APPROXIMATE EIGENVALUES OF INFINITE MATRICES

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Introduction. Let X and Y be two Banach spaces whose points are sequences of complex numbers and whose norms are $\|\ \|$ and $|||\ |||$ respectively. Also, let $A=(a_{i,j})_{i,j=1}^{\infty}$ be an infinite matrix of complex numbers. A complex number λ is called an approximate eigenvalue of the matrix A relative to the pair (X, Y) if there exists a sequence $x^{(n)}$ in X with $\|x^{(n)}\| = 1$ such that $Ax^{(n)}-\lambda x^{(n)} \in Y$ for $n = 1, 2, \ldots$ and $|||Ax^{(n)}-\lambda x^{(n)}||| \to 0$ as $n \to \infty$.

An eigenvalue of A relative to X (which was defined in [2] as a complex number λ which satisfies $Ax - \lambda x = 0$ for a point $x \neq 0$ in X) is obviously a special case of an approximate eigenvalue relative to (X, Y) for all Y.

Conditions under which 0 is not an eigenvalue of A relative to various Banach spaces X were given in [2]. Similar methods will be used in this paper to obtain conditions under which 0 is not an approximate eigenvalue of a matrix A relative to various pairs (X, Y).

The following usual notations will be needed. The spaces of sequences of complex numbers $x = (x_1, x_2, \ldots)$ with finite norm $\|x\| = \sup_i |x_i|$, $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ and $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$ $(p > 1)$ will be denoted by m, l and l_p, respectively.

In section 1 two examples will be given. In the first 0 will be an approximate eigenvalue of a matrix A relative to (m,m) and will not be an approximate eigenvalue of A relative to (l, l). In the second 0 will be an approximate eigenvalue of A relative to (l, m) and will not be an approximate eigenvalue of A relative to (l, l).

Conditions under which 0 is not an approximate eigenvalue of a matrix relative to (l, l), (l_p, l_p), (m, m), (l, m), (l_p, m) and (l_p, l), $(p > 1)$ will be formulated in section 2.

Let A be a matrix which maps continuously the Banach space X into itself. The spectrum of A is defined as the set of complex numbers λ for which the operator $A - \lambda I$ has not a continuous inverse.

In section 3 we will show that the spectrum of A and the set of approximate eigenvalues of A relative to (X, X) are identical for certain Banach spaces X if A is symmetric (it satisfies $a_{ij} = \overline{a}_{ji}$ for $i, j = 1, 2 \ldots$) and continuous.

1. Example 1. Let

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \cdots \end{pmatrix}$$

Also, let $x = (x_1, x_2, \ldots)$ be the point in m defined by

$$x_1 = 1,$$

$$x_k = (-1)^i \text{ for } 2^0 + 2^1 + \ldots + 2^{i-1} < k \leqslant 2^0 + 2^1 + \ldots + 2^{i-1} + 2^i, \quad i = 1, 2 \ldots$$

It is easily seen that $Ax = 0$. 0 is therefore an approximate eigenvalue of A relative to (m, m). We also see that A satisfies the conditions of theorem 1 (section 2). Hence, 0 is not an approximate eigenvalue of A relative to (l, l).

Example 2. Let $x^{(n)}$ be the sequence in l defined by

$$x_k^{(n)} = \frac{1}{2^n} \text{ for } 1 \leqslant k \leqslant 2^n - 1$$

$$x_{2^n+l}^{(n)} = \frac{1}{2^{n+l+1}} \text{ for } l = 0, 1, 2 \ldots$$

We have $\|x^{(n)}\|_1 = 1$ and $\|x^{(n)}\| \to 0$ whenever $n \to \infty$.

Therefore, 0 is an approximate eigenvalue of the identity matrix I relative to (l, m), though it is not relative to (l, l).

2. We formulate here conditions under which 0 is not an approximate eigenvalue of a matrix relative to (l, l), (l_p, l_p) (m, m), (l, m) (l_q, m) and (l_p, l) $(p > 1)$.

We begin with the (l, l) case.

