}} - \frac{X_{...} Y_{...}}{n_{..}} \\
R_{xx} &= \sum_i \sum_j \sum_h \left(\frac{X_{.j.}}{n_{.j}} - \frac{X_{...}}{n_{..}}\right)^2 = \sum_{j=1}^{r} \frac{X_{.j.}^2}{n_{.j}} - \frac{X_{...}^2}{n_{..}}
\end{aligned}
$$

have $r - 1$ degrees of freedom each.

The sums corresponding to the interaction between the factors A and B

$$
(55.\mathrm{I})\qquad
\begin{aligned}
I_{yy} &= \sum_i \sum_j \sum_h \left(\frac{Y_{ij.}}{n_{ij}} - \frac{Y_{i..}}{n_{i.}} - \frac{Y_{.j.}}{n_{.j}} + \frac{Y_{...}}{n_{..}}\right)^2 \\
I_{xy} &= \sum_i \sum_j \sum_h \left(\frac{X_{ij.}}{n_{ij}} - \frac{X_{i..}}{n_{i.}} - \frac{X_{.j.}}{n_{.j}} + \frac{X_{...}}{n_{..}}\right)\left(\frac{Y_{ij.}}{n_{ij}} - \frac{Y_{i..}}{n_{i.}} - \frac{Y_{.j.}}{n_{.j}} + \frac{Y_{...}}{n_{..}}\right) \\
I_{xx} &= \sum_i \sum_j \sum_h \left(\frac{X_{ij.}}{n_{ij}} - \frac{X_{i..}}{n_{i.}} - \frac{X_{.j.}}{n_{.j}} + \frac{X_{...}}{n_{..}}\right)^2
\end{aligned}
$$

have $rv - r - v + 1$ degrees of freedom each.

The residual sums corresponding to the action of random factors

$$
(55.\mathrm{R})\qquad
\begin{aligned}
S_{yy} &= \sum_i \sum_j \sum_h \left(Y_{ijh} - \frac{Y_{ij.}}{n_{ij}}\right)^2 = \sum_i \sum_j \left(\sum_h \frac{Y_{ij.}^2}{n_{ij}}\right) \\
S_{xy} &= \sum_i \sum_j \sum_h \left(X_{ijh} - \frac{X_{ij.}}{n_{ij}}\right)\left(Y_{ijh} - \frac{Y_{ij.}}{n_{ij}}\right) = \\
&= \sum_i \sum_j \left(\sum_h X_{ijh} Y_{ijh} - \frac{X_{ij.} Y_{ij.}}{n_{ij}}\right) \\
S_{xx} &= \sum_i \sum_j \sum_h \left(X_{ijh} - \frac{X_{ij.}}{n_{ij}}\right)^2 = \sum_i \sum_j \left(\sum_h X_{ijh}^2 - \frac{X_{ij.}^2}{n_{ij}}\right)
\end{aligned}
$$

have $n_{..} - rv$ degrees of freedom each.

The justification of equalities from (55) to (55.R) is ensured by the existence of decompositions:

$$
(56)\qquad
\begin{aligned}
&\sum_i \sum_j \sum_h \left(Y_{ijh} - \frac{Y_{...}}{n_{..}}\right)^2 = \sum_i \sum_j \sum_h \left(\frac{Y_{i..}}{n_{i.}} - \frac{Y_{...}}{n_{..}}\right)^2 + \\
&+ \sum_i \sum_j \sum_h \left(\frac{Y_{.j.}}{n_{.j}} - \frac{Y_{...}}{n_{..}}\right)^2 + \sum_i \sum_j \sum_h \left(Y_{ijh} - \frac{Y_{ij.}}{n_{ij}}\right)^2 + \\
&+ \sum_i \sum_j \sum_h \left(\frac{Y_{ij.}}{n_{ij}} - \frac{Y_{i..}}{n_{i.}} - \frac{Y_{.j.}}{n_{.j}} + \frac{Y_{...}}{n_{..}}\right)^2,
\end{aligned}
$$

which are true under the conditions

$$
(56')\qquad \sum_i \sum_j \sum_h \frac{Y_{i..} Y_{.j.}}{n_{i.}\, n_{.j}} = \frac{Y_{...}^2}{n_{..}}, \quad \sum_i \frac{Y_{i..}^2}{n_{i.}} = \sum_i \sum_j \frac{Y_{i..}}{n_{i.}} Y_{ij.}
$$

and

$$\sum_i\sum_j\sum_h\left(Y_{ijh}-\frac{Y_{...}}{n_{..}}\right)\left(X_{ijh}-\frac{X_{...}}{n_{..}}\right)=\sum_i\sum_j\sum_h\left(\frac{Y_{.j.}}{n_{.j}}-\frac{Y_{...}}{n_{..}}\right)\left(\frac{X_{.j.}}{n_{.j}}-\frac{X_{...}}{n_{..}}\right)+$$

