# ON THE REGULARITY OF SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

BY

V. BARBU

In a preceding Note [1] we have studied the propagation of hypoanalyticity of solutions of differential equations with constant coefficients, excluding the analytic case. In the present paper, using a result of Boman [3] on the intersection of non-quasianalytic classes of functions, we consider this case. We use the notation and definition of [4] and [6]. We denote the points of $R^n$ by $x = (x_1, \ldots, x_n)$ and we set $D_j = -i\partial/\partial x_j$. For each $n$-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha|$ denotes $\sum_{i=1}^{n} \alpha_i$ and we set $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$. If $K$ is any compact set of $R^n$ we set

$$\| u, K \|_\infty = \operatorname*{ess\,sup}_K | u(x) | .$$

If $u$ is a distribution with compact support, $u \in \mathcal{E}'(R^n)$ then $\hat{u}(\zeta)$ denotes its Fourier-Laplace transform.

Let $\Omega$ be an open set of $R^n$ and let $L = \{L_k\}$ be an increasing sequence of positive numbers. We shall denote by $C^L(\Omega)$ the space of infinitely differentiable functions in $\Omega$, such that for any compact set $K \subset \Omega$ there exists a constant $M$ such that the following inequality be true

$$\| D^\alpha u, K \|_\infty \leqslant M^{|\alpha|+1} L_{|\alpha|}^{|\alpha|} . \tag{1}$$

A class $C^L(\Omega)$ is called non-quasianalytic if it contains a non-trivial function with compact support contained in $\Omega$. The Carleman-Denjoy criterion states that $C^L$ is non-quasianalytic if and only if $\sum_{k=0}^{\infty} L_k^{-1} < \infty$. If $L_k = k^\rho$ with $1 \leqslant \rho \leqslant \infty$, then $C^L(\Omega)$ is the Gevrey class $G^\rho(\Omega)$. We denote the analytic class $G^1(\Omega)$ by $A(\Omega)$.

**Theorem 1.** (Boman [3]). *Let $R = (R_k)$ be a positive increasing sequence such that $\sum R_k^{-1} < \infty$. Then the intersection of all non-quasianalytic classes $C^L$, where $L$ is increasing and $L_k/R_k$ is decreasing, is equal to the analytic class $A$ if and only if $\sum_k^{\infty} R_j^{-1} \leq Ck/R_k$, $k = 1, 2, \ldots$ for some constant $C$.*

If $C^L$ is a non-quasianalytic class, we denote by $F_L(\xi)$ the function

$$(2)\qquad F_L(\xi) = \sum_{k=0}^{\infty} (|\xi|/L_k)^k .$$

Concerning the sequence $L = \{L_k\}$ we shall assume that, for any positive integer $m$, there exists a constant $C$ such that

$$(3)\qquad L_{mk} \leq C^m L_k, \qquad k = 1, 2, \ldots$$

We shall prove

**Theorem 2.** *Let $u \in \mathcal{E}'(R^n)$ be a distribution and let $C^L$ be a non-quasianalytic class with $L$ satisfying (3). Let $K$ be a compact convex set in $R^n$ with support function $H$. In order that $u \in C^L(R^n \setminus K)$, it is necessary and sufficient that there exist constants $N, A, A_1$, and a sequence of constants $C_k$ such that for $k$ large enough $|\operatorname{Im}\zeta| \leq A_1 \log F_L(C_k \operatorname{Re}\zeta) + A$, implies*

$$(4)\qquad |\hat{u}(\zeta)| \leq C_k (1 + |\zeta|)^N \exp(H(\operatorname{Im}\zeta) + 1/k|\operatorname{Im}\zeta|); \quad \zeta \in C^n.$$