Theorem 1. *Let $A = (a_{ij})_{i,j=1}^{\infty}$ be an infinite matrix such that*

$$(1) \qquad |a_{ii}| > \alpha \qquad i = 1, 2, \ldots$$

for a positive α and

$$(2) \qquad A_j = \sum_{i=1, i \neq j}^{\infty} \left| \frac{a_{ij}}{a_{ii}} \right| < \delta \qquad j = 1, 2, \ldots$$

for a real number δ, satisfying $0 < \delta < 1$. Then 0 is not an approximate eigenvalue of A relative to (l, l).

Proof. Suppose to the contrary, that there exists a point $x = (x_1, x_2, \ldots)$, in l such that

$$(3) \qquad \sum_{i=1}^{\infty} |x_i| = 1,$$

$$(4) \qquad \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{ij} x_j \right| < \varepsilon,$$

where

$$(5) \qquad 0 < \varepsilon < \alpha (1 - \delta).$$

From (4) we have

$$\sum_{i=1}^{\infty} |a_{ii}| \left| \left(|x_i| - \left| \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{a_{ij} x_j}{a_{ii}} \right| \right) \right| \leqslant \sum_{i=1}^{\infty} |a_{ii}| \left| \sum_{j=1}^{\infty} \frac{a_{ij} x_j}{a_{ii}} \right| < \varepsilon,$$

Using (1) we obtain

$$\sum_{i=1}^{\infty} \left(|x_i| - \left| \sum_{j=1, j \neq i}^{\infty} \frac{a_{ij} x_j}{a_{ii}} \right| \right) < \frac{\varepsilon}{\alpha}.$$

Hence, by (3)

$$1 < \frac{\varepsilon}{\alpha} + \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} \frac{a_{ij} x_j}{a_{ii}} \right| \leqslant \frac{\varepsilon}{\alpha} + \sum_{i=1}^{\infty} \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \left| \frac{a_{ij}}{a_{ii}} \right| |x_j|.$$

Using (2) and (3) we obtain the inequality $1 < \varepsilon/\alpha + \delta$, which contradicts (5). Theorem 1 is proved.

Remark. It was shown in [2] that 0 is not an eigenvalue of A relative to l if the condition $a_{ii} \neq 0$ which is weaker than (1), and condition (2) with $\delta = 1$ hold.

It is easily seen that 0 is not an eigenvalue and is an approximate eigenvalue relative to (l, l) of a diagonal matrix for which the diagonal numbers are other than 0 and form a sequence converging to 0. Furthermore, we show now that there exist matrices satisfying (1) and (2) with $\delta = 1$ for which 0 is an approximate eigenvalue relative to (l, l).

Let a_n be a sequence of real numbers such that $0 < a_n < 1$ and $a_n \to 1$ as $n \to \infty$. Let

$$A = \begin{pmatrix} 1 & 0 & -a_1 & 0 & 0 & \dots & \dots & \\ 0 & 1 & 0 & -a_2 & 0 & 0 & \dots & \\ 0 & 0 & 1 & 0 & -a_3 & 0 & \dots & \\ 0 & 0 & 0 & 1 & 0 & -a_4 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & -a_5 & 0 \; \dots \\ & & . & & & . & & . \end{pmatrix}$$

It is clear that A satisfies (1) and (2) with $\delta = 1$. Let $x^{(n)}$ be the sequence of points in l defined as follows

$$x_k^{(n)} = \frac{1}{n} \quad \text{for } k = n,\ n+2,\ n+4, \dots, n+2(n-1)$$

$$x_k^{(n)} = 0 \quad \text{for } k = 1, 2, 3, \dots, n-1,\ n+3,\ n+5 \dots n+2(n-1)-1,$$
$$\text{and } k > n+2(n-1),\ n > 1$$

$$x_k^{(1)} = 0 \quad \text{for } k > 1.$$

We have obviously $\|x^{(n)}\|_1 = 1$ for $n = 1, 2, \dots$. Moreover, for $n > 2$ we have

$$(5) \qquad \|Ax^{(n)}\|_1 = \frac{1}{n} a_{n-2} + \frac{1}{n}\left((1-a_n) + (1-a_{n+2}) + \dots + (1-a_{n+2(n-2)})\right) + \frac{1}{n}.$$

Let $\varepsilon > 0$. Since $a_n \to 1$ there exists an integer $N \geq 2$, such that $\max\left(\frac{2}{n},\ 1 - a_n\right) < \frac{\varepsilon}{2}$ for $n > N$. Hence, by (5) $\|Ax^{(n)}\|_1 < \varepsilon$ for $n > N$. Hence 0 is an approximate eigenvalue of A relative to (l, l).

