ON F-DIFFERENTIABILITY OF THE SOLUTIONS OF VOLTERRA NONLINEAR INTEGRAL OPERATORS

BY

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1. In this paper we shall bring to evidence the properties of continuity and differentiability in the Fréchet sense (F-differentiability) for the Volterra nonlinear integral operators, defined in $L_p(X, \mu)$, where X is a Banach space and μ is the Borel-Lebesgue measure defined in the set of real numbers.

In this order we shall adopt the following notations: I is a certain interval of the set of real numbers,

$$\Delta = \{(t, s)\,;\ a \leqslant s \leqslant t \leqslant b,\ I = (a, b)\},$$

$(X, \|\cdot\|)$ is a Banach space according to the indicated norm in the brackets. By $L_p(X, \mu)$ we shall understand, as is natural, the set of all mappings $x(t)$ defined on I with values in X measurable in the Bochner sense with the property that

$$\int_I |x(t)|^p \, d\mu(t) < +\infty,$$

where μ is the Borel-Lebesgue measure on I and $p \geqslant 1$. In the case $X \equiv R$, we shall denote this space by L_p.

For every pair of elements $u(t) \in L_p(X, \mu)$ and $\delta(t) \in L_p$, $\delta(t) \geqslant 0$ we shall denote

$$W = \{x\,;\ x \in L_p(X, \mu),\quad |x(t) - u(t)| \leqslant \delta(t)\}.$$

Obvioulsy, from the given definition it results that $W \subset L_p(X; \mu)$, but in accordance with the induced topology by $L_p(X; \mu)$, it is a metric subspace of $L_p(X; \mu)$.

It can be remarked that the W set is convexe. because if

$\lambda_i \in [0,1]$, $\sum_{i=1}^{n} \lambda_i = 1$ and $x_i \in W$ with $1 \leqslant i \leqslant n$ then $\left| \sum_{i=1}^{n} \lambda_i x_i(t) - \right.$

$$\left. - u(t) \right| = \left| \sum_{i=1}^{n} \lambda_i x_i(t) - \sum_{i=1}^{n} \lambda_i u(t) \right| \leqslant \sum_{i=1}^{n} \lambda_i \left| x_i(t) - u(t) \right| \leqslant \delta(t)$$

and $\sum_{i=1}^{n} \lambda_i x_i(t) \in W$.

It can be seen very easily that W is not a vectorial subspace of $L_p(X; \mu)$.

The set W being closed and $L_p(X; \mu)$ a Banach space, it results that W regarded as a metric space, is complete.

In conclusion the set W is convexe and according to the induced metric, it is a complete metric space.

In the following, by $\varphi(t, s, u)$ we shall denote mappings defined on $\Delta \times W$ taking values in X or R, with the property that

$$\int_a^b \left\{ \int_a^t \left| \varphi(t, s; u(s)) \right|^q ds \right\}^{p/q} dt < +\infty.$$

The set of these mappings forms a normed linear space (see [6]) with regard to the application:

$$|||\varphi||| = \left(\int_a^b \left\{ \int_a^t \left| \varphi(t, s; u(s)) \right|^q d\mu(s) \right\}^{p/q} dt \right)^{1/p}.$$

We remind also the definitions of differentiability in the Fréchet and Gâteaux sense.

Let E and F be two normed linear spaces and T an operation defined on E taking values in F.

We shall say that the operation T is F-differentiable in the point $x \in E$ if there is a continuous linear application denoted by $DT(x, h)$ such that

$$\| T(x+h) - T(x) - DT(x, h) \| = O(\| h \|).$$

The operator $DT(x, *) = T'_{(x)}(*)$ will be called the derivative operator, in the Fréchet sense, of T in the point X, and evidently it is connected with the differential by the realtion $DT(x, h) = T'_{(x)}(h)$.

The T operation being F-differentiable in x, is also continuous.

In a similar way the T operation is G-differentiable in the point x, and the direction h if there exists $\lim_{t \to 0} \frac{1}{t} (T(x+th) - T(x))$, h — fixed.

