for the global extremum $x = 0$. For $n = 1, 2, 3, \ldots$, and $0 < x$, the following can be verified by simple algebraic substitution:

(13), (15) and (16) imply (10), while (10), (13) and (16) imply (15).

Thus, since (13) is an identity and (10) determines the local extrema for positive x, the statement of the theorem follows.

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DESPRE MEDIA ARITMETICĂ A FUNCȚIILOR LAGUERRE CU DOMENIU TRUNCHIAT

Rezumat

Rezultatul esențial din această Notă este acela că dacă $m \neq 1, 2$ și $n \neq$ întreg negativ, x_l este un extremum local pentru funcția Laguerre $L_n(mx)e^{-x}$ dacă satisface (15) și (16).

DOUBLE INTEGRALS INVOLVING BESSEL FUNCTIONS

BY

B. L. SHARMA

1. Introduction. A function $\Phi(p)$ is operationally related with $h(t)$ when they satisfy the integral equation

$$\Phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt\,, \tag{1}$$

provided the integral is convergent and $R(p) > 0$.

As usual, we shall denote (1) by the symbol

$$\Phi(p) \doteqdot h(t)\,. \tag{2}$$

The object of this paper is to prove a theorem in operational calculus and use it to evaluate double integrals involving products of six Bessel functions of different arguments. The results obtained are believed to be new.

2. Theorem. *If* $\Phi(p) \doteqdot h(1/t)$ and $\psi(p) = I_\mu(2abt)\, I_\nu(2cet)\, h(t)$ *then*

$$\int_0^\infty \int_0^\infty J_\mu(2a\sqrt{x})\, J_\mu(2b\sqrt{x})\, J_\nu(2c\sqrt{y})\, J_\nu(2e\sqrt{y})\, (x+y)^{-1}\, \Phi(x+y)\, dx\, dy = (a^2+b^2+c^2+e^2)^{-1}\, \psi(a^2+b^2+c^2+e^2)\,, \tag{3}$$

provided the integrals are absolutely convergent and $a > 0$, $b > 0$, $c > 0$, $e > 0$.

Proof. By definition, we have

$$\Phi(p) = p \int_0^\infty e^{-pt}\, h\left(\frac{1}{t}\right) dt\,;$$

then

$$\int_0^\infty\int_0^\infty J_\mu(2a\sqrt{x})\, J_\nu(2b\sqrt{x})\, J_\rho(2c\sqrt{y})\, J_\rho(2e\sqrt{y})\,(x+y)^{-1}\Phi(x+y)\,dx\,dy =$$

$$= \int_0^\infty\int_0^\infty J_\mu(2a\sqrt{x})\, J_\mu(2b\sqrt{x})\, J_\rho(2c\sqrt{y})\, J_\rho(2e\sqrt{y}) \left[\int_0^\infty e^{-(x+y)t}\, h\left(\frac{1}{t}\right) dt\right] dx\,dy =$$

$$= \int_0^\infty h\left(\frac{1}{t}\right)\left[\int_0^\infty J_\mu(2\sqrt{x})\, J_\mu(2b\sqrt{x})\, e^{-xt}\,dx \int_0^\infty J_\rho(2c\sqrt{y})\, J_\rho(2e\sqrt{y})\, e^{-yt}\,dy\right] dt =$$

$$= \int_0^\infty e^{-(a^2+b^2+c^2+e^2)t}\, I_\mu(2abt)\, I_\rho(2cet)\, h(t)\,dt =$$

$$= (a^2+b^2+c^2+e^2)^{-1}\psi\,(a^2+b^2+c^2+e^2)\,,$$

on changing the order of integration and evaluating the inner integral by means of formula [2, p. 281, Equ. 23]

$$(4)\qquad \int_0^\infty J_\mu(2\sqrt{at})\, J_\mu(2\sqrt{bt})\,e^{-pt}\,dt = \frac{1}{p}\,\exp\left(-\frac{a+b}{p}\right)\, I_\mu\left(\frac{2\sqrt{ab}}{p}\right)$$

where $R(\mu) > -1$.

The change of the order of integration can be justified by the application of de la Vallée Poussin's theorem [1, p. 504], when the integrals involved are absolutely convergent.