Corollary. If λ is an approximate eigenvalue of a matrix $A = (a_{ij})_{i,j=1}^{\infty}$ relative to (l, l), then one of the following three conditions holds:

1) There exists an i such that $a_{ii} = \lambda$;
2) There exists a sequence n_k of natural numbers such that $a_{n_k, n_k} \to \lambda$;
3) There exists a i such that

$$\sum_{j=1,\, i \neq j}^{\infty} \left| \frac{a_{ij}}{a_{ii} - \lambda} \right| \geq \delta ,$$

for an arbitrary δ with $0 < \delta < 1$.

Now we treat the case in which 0 is not an approximate eigenvalue of a matrix relative to (l_p, l_p) $p > 1$.

Theorem 2. *Let $A = (a_{ij})_{i,j=1}^{\infty}$ be an infinite matrix such that*

$$(1) \qquad |a_{ii}| > \alpha \qquad i = 1, 2 \dots$$

for a positive α and

$$B = \sum_{i=1}^{\infty} \left(\sum_{\substack{j=1 \\ j \neq i}}^{\infty} \left| \frac{a_{ij}}{a_{ii}} \right|^q \right)^{p/q} < 2^{-p/q}$$

where $p > 1$ and q is such that $1/p + 1/q = 1$. Then 0 is not an approximate eigenvalue of A relative to (l_p, l_p).

Proof. Let us suppose, that 0 is an approximate eigenvalue of A relative to (l_p, l_p). Then there exists a point $x = (x_1, x_2 \dots) \in l_p$ such that

$$(6) \qquad \sum_{i=1}^{\infty} |x_i|^p = 1 ,$$

and

$$(7) \qquad \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{ij} x_j \right|^p < \varepsilon ,$$

where

$$(8) \qquad 0 < \varepsilon < \alpha^p \left(2^{-p/q} - B\right).$$

By Hölder's inequality we have

$$|x_i| \leq 2^{1/q} \left(\left| x_i - \sum_{j \neq i} \frac{a_{ij} x_j}{a_{ii}} \right|^p + \left| \sum_{j=1}^{\infty} \frac{a_{ij} x_j}{a_{ii}} \right|^p \right)^{1/p}.$$

Hence

$$\sum_{i=1}^{\infty}|x_i|^p \leq 2^{p/q}\left(\sum_{i=1}^{\infty}\left|x_i+\sum_{j=1}^{\infty}\frac{a_{ij}x_j}{a_{ii}}\right|^p+\sum_{i=1}^{\infty}\left|\sum_{\substack{j=1\\ j\neq i}}^{\infty}\frac{a_{ij}x_j}{a_{ii}}\right|^p\right)=$$

(9)

$$=2^{p/q}\left(\sum_{i=1}^{\infty}\frac{1}{|a_{ii}|^p}\left|\sum_{j=1}^{\infty}a_{ij}x_j\right|^p+\sum_{i=1}^{\infty}\left|\sum_{\substack{j=1\\ j\neq i}}^{\infty}\frac{a_{ij}x_j}{a_{ii}}\right|^p\right).$$

Using (1) and (7) we obtain

$$\sum_{i=1}^{\infty}\frac{1}{|a_{ii}|^p}\left|\sum_{j=1}^{\infty}a_{ij}x_j\right|^p<\frac{\varepsilon}{\alpha^p}.$$

Moreover, by Hölder's inequality,

$$\sum_{i=1}^{\infty}\left|\sum_{\substack{j=1\\ j\neq i}}^{\infty}\frac{a_{ij}x_j}{a_{ii}}\right|^p\leq B\sum_{\substack{j=1\\ j\neq i}}^{\infty}|x_j|^p.$$

Hence, (1) and (8) imply the inequality $1<2^{p/q}(\varepsilon/\alpha^p+B)$ which contradicts (8). Theorem [2] is proved.