It is known that if T is F-differentiable then it is also G-differentiable. and the differentials are equal.

If T is G-differentiable in a convexe neighborhood of x, then for any point $x+h$ belonging to the neighborhood is accomplished the inequality

$$\| T(x+h) - T(x) - DT(x, h) \| \leqslant \sup_{0 \leq t \leq 1} \| DT(x+th, h) - DT(x, h) \|.$$

2. Let us consider the Volterra nonlinear integral equation

$$(2.1) \qquad y(t) = x(t) + \int_a^t K(t, s; y(s))\, ds; \quad a \leqslant t \leqslant b,$$

where $x(t)$ and $y(t)$ are mappings defined on I taking values in X or R.

D. Willet shows in [5] that the equation (2.1) admits a unic solution if the following sufficient conditions are satisfied:

a) $K(t, s; w(s))$ is Bochner measurable on Δ for every $w \in W$;

b) there exist two functions $\lambda(t,s)$ and $\alpha(t)$ measurable on Δ respectively on I, nonnegative, with $|||\lambda||| < +\infty$ and $\| \alpha \|_p < +\infty$ such that the following inequalities are satisfied:

$$| K(t, s; u(s)) | \leqslant \lambda(t, s)\, \alpha(s), \qquad (t, s) \in \Delta;$$

$|K(t, s; w(s)) - K(t, s; v(s))| \leqslant \lambda(t,s) |w(s) - v(s)|$, for every $(t, s) \in \Delta$ and $w, v \in W$;

c) $\delta(t)$ is sufficiently large, satisfying for instance

$$\delta(t) \geqslant |x(t) - u(t)| + \| \lambda(t, s) \|_q \left(\sum_{K=0}^{\infty} \left[\left(\int_a^t \| \lambda(t, .) \|_p^q ds \right)^K : K! \right]^{1/p} \right) \cdot \left(\int_a^t \alpha^p(s)\, ds \right)^{1/p}$$

almost everywhere in $I \subset R$.

Evidently, imposing more restrictive conditions the above inequalities become simpler.

The solution $y(t)$ of the equation (2.1), when it exists depends from the free term. If we shall assume as permanently satisfied the existence and unicity conditions and we shall let $x(t)$ vary in $L_p(X; \mu)$ then we shall define punctually an univoque mapping from $x(t)$ to $y(t)$ which will be a nonlinear operator defined in $L_p(X; \mu)$ with values in $L_p(X; \mu)$ and which we denote by $V(t; x) = y(t)$.

In this way we have defined an operator $V(t; x)$ by the equality

$$(2.2) \qquad V(t; x) = x(t) + \int_a^t K(t, s; V(s; x))\, ds.$$

In the following we shall bring to evidence some properties, as F-differentiability and continuity, for this operator $V(t; x)$. Finally, for the following it is strictly necessary to remind an inequality that which frequently be used.

Let $\varphi, \psi, \zeta \in L_p$ be almost everywhere nonnegative and satisfy the inequality

$$\varphi(t) \leqslant \zeta(t) + \psi(t)\left(\int_a^t \varphi^p(s)\, ds\right)^{1/p} \tag{2.3}$$

almost everywhere in I, then the Gronwal inequality:

$$\left(\int_a^t \varphi^p(s)\, ds\right)^{1/p} \leqslant \frac{1}{1-[1-\varepsilon(t)]^{1/p}}\left(\int_a^t \zeta^p(s)\,\varepsilon(s)\, ds\right)^{1/p} \tag{2.4}$$

takes place everywhere in I, where

$$\varepsilon(t) = \exp\left(-\int_a^t \psi^p(s)\, ds\right).$$

We observe that $\varepsilon(t)$ is a decreasing function and lower bounded by $\varepsilon(b) = \exp(-\|\psi\|^p)$.

3. Before showing the results, we shall prove;

Lemma 3.1. In conditions of existence and unicity for the equation (2.1) the operator (2.2) satisfies the inequality

$$\| V(t\,;x+h) - V(t\,;x)\| \leqslant H\,\|h\|, \tag{3.1}$$

for any $h \in L_p(X\,;\mu)$ with the property that $h + x \in W$, H being a constant.

