SEMI-PARALLEL TENSORS

BY

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1. Introduction

Let M be a C^∞ differentiable manifold of dimension n. Given a linear connection Γ on M, a tensor field K on M is said to be recurrent if

$$\nabla K = W \otimes K,$$

where W is a covector on M and ∇K denotes covariant differentiation with respect to Γ [2, 4].

The tensor field K is said to be parallel if $\nabla K = 0$. A tensor field S on M is defined to be *semi-parallel* if there exists a tensor K on M and a differentiable function σ on M such that

(i) $S = \sigma K$,

(ii) K is parallel i. e. $\nabla K = 0$.

The purpose of this paper is to study semi-parallel tensors. Such tensors correspond to semi-covariant tensors in Riemannian Geometry [1, 3]. For a tensor to be semi-covariant constant conditions were expressed in some restricted cases. In this paper using methods of Wong [4, 5] a more general and simple result has been given.

Let $T_x(M) = T_x$ be the tangent space to M at $x \in M$. A tensor field K of type (r, s) over M is an assignment of a tensor $K_x \in T^r_s(x)$ to each point $x \in M$, where $T^r_s(x)$ denotes the tensor space of type (r,s) over T_x. Given a basis $\{X_i\}$, $i = 1, \dots n$, of T_x the tensor field K admits an unique expansion with respect to this basis given by [2]

$$K_x(X) = K^{i_1 \dots i_r}_{j_1 \dots j_s} X_{i_1} \otimes \dots \otimes X_{i_r} \otimes X^{j_1} \oplus \dots \otimes X^{j_s}, \tag{1.1}$$

where $\{X^i\}$ is the dual basis of $\{X_i\}$. $K^{i_1 \dots i_r}_{j_1 \dots j_s}$ are called the components of K_x relative to $\{X_i\}$.

Corresponding to each $a = A^i_j \in GL(n, R)$ we get another basis $\{aX\} = \{A^i_j X_j\}$ of T_x. The components $K'^{i_1 \dots i_r}_{j_1 \dots j_s}$ of K_x with respect to $\{aX\}$ are related to those with respect to $\{X\}$ by

$$K'^{i_1 \dots i_r}_{j_1 \dots j_s} = K^{p_1 \dots p_r}_{q_1 \dots q_r} B^{i_1}_{p_1} \cdots B^{i_r}_{p_r} A^{q_1}_{j_1} \cdots A^{q_s}_{j_s}, \tag{1.2}$$

where B^i_j is the inverse matrix of A^i_j. We shall write the above equation as:

$$K(aX) = a^{-1} \cdot K(X). \tag{1.3}$$

Let $L(M)$ be the bundle of linear frames over the manifold M and with the structure group $GL(n,R)$ and let π be the projection mapping $L(M) \to M$. An element $z \in L(M)$ is given by a point $x \in M$ such that $\pi z = x$ and a set of n linearly independent tangent vectors at x. We shall denote $z \in L(M)$ by $(\pi z, X_i)$, where X_i $(i = 1, \dots, n)$ are the n corresponding tangent vectors at x. The set $\{X_i\}$ $(i = 1, \dots, n)$ is a basis of $T_{\pi z}(M)$; we shall refer to this basis simply as the frame z.

Given a tensor field K on M of type (r,s) we can associate a set of n^{r+s} functions denoted by $\widetilde{K}$ on $L(M)$ as follows. At each $z \in L(M)$, $\widetilde{K}(z)$ is simply the component of K at z relative to the frame z. Thus

$$\widetilde{K}(z) = K^{i_1 \dots i_r}_{j_1 \dots j_s}(\pi z) \text{ if } K_{\pi z}(X) = K^{i_1 \dots i_p}_{j_1 \dots j_s} X_{i_1} \otimes X_{i_2} \oplus \dots \otimes X^{j_s}.$$

Given $a = A^i_j \in GL(n, R)$ we know that $\pi(za) = x$ and $za = (\pi z, a^i_j X_i)$. Hence $\widetilde{K}(za)$ are simply the components of $K_{\pi z}$ relative to the basis $\{aX\}$. Thus from (1.3),

$$\widetilde{K}(za) = a^{-1} \cdot \widetilde{K}(z). \tag{1.4}$$

It can be easily verified that if σ is a real valued function on M then σK is also a tensor field of type (r, s) on M and if the components of K relative to a basis $\{X_i\}$ at $x \in M$ are $K^{i_1 \dots i_r}_{j_1 \dots j_s}$ then the components of $(\sigma K)(x)$ w.r.t. $\{X_i\}$ are simply $\sigma(x) K^{i_1 \dots i_r}_{j_1 \dots j_s}$. Thus the functions on $L(M)$ associated with σK are given by

$$\widetilde{\sigma K}(z) = \sigma(\pi z) \widetilde{K}(z).$$

2. Semi-parallel tensors

Let now $S = \sigma K$ be a semi-parallel tensor. Since K is parallel, it follows from theorems of W o n g [4, 5] that the restrictions of $\widetilde{K}$ for the holonomy bundle $P(z_0)$ through any $z_0 \in L(M)$ are all constants but non zero, i.e.

$$\widetilde{K}(z) = \widetilde{C} \text{ for } z \in P(z_0),$$

where $\widetilde{C}$ denotes a set of constants not all zero.