Corollary. An approximate eigenvalue λ relative to (l_p, l_p), $p>1$ of a matrix $A=(a_{ij})_{i,j=1}^{\infty}$ satisfies either one of the conditions 1) and 2) from the preceding corollary, or the condition

a)
$$\sum_{i=1}^{\infty}\left(\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left|\frac{a_{ij}}{a_{ii}-\lambda}\right|^q\right)\geq 2^{-p/q},$$

where q is such that $1/p+1/q=1$.

It was shown in [2] that 0 is not an eigenvalue of an infinite matrix $A=(a_{ij})$, relative to m if $a_{ii}\neq 0$, for $i=1,2,\dots$ and

(10)
$$C=\sup_i\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right|<1.$$

Next we obtain a similar result.

Theorem 3. *If $A=(a_{ij})_{i,j=1}^{\infty}$ satisfies (1) and (10) then 0 is an approximate eigenvalue of A relative to (m, m).*

Proof. If 0 is an appeoximate eigenvalue of A relative to (m, m) then there exists a point $x=(x_1,x_2,\dots)$ in m such that

(11)
$$\sup_i|x_i|=1,$$

and

(12)
$$\left|\sum_{j=1}^{\infty}a_{ij}x_j\right|<\varepsilon,$$

where

(13)
$$0<\varepsilon<\alpha(1-C).$$

By (12), we have

(14)
$$|x_i|-\left|\sum_{j=1}^{\infty}\frac{a_{ij}x_j}{a_{ii}}\right|<\frac{\varepsilon}{|a_{ii}|}\qquad i=1,2,\dots$$

Using (1) and (11) we obtain

$$|x_i|\leq\frac{\varepsilon}{\alpha}+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right|.$$

Finally, using (11) we get $1\leq\varepsilon/\alpha+C$ which contradicts (13). Theorem 3 is proved.

Corollary. An approximate eigenvalue λ relative to (m, m) of a matrix $A=(a_{ij})_{i,j=1}^{\infty}$ satisfies either one of the conditions 1) and 2) of the corollary of theorem 1 or the condition

$$\sup_i\sum_{\substack{j=1\\ j\neq i}}\left|\frac{a_{ij}}{a_{ii}-\lambda}\right|\geq 1.$$

Remark. Theorem 3 can also be proved by a similar method to that used in the proof of Theorem 7 in [1].

Next, we obtain, theorems concerning the (l, m) and (l_p, m) cases replacing the condition (1) by stronger conditions in the statements of theorems 1 and 2 respectively.

Theorem 4. *Let $A=(a_{ij})_{i,j=1}^{\infty}$ be a matrix such that*

(15)
$$\sum_i\left|\frac{1}{a_{ii}}\right|=M<\infty,$$

and

(2)
$$A_j=\sum_{\substack{i=1\\ i\neq j}}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right|\delta<.$$

where $0<\delta<1$. Then 0 is not an approximate eigenvalue of A relative to (l, m)

Proof. Let us suppose that 0 is an approximate eigenvalue of A relative to (l, m). Then there exists a point $x = (x_1, x_2, \ldots)$ in l such that

$$\sum_{i=1}^{\infty} |x_i| = 1, \tag{16}$$

and (14) with

$$0 < \varepsilon < \frac{1-\delta}{M} \tag{17}$$

hold. (14) implies

$$\sum_{i=1}^{\infty} |x_i| \leq \sum_{i=1}^{\infty} \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \left| \frac{a_{ij}}{a_{ii}} \right| |x_j| + \sum_{i=1}^{\infty} \frac{\varepsilon}{|a_{ii}|}.$$

Using (15) we get

$$\sum_{i=1}^{\infty} |x_i| \leq \sum_{j=1}^{\infty} |x_j| \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \left| \frac{a_{ij}}{a_{ii}} \right| + M\varepsilon.$$

Therefore, by (2) and (16), $1 \leq \delta + M\varepsilon$. This inequality contradicts (17). Theorem 4 is proved.