Proof. Evidently

$$V(t\,;x+h) - V(t\,;x) = h(t) + \int_a^t \left\{K(t,s\,;V(s\,;x+h)) - K'(t,s\,;V(s\,;x))\right\} ds$$

from this equality using the hypotheses we obtain:

$$|V(t\,;x+h) - V(t\,;x)| \leqslant |h(t)| + \int_a^t \lambda(t\,;s)\,|V(s\,;x+h) - V(s\,;x)|\, ds.$$

Using the properties of $\lambda(t,s)$ and the definition of the operator $V(t\,;x)$, we can apply to the second member the Hölder inequality for $\lambda(t,.)\in L_q(X;\mu)$ and $V(t\,;x) \in L_p(X\,;\mu)$; we shall obtain

$$|V(t\,;x+h) - V(t\,;x)| \leqslant |h(t)| + \left(\int_a^t \lambda^q(t,s)\, ds\right)^{1/q}\left(\int_a^t |V(s\,;x+h) - V(s\,;x)|^p\, ds\right)^{1/p}. \tag{3.2}$$

Remarking in the inequality (3.2) that the functions: $|V(t\,;x+h) - V(t\,;x)|$, $|h(t)|$, $\left(\int_a^t \lambda^q(t,s)\, ds\right)^{1/q}$ are nonnegative and belong to L_p we may conclude

$$\left(\int_a^t |V(s\,;x+h) - V(s\,;x)|^p\, ds\right)^{1/p} \leqslant \frac{1}{1-[1-\varepsilon(t)]^{1/p}}\left(\int_a^t |h(s)|^p\,\varepsilon(s)\, ds\right)^{1/p}$$

everywhere in I, with

$$\varepsilon(t) = \exp\left(-\int_a^t\left\{\int_a^s \lambda^q(s,\tau)\, d\tau\right\}^{p/q} ds\right).$$

Using the properties of $\varepsilon(t)$ we obtain

$$\left(\int_a^t |V(s\,;x+h) - V(s\,;x)|^p\, ds\right)^{1/p} \leqslant \frac{\left(\int_a^t |h(s)|^p\, ds\right)^{1/p}}{1-[1-\varepsilon(b)]^{1/p}},$$

that is

$$\| V(t\,;x+h) - V(t\,;x)\| \leqslant H\,\|h\|,$$

where

$$H = (1-[1-\varepsilon(b)]^{1/p})^{-1} < +\infty$$

because $|||\lambda||| < +\infty$. With this, the lemma is proved.

Consequence 1. The $V(t\,;x)$ nonlinear operator is lipschitzian in the norm topology of $L_p(X\,;\mu)$.

Consequence 2. The $V(t\,;x)$ nonlinear operator is continuous in the same topology.

It must be remarked that these properties of the operator $V(t, x)$ are established only in the conditions of existence and unicity exposed above.

In the following we are compelled to suppose that the functions which occur are numerical, that means we shall work only in L_p. But all the reasoning remains true if the functions take values in a normed algebra over the field of real numbers.

Theorem 3.1. Let us admit the following conditions:

a) conditions of existence and unicity,

b) there exist $K_u(t\,;s\,;u(s))$ in I in the L_p norm,

c) $|||K_u||| < +\infty$,

d) $K_u(t,s\,;u(s))$ is continuous (in the norm topology) with regard to the third variable, that is $|||K_u(t,s,v(s)) - K_u(t,s\,;w(s))||| < \varepsilon$ when $\|v - w\| < \delta(\varepsilon)$.

In these conditions the $V(t\,;x)$ nonlinear integral operator is *F*-differentiable, and its F-differential satisfies the equation

$$DV(t,x;h)=h(t)+\int_a^t K_u(t,s;V(s;x))\,DV(s,x;h)\,ds \tag{3.3}$$

called equation in variations.

We must remark from the begining that the linear integral operator (3.3) allows an unic solution in hypotheses assumed for the kernel, hypotheses which together with the existence of the solution ensure its continuity.