(57)

$$+\sum_i\sum_j\sum_h\left(\frac{X_{i..}}{n_{i.}}-\frac{X_{...}}{n_{..}}\right)\left(\frac{Y_{i..}}{n_{i.}}-\frac{Y_{...}}{n_{..}}\right)+\sum_i\sum_j\sum_h\left(Y_{ijh}-\frac{Y_{ij.}}{n_{ij}}\right)\left(X_{ijh}-\frac{X_{ij.}}{n_{ij}}\right)+$$

$$+\sum_i\sum_j\sum_h\left(\frac{X_{ij.}}{n_{ij}}-\frac{X_{i..}}{n_{i.}}-\frac{X_{.j.}}{n_{.j}}+\frac{X_{...}}{n_{..}}\right)\left(\frac{Y_{ij.}}{n_{ij}}-\frac{Y_{i..}}{n_{i.}}-\frac{Y_{.j.}}{n_{.j}}+\frac{Y_{...}}{n_{..}}\right)$$

which are true under the conditions

$$\sum_i\sum_j\sum_h\left(\frac{X_{.j.}}{n_{ij}}-\frac{X_{i..}}{n_{i.}}\right)=\sum_i\sum_j\sum_h\frac{X_{i..}\,Y_{.j.}}{n_{i.}\,n_{.j}}=\frac{X_{...}\,Y_{...}}{n_{..}} \tag{57'}$$

and

$$\sum_i\sum_j\sum_h\left(X_{ijh}-\frac{X_{...}}{n_{..}}\right)^2=\sum_i\sum_j\sum_h\left(\frac{X_{i..}}{n_{i.}}-\frac{X_{...}}{n_{..}}\right)^2+$$

(58)

$$+\sum_i\sum_j\sum_h\left(\frac{X_{.j.}}{n_{.j}}-\frac{X_{...}}{n_{..}}\right)^2+\sum_i\sum_j\sum_h\left(X_{ijh}-\frac{X_{ij.}}{n_{ij}}\right)+$$

$$+\sum_i\sum_j\sum_h\left(\frac{X_{ij.}}{n_{ij}}-\frac{X_{i..}}{n_{i.}}-\frac{X_{.j.}}{n_{.j}}+\frac{X_{...}}{n_{..}}\right)^2$$

which are true under the conditions

$$\sum_i\sum_j\sum_h\frac{X_{i..}\,X_{.j.}}{n_{.i}\,n_{.j}}=\frac{X^2_{...}}{n_{..}},\quad \sum_i\frac{Y^2_{i..}}{n_{i.}}=\sum_i\sum_j\frac{Y_{i..}\,Y_{ij.}}{n_{i.}}. \tag{58'}$$

Between the sums taken into account before, the following relations

$$\begin{aligned} T_{yy}&=V_{yy}+R_{yy}+I_{yy}+S_{yy},\\ T_{xy}&=V_{xy}+R_{xy}+I_{xy}+S_{xy},\\ T_{xx}&=V_{xx}+R_{xx}+I_{xx}+S_{xx}\end{aligned} \tag{59}$$

take place.