*Proof.* Here $H$ denotes the function $H(\xi) = \sup_{x \in K} (x, \xi)$. Let $K_\varepsilon$ be an $\varepsilon$-neighborhood of the compact $K$ and let $\chi_\varepsilon$ be a $C_0^\infty$ function with its support in $K$ and equal to 1 on $K_{\varepsilon/2}$. Since the class $C^L(\Omega)$ is non-quasianalytic, we choose the function $\chi_\varepsilon(x)$ such that

$$(5)\qquad \|D^\alpha \chi_\varepsilon\|_\infty \leq M \varepsilon^{-|\alpha|+1} L_{|\alpha|}^{|\alpha|}$$

with a certain constant $M$ independent of $\alpha$. Thus by Paley-Wiener theorem we have

$$(6)\qquad |\widehat{\chi_\varepsilon u}(\zeta)| \leq C_\varepsilon (1 + |\zeta|)^N \exp(H(\operatorname{Im}\zeta) + \varepsilon|\operatorname{Im}\zeta|), \qquad \zeta \in C^n .$$

On the other hand from the hypothesis it follows

$$\|D^\alpha(1 - \chi_\varepsilon)u\|_\infty \leq M_\varepsilon^{|\alpha|+1} L_{|\alpha|}^{|\alpha|}.$$



Hence

$$(7)\qquad |\widehat{(1-\chi_\varepsilon) u}(\zeta)| \leq (M_\varepsilon L_k)^k / |\operatorname{Re}\zeta|^k \exp(B|\operatorname{Im}\zeta|).$$

Then by an elementary calculus it follows

$$(8)\qquad |\widehat{(1-\chi_\varepsilon) u}(\zeta)| \leq M \exp(B|\operatorname{Im}\zeta|)/F_L(A_2\varepsilon \operatorname{Re}\zeta).$$

Now (4) follows from (6) and (8) if we take $\varepsilon = 1/k$.

To prove the sufficiency of (4) it is enough to show that $u \in C^L$ in any set of the form

$$\Omega_a = \{x \in R^n;\ x_n > a > H(\eta)\}.$$

where $\eta = (\eta_j)$; $\eta_j = \delta_j^n$. In the complex space $C^n$, we consider the variety

$$(9)\qquad \Gamma_k = \{\zeta \in C^n;\ \operatorname{Im}\zeta_n = A_1 K^{-1} \log F_L(C_k(|\operatorname{Re}\zeta_n|^2 + \sum_{j=1}^{n-1} \xi_j^2)/k)\}.$$

Remark now that, if $\varphi \in C_0^\infty(\Omega_a) \cap C^L$, then for $k$ so large,

$$|\hat{u}(\zeta)\hat{\varphi}(-\zeta)| \to \infty, \text{ for } |\zeta| \to \infty, \ \zeta \in \Gamma_k .$$

Hence we may write

$$(10)\qquad u(\varphi) = (2\pi)^{-n} \int d\xi' \int \hat{u}(\xi', \zeta_n)\, \hat{\varphi}(-\xi', -\zeta_n) d\zeta_n ,$$

where the integration with respect to $\zeta_n$ is over the boundary $\Gamma_k$ defined by (9). Since the space $C_0^\infty \cap C^L$ is dense in $C_0^\infty$, this implies that for $x_n > a$, the distribution $u$ is equal to the function

$$u(x) = (2\pi)^{-n} \int \exp i < x', \xi' > d\xi' \int_{\Gamma_k} \exp(ix_n\zeta_n)\hat{u}(\xi', \zeta_n) d\zeta_n .$$

If we denote $p = -(H(\eta) - a + 1/k)$, then for $k$ large enough it follows the estimate

$$(11)\qquad \|D^\alpha u, \Omega_a\| \leq M \int |\xi|^\alpha F_L^{-p}(A\xi)\, d\xi .$$

Since $F_L(\xi) \geq |\xi|^j L_j^{-j}$ for every $j = 1, 2, \ldots$, then if we choose $j$ such that $jp - n - 2 \leq |\alpha| \leq jp - n - 1$, from (11) we obtain

$$\|D^\alpha u, \Omega_a\|_\infty \leq M^{|\alpha|+1} L_{q|\alpha|}^{|\alpha|},$$

with a certain non-negative integer $q$ independent of $\alpha$. Now from (3) it follows that $u \in C^L(\Omega_a)$ for each $a > H(\eta)$ and this proves the theorem.