Proof. Taking in account the definition of F-differential we shal deal with:

$$V(t;x+h)-V(t;x)-DV(t,x;h)=\int_a^t\{K(t,s;V(s;x+h))-$$

$$-K(t,s;V(s;x))-K_u(t,s;V(t;x))\,DV(s,x;h)\}\,ds=$$

$$=\int_a^t K_u(t,s;V(s))\{V(s;x+h)-V(s;x)\}\,ds-\int_a^t K_u(t,s;V(s;x))\,DV(s,x;h)\,ds=$$

$$=\eta(t)+\int_a^t K_u(t,s;V(s;x))\{V(s;x+h)-V(s;x)-DV(s,x;h)\}\,ds,$$

where

$$\eta(t)=\int_a^t\{K_u(t,s;V(s))-K_u(t,s;V(s;x))\}\{V(s;x+h)-V(s;x)\}\,ds.$$

In the admitted hypotheses we can establish for $\eta(t)$ the following inequality

$$|\eta(t)|\leqslant\int_a^t|K_u(t,s;\widetilde{V}(s))-K(t,s)\,V(s;x))|\,|V(s;x+h)-V(s;x)|\,ds$$

which, thanks to the Hölder inequality, gives

$$|\eta(t)|\leqslant\left(\int_a^t|K_u(t,s;V(s))-K_u(t,s;V(s;x))|^q\,ds\right)^{1/q}\left(\int_a^t|V(s;x+h)-\right.$$

$$\left.-V(s;x)|^p\,ds\right)^{1/p}$$

or

$$|\eta(t)|^p\leqslant\left(\int_a^t|K_u(t,s;V(s))-K_u(t,s;V(s;x))|^q\,ds\right)^{1/p}\left(\int_a^t|V(s;x+h)-\right.$$

$$\left.-V(s;x)|^p\,ds\right)^{1/p}.$$

Using the monotony of the integral and making convenient evaluations we find

$$\|\eta\|_p\leqslant\|V(x+h)-V(x)\|\,|||K_u(\widetilde{V})-K_u(V)|||<+\infty$$

whence it results that $\eta(t)\in L_p$.

Using the results of the lemma (3.1) we find for $\eta(t)$ the bound

$$\|\eta\|_p\leqslant H\|h\|\,|||K_u(\widetilde{V})-K_u(V)||| \tag{3.4}$$

but

$$|\widetilde{V}-V|=\widetilde{\omega}(V(x+h)-V(x)),\ \text{where}\ \widetilde{\omega}\in[0,1],$$

therefore $\|\widetilde{V}-V\|\leqslant\widetilde{\omega}H\|h\|$.

From the properties imposed on K_u and from the last result we find that

$$|||K_u(\widetilde{V})-K_u(V)|||\leqslant\Lambda(\|h\|)\,, \tag{3.5}$$

where Λ is a monoton increasing continuous and positive function, with the property that it tends to zero when the argument tends to zero.

By combining (3.4) and (3.5) we find

$$\|\eta\|\leqslant H\|h\|\,\Lambda(\|h\|) \tag{3.6}$$

necessary for the proof of the theorem.

In the new notation we obatin

$$|V(t;x+h)-V(t;x)-DV(t,x;h)|\leqslant|\eta(t)|+\int_a^t|K_u(t,s;V(s;x))|\,.$$

$$.\,|V(s;x+h)-V(s;x)-DV(s,x;h)|\,ds;$$

if we apply here Hölder's inequality we obtain

$$|V(t;x+h)-V(t;x)-DV(t,x;h)|\leqslant|\eta(t)|+$$

$$+\left(\int_a^t|K_u(t,s;V(s;x))^q\,ds\right)^{1/q}\left(\int_a^t|V(s;x+h)-V(s;x)-DV(s,x;h)|^p\,ds\right)^{1/p}.$$