Corollary. An approximate eigenvalue λ of a matrix $A = (a_{ij})_{i,j=1}^{\infty}$ relative to (l, m) satisfies either the condition 3) of the corollary to Theorem 1, or

$$\sum_{i=1}^{\infty} \frac{1}{|a_{ii} - \lambda|} = \infty.$$

An analogous condition to (15) is formulated in the following:

Theorem 5. *Let $A = (a_{ij})_{i,j=1}^{\infty}$ be a matrix such that*

$$\sum_{i=1}^{\infty} \frac{1}{|a_{ii}|^p} = P < \infty \tag{18}$$

and

$$B = \sum_{i=1}^{\infty} \left(\sum_{j \neq i}^{\infty} \left| \frac{a_{ij}}{a_{ii}} \right|^q \right)^{p/q} < 2^{-p/q}$$

where $p > 1$ and q is such that $1/p + 1/q = 1$.

Then 0 is not an approximate eigenvalue of A relative to (l_p, m).

Proof. If 0 is an approximate eigenvalue of A relative to (l_p, m) then there exists a point $x = (x_1, x_2, \ldots)$, such that

$$\sum_{i=1}^{\infty} |x_i|^p = 1, \tag{19}$$

and (9) with

$$0 < \varepsilon < \frac{2^{-\frac{p}{q}} - B^{1/p}}{M} \tag{20}$$

are satisfied.

By (9) and Hölder's inequality we have

$$|x_i| \leq 2^{1/q} \left(\left| \sum_{\substack{j=1 \\ j \neq i}} \frac{a_{ij} x_j}{a_{ii}} \right|^p + \frac{\varepsilon^p}{|a_{ii}|^p} \right)^{1/p}.$$

Hence

$$|x_i|^p \leq 2^{p/q} \left(\left| \sum_{j \neq i} \frac{a_{ij} x_j}{a_{ii}} \right|^p + \frac{\varepsilon^p}{|a_{ii}|^p} \right).$$

It follows that

$$1 = \sum_{i=1}^{\infty} |x_i|^p \leq 2^{p/q} \left(\sum_{i=1}^{\infty} \left| \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{a_{ij} x_j}{a_{ii}} \right|^p + \sum_{i=1}^{\infty} \frac{\varepsilon^p}{|a_{ii}|^p} \right).$$

But by (19) and Hölder's inequality

$$\sum_{i=1}^{\infty} \left| \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{a_{ij} x_j}{a_{ii}} \right|^p \leq B \sum_{\substack{j=1 \\ j \neq i}}^{\infty} |x_j|^p \leq B,$$

and by (18)

$$\sum_{i=1}^{\infty} \frac{\varepsilon^p}{|a_{ii}|^p} < M\varepsilon^p.$$

Hence $1 \leq 2^{p/q}(B + M\varepsilon^p)$. This inequality contradicts (20). Theorem 5 is proved.

Corollary. An approximate eigenvalue of a matrix $A = (a_{ij})_{i,j=1}^{\infty}$ relative to (l_p, m) satisfies either the condition (a) of the corollary to Theorem 2 or

$$\sum_{i=1}^{\infty} \frac{1}{|a_{ii} - \lambda|^p} = \infty.$$

In the sequel, the case (l_p, l) $p > 1$ is considered.

It is easily seen that an approximate eigenvalue of a matrix relative to (l_p, l) is also an approximate eigenvalue of the same matrix relative to (l_p, l_p). Hence 0 is not an approximate eigenvalue relative to (l_p, l) of a matrix which satisfies the two conditions of theorem 2. Next we show that we may substitute the second condition by another if the matrix A maps continuously l into l.