Among the numbers of degrees of freedom there exists the relation

$$n_{..}-1=(r-1)+(v-1)+(rv-r-v+1)+(n_{..}-rv). \tag{60}$$

According to Cochran's theorem we may state that the sums T_{yy} and T_{xx}, V_{yy} and V_{xx}, R_{yy} and R_{xx}, I_{yy} and I_{xx}, S_{yy} and S_{xx} have independent $\sigma^2\chi^2$-distribution with $n_{..}-1$, $v-1$, $r-1$, $rv-r-v+1$, $n_{..}-rv$ degrees of freedom, respectively. The results obtained may be arranged in the fololwing tabular form

Sums / Variation source	Y^2	XY	X^2	Degrees of freedom
A factor	V_{yy}	V_{xy}	V_{xx}	$v-1$
B factor	R_{yy}	R_{xy}	R_{xx}	$r-1$
Interaction	I_{yy}	I_{xy}	I_{xx}	$rv-r-v+1$
Random factors	S_{yy}	S_{xy}	S_{xx}	$n_{..}-rv$
Total	T_{yy}	T_{xy}	T_{xx}	$n_{..}-1$

We introduce the sampling variance for the random variable Y

$$S^2_y=\frac{T_{yy}}{n_{..}-1},\quad S^2_{y_A}=\frac{V_{yy}}{v-1},\quad S^2_{y_B}=\frac{R_{yy}}{r-1},\quad S^2_{y_I}=\frac{I_{yy}}{rv-r-v+1}, \tag{61}$$

$$S^2_{y_R}=\frac{S_{yy}}{n_{..}-rv}.$$

The sums adjusted by linear regression are given by

$$T'_{yy}=T_{yy}-\frac{T^2_{xy}}{T_{xx}},\quad V'_{yy}=V_{yy}-\frac{V^2_{xy}}{V_{xx}},\quad R'_{yy}=R_{yy}-\frac{R^2_{xy}}{R_{xx}}, \tag{62}$$

$$I'_{yy}=I_{yy}-\frac{I^2_{xy}}{I_{xx}},\quad S'_{yy}=S_{yy}-\frac{S^2_{xy}}{S_{xx}}$$

with $n_{..}-2$, $v-2$, $r-2$, $rv-r-v$, $n_{..}-rv-1$ degrees of freedom, respectively.

We introduce the sampling variances adjusted by linear regression

$$S'^2_y=\frac{T'_{yy}}{n_{..}-2},\quad S'^2_{y_A}=\frac{V'_{yy}}{v-2},\quad S'^2_{y_B}=\frac{R'_{yy}}{r-2},\quad S'^2_{y_I}=\frac{I'_{yy}}{rv-r-v}, \tag{62}$$

$$S'^2_{y_R}=\frac{S'_{yy}}{n_{..}-rv-1},\quad S'^2_{y\,ABIR}=\frac{T'_{yy}-V'_{yy}-R'_{yy}-I'_{yy}-S'_{yy}}{1}$$

which are unbiased estimations of the σ^2 general variance.

Taking into account the hypothesis (i_2) we may state that the quantities

$\dfrac{T'_{yy}}{\sigma^2}$, $\dfrac{V'_{yy}}{\sigma^2}$, $\dfrac{R'_{yy}}{\sigma^2}$, $\dfrac{I'_{yy}}{\sigma^2}$, $\dfrac{S'_{yy}}{\sigma^2}$ have χ^2-distribution with $n_{..}-2$, $v-2$, $r-2$, $rv-r-v$, $n_{..}-rv-1$ degrees of freedom, respectively.

$$
(64)\quad
\begin{aligned}
&F_A = \frac{S'^2_{y_A}}{S'^2_{y_R}} \text{ has } F\text{-distribution with } k_1 = v - 2,\quad k_2 = n_{..} - rv - 1 \text{ degrees of freedom,}\\
&F_B = \frac{S'^2_{y_B}}{S'^2_{y_R}} \text{ has } F\text{-distribution with } k_1 = r - 2,\ k_2 = n_{..} - rv - 1 \text{ degrees of freedom,}\\
&F_I = \frac{S'^2_{y_I}}{S'^2_{y_R}} \text{ has } F\text{-distribution with } k_1 = rv - r - v,\ k_2 = n_{..} - rv - 1 \text{ degrees of freedom,}\\
&F_{ABIR} = \frac{S'^2_{y_{ABIR}}}{S'^2_{y_R}} \text{ has } F\text{-distribution with } k_1 = 1,\ k_2 = n_{..} - rv - 1 \text{ degrees of freedom.}
\end{aligned}
$$

By means of (64) we may verify hypotheses concerning the A and B factor influence or their interaction over the random variable Y.

BIBLIOGRAPHY

1. Шеффе Г. — *Дисперсионный анализ*, Москва, 1963.
2. Романовский В. И. — *Математическая статистика*, Ташкент 1963.

