ON THE ARITHMETIC MEAN OF LAGUERRE FUNCTIONS WITH TRUNCATED DOMAIN

BY

H. L. MANOCHA

1. Let

$$(1)\qquad \bar{L}_{n,x,m} = \int_x^\infty u\, L_n(mu)\, e^{-u}\, du \Big/ \left[\int_x^\infty L_n(mu)\, e^{-u}\, du\right],$$

$$0 \leq x \leq u < \infty, \quad 0 < m, \quad n = 0, 1, 2, \dots .$$

where L_n is the n-th Laguerre polynomial.

In this paper we propose to prove a theorem for $\bar{L}_{n,x,m}$, similar to the one proved by Bayard Rankin in his paper [4]. Rankin has proved for $m = 2$, while we give the proof for values of m other than 2, and also 1.

It may be mentioned that statements of the type $\bar{L}_{n,0,2} = 1 + 2n$ have occured in the phase-space analysis of the quantized harmonic oscillator [1, 2, 3] and have been shown, in that analysis, to supplement the more familiar energy eigenvalue equations: $2H\Phi_n = (1 + 2n)h\nu\Phi_n$, $n = 0, 1, 2, \dots$, where H is the Hamiltonian operator for the oscillator, h is Planck's constant, ν is the frequency of oscillation, and Φ_n, $n = 0, 1, 2, \dots$ are the stationary wave functions.

2. The generalized Laguerre polynomial of order α is defined as

$$(2)\qquad L_n^{(\alpha)}(x) = (n!)^{-1}\, e^x x^{-\alpha} \frac{d^n}{dx^n}\left[e^{-x} x^{n+\alpha}\right].$$

For zero order, we write $L_n(x) = L_n^{(0)}(x)$, and for $n = 0$, $L_0(x) = 1$.

We require the following known identities with a view to proving some lemmas:

$$(3)\qquad L_n(xy) = \sum_{k=0}^{n} \frac{n!\,(1-y)^{n-k}\, y^k L_k^{(\alpha)}}{(n-k)!\; k!}.$$

(4) $$\int_0^\infty e^{-u} L_m(u)\, L_n(u)\,du = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

(5) $$mx\, L^{(1)}_{n-1}(mx) = nL_{n-1}(mx) - nL_n(mx), \quad (6)\ \frac{d}{dx} L_n(x) = -L^{(1)}_{n-1}(x),$$

(7) $$\int_x^\infty L_n(mu)\, e^{-u}\,du = e^{-x}[L_n(mx) - L_{n-1}(mx)] + (1-m)\int_x^\infty L_{n-1}(mu)\, e^{-u}\,du.$$

(8) $$nL_n(mx) = (2n-1-mx)\, L_{n-1}(mx) - (n-1)\, L_{n-2}(mx).$$

The lemmas that will be necessary for our main result are as below:

Lemma 1. *For values of m other than 1 and n non-negative integer, the following evaluation holds:*

(9) $$L_{n,0,m} = 1 - \frac{mn}{1-m}.$$

This can be proved easily by using (3) and (4).

Lemma 2. *For each non-negative integer n, the global extremum of* $L_n(mx)e^{-x}$ *occurs at* $x = 0$. *For* $0 < x$, there are n local extrema *which are roots of the equation:*

(10) $$xL_n(mx) = n[L_n(mx) - L_{n-1}(mx)].$$

This follows from definition (2) and identities (5) and (6).

Lemma 3. *For* $0 \le x$, $m \neq 1$ *and* $n = 1, 2, 3, \ldots$, *the following formulas hold:*

(11) $$\int_x^\infty L_n(mu)\, e^{-u}\,du = e^{-x} L_n(mx) - me^{-x}\sum_{k=0}^{n} (1-m)^{k-1} L_{n-k}(mx),$$

(12) $$\int_x^\infty uL_n(mu)\, e^{-u}\,du = (x+1)e^{-x} L_n(mx) - \frac{m(x+2)}{1-m}\, e^{-x}\sum_{k=1}^{n} (1-m)^k L_{n-k}(mx) + \frac{m^2}{(1-m)^2}\, e^{-x}\sum_{k=1}^{n} (k-1)(1-m)^k L_{n-k}(mx).$$

This can be proved by repeated application of identity (7) and integration by parts.