Theorem 6. *Let $A=(a_{ij})_{i,j=1}^{\infty}$ be a matrix which maps continuously l into l. If there exists a positive α such that*

$$|a_{ii}|>\alpha \qquad i=1,2\ldots \tag{21}$$

and

$$\sum_{i=1}^{\infty}\Big(\sum_{\substack{j=1\\ j\neq i}}^{\infty}\Big|\frac{a_{ij}}{a_{ii}}\Big|^{q}\Big)^{1/q}\leqslant 1\,, \tag{22}$$

where $q>1$, then 0 is not an approximate eigenvalue of A relative to (l_p,l), where p is such that $1/p+1/q=1$.

Proof. Let us denote by L the set of points $x=(x_1,x_2,x_3,\ldots)$ in l satisfying

$$\sum_{i=1}^{\infty}|x_i|^p=1\,, \tag{23}$$

and

$$\sum_{i=1}^{\infty}\Big|\sum_{j=1}^{\infty}a_{ij}x_j\Big|\leqslant\alpha\,. \tag{24}$$

Every point in L has at least two coordinates different from 0. In fact, if there exists a point x in L which has at most one coordinate other than 0 then by (23) it has exactly one coordinate $x_k\neq 0$ and this coordinate satisfies $|x_k|=1$. So that $\sum_{i=1}^{\infty}|a_{ik}|\leqslant\alpha$ by (24). This inequality contradicts (21).

Let $x=(x_1,x_2\ldots)$ be a point in L and let x_l, x_m be two coordinates other than 0 or x. We may write

$$1=\sum_{i=1}^{\infty}|x_i|^p=|x_l|^p+|x_m|^p+\sum_{i=1}^{\infty}|x_i|^p<(|x_l|+|x_m|)^p+$$
$$+\Big(\sum_{\substack{i=1\\ i\neq l,\,i\neq m}}^{\infty}|x_i|\Big)^p\leqslant\Big(\sum_{i=1}^{\infty}|x_i|\Big)^p.$$

Hence

$$\|x\|_1=\sum_{i=1}^{\infty}|x_i|>1\,.$$

Since A is continuous, L is a closed set in l. Hence

$$\inf_{x\in L}\|x\|_1>1$$

$$0<\varepsilon<\inf\big(\alpha,\ \alpha\,(\inf_{x\in L}\|x\|_1-1)\big) \tag{25}$$

Let us suppose now that 0 is an approximate eigenvalue of A relative to (l_p,l). Then there exists a point $y=(y_1,y_2,\ldots)$ satisfying

$$\sum_{i=1}^{\infty}|y_i|^p=1\,, \tag{26}$$

and

$$\sum_{i=1}^{\infty}\Big|\sum_{j=1}^{\infty}a_{ij}y_j\Big|<\varepsilon\,. \tag{27}$$

Hence y satisfies

$$\sum_{i=1}^{\infty}|y_i|-\frac{\varepsilon}{\alpha}<\sum_{i=1}^{\infty}\Big|\sum_{j\neq i}\frac{a_{ij}\,y_j}{a_{ii}}\Big|\,, \tag{28}$$

(see the proof of Theorem 2).

But by (22), (26) and Hölder's inequality

$$\Big|\sum_{\substack{j=1\\ j\neq i}}^{\infty}\frac{a_{ij}\,y_j}{a_{ii}}\Big|\leqslant\sum_{i=1}^{\infty}\Big(\sum_{\substack{j=1\\ j\neq i}}^{\infty}\Big|\frac{a_{ij}}{a_{ii}}\Big|^{q}\Big)^{1/q}\Big(\sum_{\substack{j=1\\ j\neq i}}^{\infty}|y_j|^p\Big)^{1/p}\leqslant 1\,. \tag{29}$$

Hence

$$\sum_{i=1}^{\infty}|y_i|-\frac{\varepsilon}{\alpha}\leqslant 1\,. \tag{28}$$

It follows that $y\in l$. Since (25) (26) and (27) are satisfied $y\in L$. Therefore (28) implies,

$$\inf_{x\in L}\|x\|-\frac{\varepsilon}{\alpha}\leqslant 1\,,$$

which contradicts (25). Theorem 6 is proved.