ANALIZA COVARIAŢIONALĂ ÎN SCHEME NEECHILIBRATE

Rezumat

Analiza covariaţională creată de R. Fisher pune următoarea problemă: a studia simultan cîţiva indici, legaţi unul de altul şi aflaţi sub acţiunea mai multor factori.

În Nota de faţă, pornind de la ecuaţia unui model liniar, cu factori constanţi, aflată în [1], se arată o cale de efectuare a analizei covariaţionale într-o schemă neechilibrată în două cazuri:

a) cazul lipsei interacţiunii între factori,

b) cazul prezenţei interacţiunii între factori.

Mai întîi, în ambele cazuri, s-au găsit estimaţii pentru parametri necunoscuţi şi apoi s-au dat criterii de verificare a diferitelor ipoteze relativ la acţiunea factorilor ce acţionează asupra caracteristicii de cercetat. În cazul particular, cînd coeficientul de regresie γ este egal cu zero, se obţine analiza dispersională neechilibrată.

Considerarea analizei covariaţionale după o schemă neechilibrată este cerută de aplicaţiile practice în scopul economisirii numărului de experienţe.

UNE MESURE DANS LE PLAN ASSOCIÉE A UNE CLASSE D'APPLICATIONS ISOTONES

PAR

OLGA COSTINESCU

Dans le cas linéaire, on connaît le rapport qui existe entre les mesures Stieltjes et les fonctions réelles à variation finie (voir [3], p. 242 – 249).

Il est évident qu'en général on ne peut pas poser un tel problème dans R_n, $n > 1$; pourtant, si on introduit une relation d'ordre, „ $<$ ", par le procédé indiqué dans [1] et si on considère des fonctions définies sur $(R_n, <)$ à valeurs dans R, isotones dans le sens de l'ordre indiqué plus haut, alors on peut associer à une telle fonction une mesure régulière (voir [4]). Le but de cette Note est de résoudre ce problème.

Dans ce qui suit nous considérons le cas $n = 2$; naturellement, le procédé employé peut être utilisé en général pour R_n quel que soit $n \geqq 2$.

Soit X un domaine dans R_2; supposons que les ensembles $(X_\gamma)_{\gamma \in \Gamma}$, $X_\gamma \subset X$, considérés dans la Note [1] satisfont aux conditions:

(a) $X_\gamma = \bar{X}_\gamma$, $\gamma \in \Gamma$.

(b) $\forall \gamma \in \Gamma$, X_γ est un ensemble connexe.

(c) Chaque ensemble X_γ détermine sur X deux sous-ensembles, P_γ et Q_γ, simplement connexes, de manière que

$$
(1)\quad X = X_\gamma \cup P_\gamma \cup Q_\gamma,\quad P_\gamma \cap Q_\gamma = \emptyset,\quad X_\gamma \cap P_\gamma = \emptyset,\quad X_\gamma \cap Q_\gamma = \emptyset.
$$

(d) Pour chaque $\gamma \in \Gamma$, $X_\gamma = \operatorname{Fr} P_\gamma$, $X_\gamma = \operatorname{Fr} Q_\gamma$.

(e) Quel que soit $x \in X$, il existe un ensemble unique X_γ, tel que $x \in X_\gamma$.

On considère l'ensemble X doué de la topologie τ induite par la topologie habituelle τ donnée sur R_2, et on introduit dans l'espace (X, τ) les P-voisinages et les Q-voisinages (voir [1]); en utilisant les mêmes notations que celles employées dans cette Note, on a

$$
(2)\quad \begin{cases}
\mathfrak{P}_\gamma = P_\gamma \cup X_\gamma \quad \mathfrak{Q}_\gamma = Q_\gamma \cup X_\gamma,\quad \mathfrak{P}_x = \bigcap\limits_{\gamma : x \in P_\gamma} \mathfrak{P}_\gamma,\quad \mathfrak{Q}_x = \bigcap\limits_{\gamma : x \in \mathfrak{Q}_\gamma} \mathfrak{Q}_\gamma,\\
W_x^P = (\complement \mathfrak{Q}_x) \cap W_x,\quad W_x^Q = (\complement \mathfrak{P}_x) \cap W_x,\quad W_x \in \mathfrak{V}(x).
\end{cases}
$$