Thus if S is semi-parallel then

$$\widetilde{S}(z) = \sigma(\pi z) \widetilde{C} \text{ for } z \in P(z_0).$$

If we assume σ to be nowhere zero on M and differentiable, it follows from theorem 3.9 of W o n g [4] that S is necessarily recurrent.

Conversely, we prove the following:

Theorem. *A necessary and sufficient condition for a recurrent tensor to be semi-parallel is that the recurrence vector be locally the gradient of a scalar.*

We consider a local co-ordinate neighbourhood U of $x \in M$ with local co-ordinate system $(x^1, \dots, x^n)$. The components of a tensor field S at $x \in M$ are then the components relative to the natural basis $(\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^n)$ of $T_x(M)$ and the dual basis $(dx^1, \dots, dx^n)$. Let S be of type (r, s) and let $S^{i_1 \dots i_r}_{j_1 \dots j_s}$ be its components at x; then the components of S at x are given in usual notation by

$$\nabla_h S^{i_1 \dots i_r}_{j_1 \dots j_s}.$$

If S is recurrent with $S = W \oplus S$, then

$$\nabla_h S^{i_1 \dots i_r}_{j_1 \dots j_s} = W_h S^{i_1 \dots i_r}_{j_1 \dots j_s}, \tag{2.1}$$

where W_h are the components of W at x. If S is semi-parallel with $S = \sigma K$, then since σ is nowhere zero it has the same sign everywhere. We assume σ is positive everywhere. Then

$$\nabla_h S^{i_1 \dots i_r}_{j_1 \dots j_s} = \nabla_h (\sigma K^{i_1 \dots i_r}_{j_1 \dots j_s}) = \frac{\partial \sigma}{\partial x^h} K^{i_1 \dots i_r}_{j_1 \dots j_s} = \frac{\partial \log \sigma}{\partial x^h} S^{i_1 \dots i_r}_{j_1 \dots j_s}, \tag{2.2}$$

since $\nabla_h K = 0$. From (2.1) and because S is nowhere zero on M, it follows that $W_h = \partial(\log \sigma)/\partial x^h$, which proves that the condition in the theorem is necessary.

To prove that the condition is sufficient we know from Wong's theorem that if S is recurrent then

$$\widetilde{S}(z) = \widetilde{\Phi}(z) C \text{ on } P(z_0),$$

where $\Phi(z)$ is a nowhere zero function on $P(z)$.

Let $x(t)$ be any sectionally C^∞ curve in M through $x_0 = \pi z_0$ and let $z(t)$ be the horizontal lift of $x(t)$ through z_0 such that $\pi(z(t)) = x(t)$. Then

$$\widetilde{S}(z(x)) = \Phi(z(x)) \widetilde{C} = \rho(x) \widetilde{C};$$

the function $\rho(x)$ is not in general continuous. If $\rho(x)$ were differentiable, every recurrent tensor would have been semi-parallel and with each recurrent tensor we could associate the parallel tensor $T = (1/\rho) S$.

We shall now prove that if the recurrence vector W is locally a gradient then $\rho(x(t))$ is a differentiable function of $x(t)$ showing that S in this case is semi-parallel. For, along $x(t)$ we have [4, Th. 3.4].

$$\tilde{S} = \rho(x(t))\,\tilde{C}, \text{ where } \rho(x(t)) = \exp \int f(t)\,dt \text{ and}$$

$$f(t) = W_h(x(t))\,\frac{dx^h}{dt}.$$

If W is locally a gradient, then $W_h(x(t)) = \partial\sigma(x(t))/\partial x^h$, where σ is a differentiable function on M and

$$f(t) = \frac{\partial\sigma(x(t))}{\partial x^h}\frac{dx^h}{dt} = \frac{d\sigma(x(t))}{dt}$$

and

$$\rho(x(t)) = \exp \int \frac{d\sigma(x(t))}{dt}\,dt = \exp \sigma(x(t)).$$

Thus $\rho(x(t))$ is a differentiable function on M.

In general $S = (\exp \sigma) K$ where K are the parallel tensors corresponding to the constants $\tilde{C}$ and σ is the scalar function such that the recurrence vector is the gradient of σ.

A recurrent tensor S on M is said to be *almost parallel*, if there exists a linear connection on M with respect to which S is parallel. Wong has proved that every recurrent tensor which is not of the type (r,r) is always almost parallel and for a tensor of the type (r,r) which is not almost parallel the recurrence vector is locally a gradient. In view of the theorem proved here we now have that a recurrent tensor of type (r,r) which is not almost parallel is necessarily semi-parallel.

In case of linear connection with recurrent curvature tensor and zero torsion, it was conjectured by Wong [6] that the recurrence vector is the gradient of a scalar function if and only if the Ricci-tensor is symmetric. In view of the above theorem, since for such a connection the Ricci-tensor is also recurrent with the same recurrence covector it appears that Wong's conjecture would be true if the Ricci-tensor is proportional to a symmetric parallel tensor.

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TENSORI SEMI-PARALELI

Rezumat

Un cîmp de tensori S se definește ca semi-paralel dacă există un tensor K și o funcție diferențiabilă σ așa ca să fie satisfăcute condițiile (i) și (ii) din **1.** În articol se studiază condiții de existență a unui astfel de cîmp pe o varietate diferențiabilă de clasă C^∞.