By using the properties of the functions that occur in the last inequality as well as Gronwal's inequality we obtain

$$(3.7)\quad \left(\int_a^t |V(s;x+h)-V(s;x)-DV(s,x;h)|^p ds\right)^{1/p} \leqslant \frac{\left(\int_a^t |\eta(s)|^p \varepsilon(s)\, ds\right)^{1/p}}{1-[1-\varepsilon(t)]^{1/p}},$$

for all $t \in I$, where

$$\varepsilon(t) = \exp\left(-\int_a^t \left\{\int_a^s |K_u(s,\tau;V(\tau;x))|^q d\tau\right\}^{p/q} ds\right).$$

Using in (3.7) the properties of $\varepsilon(t)$ we get

$$(3.8)\quad \left(\int_a^b |V(s;x+h)-V(s;x)-DV(s,x;h)|^p ds\right)^{1/p} \leqslant \frac{\|\eta\|}{1-[1-\varepsilon(b)]^{1/p}}.$$

By combining the inequalities (3.8) and (3.6) we find

$$(3.9)\quad \|V(t;x+h)-V(t;x)-DV(t,x;h)\| \leqslant H\|h\|\Lambda(\|h\|),$$

where $H = \|\eta\|(1-[1-\varepsilon(b)]^{1/p})^{-1}$.

The inequality (3.9) permits us to conclude that

$$\|V(t;x+h)-V(t;x)-DV(t,x;h)\| = o(\|h\|),$$

that means

$$\lim_{h\to 0}\|V(t;x+h)-V(t;x)-DV(t,x;h)\| = 0;$$

the theorem is completely proved.

Consequence 1. The F-derivative of $V(t;x)$ will be the operator $V'(t;x)$ defined by the equality $DV(t, x; h) = V'(t, x)(h)$.

Consequence 2. In the conditions of theorem (3.1) the operator $V(t;x)$ is G-differentiable and its G-differential coincides with $DV(t,x;h)$.

Consequence 3. If the conditions of the theorem (3.1) are satisfied in a convexe neighborhood of $x \in L_p$ then for any $y = x+h$ of this neighborhood, takes place the inequality

$$\|V(x+h)-V(x)-DV(x;h)\| \leqslant \sup_{0\leq t\leq 1}\|DV(x+th;h)-VD(x;h)\|.$$

Evidently these results are applicable for entire classes of nonlinear Volterra integral operators in which the kernel K has different forms.

The concrete forms of the theorems in the cases of particular kernels. will constitute the object of a further Note.

Theorem 3.2. Preserving the hypotheses of the theorem 3.1, we may conclude that the operator $DV(t,x;h) = DV(x,h)$ is a continuous operator relating to x for every fixed $h \in L_p$.

Proof. Let x_1 and x_2 be two elements of L_p and h fixed. Taking into account the usual evaluations, we find

$$|DV(x_1,h)-DV(x_2,h)| \leqslant \int_a^t |K_u(t,s;V(s,x_1))|\,|DV(x_1,h)-DV(x_2,h)|ds +$$
$$+\int_a^t |K_u(t,s;V(s;x_1))-K_u(t,s;V(s;x_2))|\,|DV(x_2;h)|ds.$$

Applying Hőlder's inequality to the first term of the second member and denoting the last term by $\theta(t)$ we obtain

$$(3.10)\quad \begin{aligned}|DV(x_1,h)-DV(x_2,h)| &\leqslant |\theta(t)| + \left(\int_a^t |K_u(t,s;V(s;x_1))|^q ds\right)^{1/q}\cdot\\ &\cdot\left(\int_a^t |DV(x_1;h)-DV(x_2;h)|^p dx\right)^{1/p}.\end{aligned}$$

But $\theta(t)$ satesfies the inequality

$$|\theta(t)| \leqslant \left(\int_a^t |K_u(t,s;V(x_1))-K_u(t,s;V(x_2))|^q ds\right)^{1/q}\left[\int_a^t |DV(x_2,h)|^p ds\right]^{1/p}$$

or

$$|\theta(t)|^p \leqslant \left[\int_a^t |K_u(t,s;V(x_1))-K_u(t,s;V(x_2))|^q ds\right]^{p/q}\int_a^t |DV(x_2;h)|^p ds.$$

Integrating in both members from a to b, taking the $1/p$ power of the result and considering that the F-differential $V(t;x)$ is a continuous linear operator, we find

$$(3.11)\quad \|\theta\|_p \leqslant M(h)\,\widetilde{\Lambda}(\|x_1-x_2\|),$$

where $\widetilde{\Lambda}(r)$ is a function of the same type as that already used.