**Corollary 1.** *Let $K$ be a compact convex set in $R^n$ and let $P(D)$ be a partial differential operator with constant coefficients. If $u \in \mathcal{E}'(R^n)$ and $P(D)u \in C^L(R^n \setminus K)$, then $u \in C^L(R^n \setminus K)$.*

*Proof* follows from preceding theorem, using the inequality (see H ö r m a n d e r [2])

$$|\hat{u}(\zeta)| \leq \sup_{|z| \leq 1} |P(\zeta + z)\, \hat{u}(\zeta + z)|, \qquad \zeta \in C^n$$

As a specialization of corollary 1, we have the following

**Corollary 2.** *Let $L = \{L_k\}$ be a sequence satisfying (3) and $\sum L_k^{-1} < \infty$. Let $u$ be a distribution in $\Omega \subset R^n$ such that $u \in C^L(\Omega \setminus Q)$, where $Q$ is a compact subset of $\Omega$. If $K$ is a compact convex of $\Omega$ with $K \subset Q$, such that $P(D)u \in C^L(\Omega \setminus K)$, then $u \in C^L(\Omega \setminus K)$.*

As a consequence we can prove the following theorem, of propagation of hypoanalyticity (see [5]).

**Theorem 3.** *Let $K$ be a compact convex contained in $\Omega$ and let $u$ be a distribution $u \in \mathcal{D}'(\Omega)$ which verifies in $\Omega \setminus K$ the equation $P(D)u = f$. We assume that $f \in G^\rho(\Omega)$ and $u \in G^\rho(\Omega \setminus \bar{\Omega}')$ where $K \subset \Omega' \subset \Omega$. Then $u \in G^\rho(\Omega \setminus K)$.*

*Proof.* If $\rho > 1$ then the Gevrey class $G^\rho$ is non-quasianalytic and the theorem is a specialization of corollary 2. Now we suppose that $\rho = 1$. If in theorem 1 we take $R_k = k^\delta$, with $\delta > 1$, then we obtain that the class of real analytic functions $A(\Omega)$ is the intersection of all non-quasianalytic classes $C^L$, where $L = \{L_k\}$ satisfies the condition (3). Thus applying corollary 2, we obtain the desired result.

**Remark 1.** Let $u \in \mathcal{D}'(\Omega)$ and $\Omega' \subset R^{n'}$, $\Omega'' \subset R^{n''}$ be two open sets such that $\Omega' \times \Omega'' \subset \Omega$, $R^n = R^{n'} \times R^{n''}$. For $\varphi \in C_0^\infty(\Omega'')$, we consider the distribution $u \cdot \varphi \in \mathcal{D}'(\Omega')$ defined by the relation $u \cdot \varphi(\psi) = u(\varphi \otimes \psi)$. If $L = \{L_k\}$ is an increasing sequence, we denote by $C_x^L(\Omega)$ the space of all distributions $u \in \mathcal{D}'(\Omega)$ such that $u \cdot \varphi \in C^L(\Omega'')$ whenever $\varphi \in C_0^\infty(\Omega'')$ and $\Omega' \times \Omega'' \subset \Omega$. Then the preceding theorem can be extended to the space $C_x^L$ (see [1]).

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ASUPRA REGULARITĂȚII SOLUȚIILOR ECUAȚIILOR
CU DERIVATE PARȚIALE LINIARE

Rezumat


În această Notă se demonstrează o teoremă de propagare a analiticității soluțiilor unui operator diferențial cu coeficienți constanți. În acest scop se stabilește o teoremă de caracterizare a distribuțiilor cu suportul non-evasianalitic într-un compact convex.