3. Integrals. If we take [1, p. 200, Equ. 4]

$$(5)\qquad h\left(\frac{1}{t}\right) = t^{-1}\exp\left(-\frac{g^2+h^2}{t}\right) I_\nu\left(\frac{2gh}{t}\right) \doteqdot 2pK_\nu(2g\sqrt{p})\, I_\nu(2h\sqrt{p}) = \Phi(p)\,,$$

$$R(g) \geq R(h) > 0,\quad R(p) > 0\,.$$

Also [3, p. 86, Equ. 3]

$$I_\mu(2abt)\, I_\rho(2cet)\, h(t) = t\,\exp\left[-(g^2+h^2)\,t\right] I_\nu(2ght)\, I_\mu(2abt)\, I_\rho(2cet) \doteqdot$$

$$\doteqdot \frac{\Gamma(\nu+\mu+\rho+2)\,(gh)^\nu\,(ab)^\mu\,(ce)^\rho\, p}{\Gamma(\nu+1)\,\Gamma(\mu+1)\,\Gamma(\rho+1)\,(p+g^2+h^2)^{\nu+\mu+\rho+1}}\quad F_c\left[\frac{1}{2}\,(\nu+\mu+\rho+2),\right.$$

$$\frac{1}{2}\,(\nu+\mu+\rho+3);\ \nu+1,\ \mu+1,\ \rho+1;\ \frac{4g^2h^2}{(p+g^2+h^2)^2},\ \frac{4a^2b^2}{(p+g^2+h^2)^2},$$

$$\left.\frac{4c^2e^2}{(p+g^2+h^2)^2}\right] = \psi\,(p),\quad R(\nu+\mu+\rho+2) > 0\,,$$

$$(6)\qquad R(p+g^2+h^2) > |R(2gh)| + |R(2ab)| + |R(2ce)|\,.$$

Using (5) and (6) in (3), we obtain

$$(7)\qquad \int_0^\infty\int_0^\infty J_\mu(2a\sqrt{x})\, J_\mu(2b\sqrt{x})\, J_\rho(2c\sqrt{y})\, J_\rho(2e\sqrt{y})\, K_\nu(2g\sqrt{x+y})$$

$$\times\, I_\nu(2h\sqrt{x+y})\,dx\,dy =$$

$$= \frac{(ab)^\mu\,(gh)^\nu\,(ce)^\rho\,\Gamma(\nu+\mu+\rho+2)}{2\Gamma(\nu+1)\,\Gamma(\mu+1)\,\Gamma(\rho+1)\,(a^2+b^2+c^2+e^2+g^2+h^2)^{\nu+\mu+\rho+1}}\times$$

$$\times\, F_c\left[\frac{1}{2}\,(\nu+\mu+\rho+2),\ \frac{1}{2}\,(\nu+\mu+\rho+3);\ \nu+1,\ \mu+1,\ \rho+1;\right.$$

$$\frac{4g^2h^2}{(a^2+b^2+c^2+e^2+g^2+h^2)^2},$$

$$\left.\frac{4a^2b^2}{(a^2+b^2+c^2+e^2+g^2+h^2)^2},\quad \frac{4c^2e^2}{(a^2+b^2+c^2+e^2+g^2+h^2)^2}\right]$$

valid for $R(\nu+\mu+\rho+2) > 0$, $R(g) > |R(h)| + |\mathrm{Im}\, a| + |\mathrm{Im}\, b|$, $R(g) > |R(h)| + |\mathrm{Im}\, c| + |\mathrm{Im}\, e|$.

By using the well known formulae

$$(8)\qquad K_\nu(z) = \frac{1}{2}\sum_{\nu,-\nu}\Gamma(-\nu)\,\Gamma(\nu+1)\, I_\nu(z)$$

and

$$K_{-\nu}(z) = K_{-\nu}(z)$$

in (7), we get

$$(9)\qquad \int_0^\infty\int_0^\infty J_\mu(2a\sqrt{x})\, J_\mu(2b\sqrt{x})\, J_\rho(2c\sqrt{y})\, J_\rho(2e\sqrt{y})\, K_\nu(2g\sqrt{x+y})\, K_\nu(2h\sqrt{x+y})\,dx\,dy =$$

$$= \sum_{\nu,-\nu}\frac{(ab)^\mu\,(gh)^\nu\,(ce)^\rho\,\Gamma(\nu+\mu+\rho+2)\,(-\nu)}{4\Gamma(\mu+1)\,\Gamma(\rho+1)\,(a^2+b^2+c^2+e^2+g^2+h^2)^{\rho+\mu+\nu+1}}\times$$