Remark. If we assume the strong inequality $<$ in (22) then it is not necessary to assume the continuity of A. In fact, in this case we can choose ε in such a way that

$$\sum_{i=1}^{\infty}\Big(\sum_{\substack{j=1\\ j\neq i}}^{\infty}\Big|\frac{a_{ij}}{a_{ii}}\Big|^{q}\Big)^{1/q}<1-\frac{\varepsilon}{\alpha}$$

and we obtain by (29), (26) and (28) $\sum_{i=1}^{\infty} |y_i| < 1$. On the other hand, from (26) follows $\sum_{i=1}^{\infty} |y_i| \geq 1$ which contradicts the preceding inequality.

This remark can also be justified by a similar method to that used in the proof of theorem 7 in [1].

3. In this section, we denote by $\mathfrak{A}$ the class of spaces $(m; l_q, q \geq 2)$. $\mathfrak{A}$ has the following two properties:

a) If $X \in \mathfrak{A}$ then $X^* \subset X$.

b) If $X \in \mathfrak{A}$ and $A = (a_{ij})_{i,j=1}^{\infty}$ is a matrix which maps continuously X into X then $A^* = (\overline{a}_{ji})_{i,j=1}^{\infty}$ maps continuously X^* into X^*.

These two properties of $\mathfrak{A}$ will be used in the proof of Theorem 7. Before we state this theorem we recall that the spectrum of a continuous operator $A: X \to X$ is the set of complex numbers λ for which $A - \lambda I$ has not a continuous inverse.

Theorem 7*. *Let $X \in \mathfrak{A}$. If $A: X \to X$ is a continuous symmetric matrix (i.e. it satisfies $A = A^*$) then the spectrum of A and the set of approximate eigenvalues of A are identical.*

Proof. Let us denote by $\| \ \|$ the norm in X. If $\lambda \notin \operatorname{sp} A$ then the condition

$$\|(A - \lambda I)\, x_n\| \to 0$$

implies

$$\|(A - \lambda I)^{-1} (A - \lambda I)\, x_n\| = \|x_n\| \to 0 .$$

Hence λ is not an approximate eigenvalue of A relative to (X, X).

For the proof of the inverse, we suppose that λ is not an approximate eigenvalue of A relative to (X, X). Then there exists a positive α such that

$$\|Ay - \lambda y\| \geq \alpha \|y\| , \tag{30}$$

for each y in X.

In the sequel we show that $z = 0$ is the only point in X^* which satisfies

$$(z, Ax - \lambda x) = 0, \tag{31}$$

for all $x \in X$.

Since A is symmetric (31) implies

$$(Az - \overline{\lambda} z, x) = (Az, x) - \lambda(z, x) = 0 . \tag{32}$$

* In the case $X = l_2$ the theorem is well known.

By property a) $z \in X$. Hence $(Az, z) - \lambda(z, z) = 0$. Since A is symmetric and continuous (Az, z) is real. Moreover $(z, z) > 0$. Hence λ is real. Therefore, (32) (which holds for all $x \in X$) implies $Az - \lambda z = 0$. It follows from (30) that $z = 0$. The range of $A - \lambda I$ is therefore dense in X (see [1], p. 64). But by (30) the range of $A - \lambda I$ is closed. The range of $A - \lambda I$ is therefore identical with X. Since λ is not an eigenvalue of A, $A - \lambda I$ is one to one. Hence, $A - \lambda I$ satisfies the conditions of Banach's theorem (see [3], p. 146). $A - \lambda I$ has therefore a continuous inverse. This means that $\lambda \notin \operatorname{sp} A$. Theorem 7 is proved.

Corollary 1. A number λ belonging to the spectrum of a continuous symmetric matrix $A: l_p \to l_p$, $p \geq 2$ satisfies one of the three conditions of the corollary to theorem 2.

Corollary 2. A number λ belonging to the spectrum of a continuous symmetric matrix $A: m \to m$ satisfies one of the three conditions of the corollary to theorem 3.