Appliyng Gronwal's inequality, from (3.10), we obtain

$$(3.12)\quad \left[\int_a^t |DV(x,h)-DV(x,h)|^p ds\right]^{1/p} \leqslant \frac{\left[\int_a^t \theta(s)^p \varepsilon(s)\, ds\right]^{1/p}}{1-[1-\varepsilon(t)]^{1/p}}$$

for all $t \in I$, where

$$z(t) = \exp\left[-\int_a^t \left\{ \int_a^s |K_u(s, \tau; V(x; \tau))|^q \, d\tau \right\}^{p/q} ds \right].$$

By combining the inequalities (3.12) and (3.11) we find

$$\|DV(x_1, h) - DV(x_2, h)\| \leqslant P \cdot \Lambda(\|x_1 - x_2\|)$$

where P is a constant which depends of h.

The affirmation is completely proved.

We remark that the operator $DV(x; h)$ will be uniformly continuous with regard to both the arguments (i.e. on the product space $L_p \times W$) if and only if the following conditions are satisfied:

1°. the family of mappings $x \to DV(x, h)$, $h \in L_p$ is uniformly equicontinuous, and

2°. the family of mappings $h \to DV(x, h)$, $x \in W$ is uniformly equicontinuous.

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ASUPRA F-DIFERENŢIABILITĂŢII SOLUŢIEI OPERATORULUI INTEGRAL NELINIAR VOLTERRA

Rezumat

Se consideră problema diferenţiabilităţii în sens Fréchet a soluţiei ecuaţiei integrale neliniare Volterra (2.1) în raport cu termenul liber, de care depinde soluţia.

Diferenţiabilitatea operatorului $V(t; x)$ este studiată în spaţiul $L_p(X; \mu)$, $p \geq 1$.

SUR L'ÉVALUATION DE L'ERREUR DANS LA RÉSOLUTION DES SYSTÈMES LINÉAIRES

PAR

C. ILIOI

Soit $Ax = f$ un système algébrique d'équations linéaires avec $\det(A) \neq 0$. Il s'agit souvent de connaître l'effet du changement des éléments de la matrice A sur la solution du système. La variation de la solution du système à une perturbation de la matrice A dépend de la variation de la matrice inverse. Pour caractériser cette dépendance on a introduit dans [4], [5] les nombres de conditionnement des matrices. En général, le nombre de conditionnement de la matrice A este $n(A) = \|A\| \cdot \|A^{-1}\|$, et le fait qu'il est une mesure du conditionnement de la matrice est une conséquence de l'inégalité

$$\frac{\|A^{-1} - (A+E)^{-1}\|}{\|A^{-1}\|} \leqslant \frac{n(A)\,\varepsilon}{1 - n(A)\,\varepsilon}, \quad \text{avec } \varepsilon = \frac{\|E\|}{\|A\|}, \; n(A)\,\varepsilon < 1, \tag{1}$$

où E est la variation de la matrice A (v. [3], p. 141). Il est évident que l'erreur relative de la matrice inverse est proportionnelle au nombre de conditionnement n de la matrice.

Le problème que nous étudierons est plus particulier que celui énoncé ci-dessus, mais il n'est pas résolu d'une manière satisfaisante par cette théorie générale. C'est pourquoi nous allons trouver d'autres méthodes pour évaluer l'erreur, plus convenables pour notre problème.

§. 1. Soit le système

$$\mathcal{A}\mathcal{X} = \mathcal{F} \text{ avec } \det(\mathcal{A}) \neq 0, \tag{2}$$

où la matrice quadratique $\mathcal{A}$ d'ordre N peut être considérée partageé en quatre sous-matrices :

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3}$$

A et D étant des matrices quadratiques d'ordre n, respectivement m, $n + m = N$;