$$\times\, F_c\left[\frac{1}{2}\,(\nu+\mu+\rho+2),\ \frac{1}{2}\,(\nu+\mu+\rho+3);\ \nu+1,\ \mu+1,\ \rho+1;\right.$$

$$\frac{4g^2h^2}{(a^2+b^2+c^2+e^2+g^2+h^2)^2},\quad \frac{4a^2b^2}{(a^2+b^2+c^2+g^2+h^2)^2},$$

$$\left.\frac{4c^2e^2}{(a^2+b^2+c^2+e^2+g^2+h^2)^2}\right]$$

valid for $R(\mu+\rho+\nu+2)>0$, $R(g+h)>|\mathrm{Im}\,a|+|\mathrm{Im}\,b|$, $R(g+h)>|\mathrm{Im}\,c|+|\mathrm{Im}\,e|$.

From (7) and (9) we obtain the following integrals

$$(10)\quad \int_0^\infty\int_0^\infty K_\nu(2g\sqrt{x+y})\, I_\nu(2h\sqrt{x+y})\, I_\mu(2a\sqrt{x})\, I_\mu(2b\sqrt{y})\, I_\rho(2c\sqrt{x})\, I_\rho(2e\sqrt{y})\, dx\, dy =$$

$$=\frac{(ab)^\mu (gh)^\nu (ce)^\rho\, \Gamma(\nu+\mu+\rho+2)}{2\Gamma(\nu+1)\,\Gamma(\mu+1)\,\Gamma(\rho+1)\,(g^2+h^2-a^2-b^2-c^2-e^2)^{\rho+\nu+\mu+1}}$$

$$\times F_C\left[\frac{1}{2}(\nu+\mu+\rho+2),\ \frac{1}{2}(\nu+\mu+\rho+3);\ \nu+1,\ \mu+1,\ \rho+1;\right.$$

$$\frac{4g^2h^2}{(g^2+h^2-a^2-b^2-c^2-e^2)^2},\quad \frac{4a^2b^2}{(g^2+h^2-a^2-b^2-c^2-e^2)^2},$$

$$\left.\frac{4c^2e^2}{(g^2+h^2-a^2-b^2-c^2-e^2)^2}\right],$$

valid for $R(\nu+\mu+\rho+2)>0$, $R(g)>|R(h)|+|R(a)|+|R(b)|$, $R(g)>|R(h)|+|R(c)|+|R(e)|$.

$$(11)\quad \int_0^\infty\int_0^\infty K_\nu(2g\sqrt{x+y})\, K_\nu(2h\sqrt{x+y})\, I_\mu(2a\sqrt{x})\, I_\mu(2b\sqrt{x})\, I_\rho(2c\sqrt{y})\, I_\rho(2e\sqrt{y})\, dx\, dy =$$

$$=\sum_{\nu,-\nu}\frac{(ab)^\mu (gh)^\nu (ce)^\rho\, \Gamma(\nu+\mu+\rho+2)\,\Gamma(-\nu)}{4\Gamma(\mu+1)\,\Gamma(\rho+1)\,(g^2+h^2-a^2-b^2-c^2-e^2)^{\rho+\nu+\mu+1}}\times$$

$$\times F_C\left[\frac{1}{2}(\nu+\mu+\rho+2),\ \frac{1}{2}(\nu+\mu+\rho+3);\ \nu+1,\ \mu+1,\ \rho+1;\right.$$

$$\frac{4g^2h^2}{(g^2+h^2-a^2-b^2-c^2-e^2)^2},\quad \frac{4a^2b^2}{(g^2+h^2-a^2-b^2-c^2-e^2)^2},$$

$$\left.\frac{4c^2e^2}{(g^2+h^2-a^2-b^2-c^2-e^2)^2}\right],$$

valid for $R(\mu+\nu+\rho+2)>0$, $R(g+h)>|R(a)|+|R(b)|$, $R(g+h)>|R(c)|+|R(e)|$.

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INTEGRALE DUBLE CUPRINZÎND FUNCŢII BESSEL

Rezumat

Se demonstrează egalitatea (3), utilă în calculul operaţional şi se utilizează în evaluarea unor integrale duble.