Lemma 4. *For* $0 \le x$ *and* $n = 1, 2, 3, \ldots$, *the following identity holds:*

(13) $$m\sum_{k=1}^{n} (x+1)(1-m)^k L_{n-k}(mx) - m^2\sum_{k=1}^{n} (x+k-n)(1-m)^k L_{n-k}(mx) = n(1-m)L_{n-1}(mx) - n(1-m)^2 L_n(mx).$$

To prove this, we have from identity (8) and the definitions $L_{-1} = 0$, $L_0 = 1$, $L_1(x) = 1 - x$

(14) $$(n+1-k)\, L_{n+1-k}(mx) - [2(n+1-k) - 1 - mx]\, L_{n-k}(mx) + (n-k)\, L_{n-k-1}(mx) = 0.$$

Multiplying both sides by $(1-m)^{k+1}$ and summing over k we have

$$m\sum_{k=1}^{n} (x+1)(1-m)^k L_{n-k}(mx) - m^2 \sum_{k=1}^{n} (x+k-n)(1-m)^k L_{n-k}(mx) =$$

$$= \sum_{k=1}^{n} \{(n-k+1)(1-m)^k L_{n-k}(mx) - (n-k)(1-m)^{k+1} L_{n-k-1}(mx)\} -$$

$$- \sum_{k=1}^{n} \{(n-k+1)(1-m)^{k+1} L_{n-k+1}(mx) - (n-k)(1-m)^{k+2} L_{n-k}(mx)\} =$$

$$= n(1-m)\, L_{n-1}(mx) - n(1-m)^2\, L_n(mx).$$

This completes the proof of lemma 3.

3. The main result is expressed in the following

Theorem. *For all the values of m, except* $m = 1, 2$, *and non-negative integer n, the number* x_i *is a point where the Laguerre function,* $L_n(mx)e^{-x}$, $0 \le x < \infty$, *assumes a local extremum, if and only if* x_i *is a solution of the equation*

(15) $$\overline{L}_{n,x,m} = 1 - \frac{mn}{1-m}$$

and also satisfies

(16) $$L_n(mx) - L_{n-1}(mx) = 0.$$

Proof. When $n = 0$ the statement of the theorem is immediate and the unique solution of (15) is $x = 0$. For $n = 1, 2, 3, \ldots$, direct calculation using (11) and (12) shows that the statement of the theorem is true for the global extremum $x = 0$. For $n = 1, 2, 3, \ldots$, and $0 < x$, the following can be verified by simple algebraic substitution:

(13), (15) and (16) imply (10), while (10), (13) and (16) imply (15).

Thus, since (13) is an identity and (10) determines the local extrema for positive x, the statement of the theorem follows.

Department of Applied Sciences,
Punjab Engineering College, Chandigarh
India

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DESPRE MEDIA ARITMETICĂ A FUNCȚIILOR LAGUERRE CU DOMENIU TRUNCHIAT

Rezumat

Rezultatul esențial din această Notă este acela că dacă $m \neq 1, 2$ și $n \neq$ întreg negativ, x_f este un extremum local pentru funcția Laguerre $L_n(mx)e^{-x}$ dacă satisface (15) și (16).

DOUBLE INTEGRALS INVOLVING BESSEL FUNCTIONS

BY

B. L. SHARMA

1. Introduction. A function $\Phi(p)$ is operationally related with $h(t)$ when they satisfy the integral equation

$$(1) \qquad \Phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt\,,$$

provided the integral is convergent and $R(p) > 0$.

As usual, we shall denote (1) by the symbol

$$(2) \qquad \Phi(p) \doteqdot h(t)\,.$$

The object of this paper is to prove a theorem in operational calculus and use it to evaluate double integrals involving products of six Bessel functions of different arguments. The results obtained are believed to be new.

2. Theorem. *If* $\Phi(p) \doteqdot h(1/t)$ and $\psi(p) = I_\mu(2abt)\, I_\rho(2cet)\, h(t)$ *then*

$$(3) \qquad \int_0^\infty \int_0^\infty J_\mu(2a\sqrt{x})\, J_\mu(2b\sqrt{x})\, J_\rho(2c\sqrt{y})\, J_\rho(2e\sqrt{y})\, (x+y)^{-1}\, \Phi(x+y)\, dx\, dy = (a^2+b^2+c^2+e^2)^{-1}\, \psi(a^2+b^2+c^2+e^2)\,,$$

provided the integrals are absolutely convergent and $a > 0$, $b > 0$, $c > 0$, $e > 0$.

Proof. By definition, we have

$$\Phi(p) = p \int_0^\infty e^{-pt}\, h\left(\frac{1}{t}\right) dt\,;$$

then