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VALORI PROPRII APROXIMATIVE LA MATRICE INFINITE

Rezumat

Fie date două spaţii Banach X şi Y ale căror puncte sînt şiruri de numere complexe şi ale căror norme sînt, respectiv $\| \ \|$ şi $\||\ |\|$. De asemenea fie $A = (a_{i,j})_{i,j=1}^{\infty}$ o matrice infinită de numere complexe. Un număr complex λ este numit valoare proprie aproximativă a matricei A în raport cu perechea de spaţii (X, Y) dacă există în X un şir de puncte $x^{(n)}$ astfel încît $\|x^{(n)}\| = 1$, $Ax^{(n)} - \lambda x^{(n)} \in Y$ şi $\||Ax^{(n)} - \lambda x^{(n)}|\| \to 0$ cînd $n \to \infty$. Un caz particular de valoare proprie aproximativă în raport cu (X, Y) este valoarea proprie în raport cu X definită în [2].

În această lucrare se folosesc notaţiile uzuale m, l şi l_p, $p \geq 1$ pentru spaţiile de şiruri $x = (x_1, x_2, \ldots)$ avînd ca normă, respectiv, $\|x\| = \sup_i |x_i|$,

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i| \quad \text{și} \quad \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}.$$

În paragraful 1 se dă un exemplu de matrice pentru care 0 este o valoare proprie aproximativă în raport cu (l, l). Se dă de asemenea un exemplu de matrice pentru care 0 este o valoare proprie aproximativă în raport cu (l, m) și nu este o valoare proprie aproximativă în raport cu (l, l).

În paragraful 2 se formulează condiții suficiente pentru ca 0 să nu fie o valoare proprie aproximativă a unei matrice în raport cu fiecare dintre perechile (l, l), (l_p, l_t), (m, m), (l, m), (l_p, m) și (l_p, l). Aceste condiții sînt asemănătoare condițiilor suficiente pentru ca 0 să nu fie o valoare proprie a unei matrice în raport cu un spațiu de tip l, m sau l_p. Aceste condiții au fost formulate în [2].

În paragraful 3 se notează cu $\mathfrak{A}$ clasa de spații $(m, l_p, p > 1)$ și se arată că pentru fiecare matrice simetrică și continuă mulțimea de valori proprii aproximative în raport cu o pereche (X, X), $X \in \mathfrak{A}$ coincide cu spectrul.

ESPACES PSEUDOMÉTRIQUES ET PSEUDONORMÉS

PAR

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L'étude d'un espace doté d'un écart prenant ses valeurs dans un ensemble ordonné a été faite pour la première fois par K u r e p a [3]. Un exposé sur les résultats principaux concernant ce sujet se trouve dans [4]. Dans cette Note, nous rappelons la notion de pseudométrique introduite dans [5], et donnons une caractérisation des topologies sur certaines structures algébriques (groupes, espaces vectoriels). Nous indiquons aussi une extension du théorème de Banach pour les espaces uniformisables. Une autre extension, à l'aide de la famille des écarts, se trouve dans [2].

Définition. *Soit D un ensemble ordonné*, avec un plus petit élément d_0 et doté d'une opération $(d', d'') \to d' + d''$ telle que :*

(D1) $\quad d' < d'' \Rightarrow d' + d < d'' + d$ *et* $d + d' < d + d'' \qquad \forall d \in D.$

On suppose qu'il y a un sous-ensemble $D_1 \subset D$, non vide et filtrant à gauche (par rapport à l'ordre induit par celui de D) tel que :

(D2) $\qquad \forall d \in D_1, \exists d' \in D_1$ *tel que* $d' + d' < d.$

Une pseudométrique sur X est une application $\rho : X \times X \to D$ telle que :

(X1) $\qquad \rho(x, y) = d_0 \Longleftrightarrow x = y,$

(X2) $\qquad \rho(x, y) = \rho(y, x), \quad x, y \in X,$

(X3) $\qquad \rho(x, y) < \rho(x, z) + \rho(z, y), \quad x, y, z \in X.$

On nomme boule de centre x_0 et de rayon $d \in D_1$ l'ensemble :

$$B(x_0, d) = \{x : \rho(x_0, x) < d\}.$$

La famille $\{B(x_0, d)\}_{d \in D_1}$ est la base d'un filtre de voisinages du point x_0. Dans ce qui suit, nous considérons chaque espace pseudométrique

*) Une relation transitive, propre et reflexive.