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ASUPRA ALGEBRELOR DE GRADUL AL III-LEA

Rezumat

În această Notă se defineşte gradul unei algebre de ordin finit ca fiind gradul maxim al polinoamelor ce sînt coordonatele produsului (1). Se studiază apoi algebrele de gradul al III-lea, determinind acele care sînt comutative (8), cele care îndeplinesc condiţia $D_1 = D_2$ şi apoi condiţiile de asociativitate date de sistemele (17)—(21). În continuare se cercetează algebre de gradul al treilea anticomutative date prin (22); printre acestea se caută acelea care îndeplinesc şi condiţia lui Jacobi (25). Se ajunge la sistemele (26)—(31). Soluţiile acestor sisteme ne dau algebre Lie de gradul al III-lea.

ON THE PRODUCT OF TWO THREE-DIMENSIONAL MATRICES

BY

CORINA REISCHER

A three-dimensional matrix A of type (p, q, r) is a system of pqr numbers a_{ijk} $(1 \leqslant i \leqslant p;\ 1 \leqslant j \leqslant q;\ 1 \leqslant k \leqslant r)$ from a certain numerical field K, placed in the points of the three-dimensional space $P(i, j, k)$. We denote $A = (a_{ijk})$ $(1 \leqslant i \leqslant p;\ 1 \leqslant j \leqslant q;\ 1 \leqslant k \leqslant r)$.

The addition of matrices of the same type and the multiplication of a matrix by scalars are similar in the bi-dimensional and three-dimensional case. For the product of two three-dimensional matrices the situation is a different one. The product is obtained generally as a 4-, 5- or even 6-dimensional matrix, e.g. the products introduced by Rice [5] and Oldenburger [3]. Using the definition given by Cayley-Scott [2, 8] we can obtain a three-dimensional product matrix in the following manner

$$c_{uvw} = a_{uvk}\, b_{kvw}, \tag{1}$$

where k is a summation index, and $A = (a_{ijk})$, $B = (b_{ijk})$ and $C = AB = (c_{ijk})$ are all three-dimensional matrices.

Sokolov's product for n-dimensional matrices [7] reduces, for $n = 3$, to the product (1). This product may be obtained as follows:

Denote by $A_j = (a_{ijk})$ $(j = 1, \ldots, q)$ the q bi-dimensional matrices. Then A can be written as $A = A_1 e_1 + \ldots + A_q e_q$, where $A_j\ (j = 1, \ldots, q)$ are regarded as bi-dimensional „leaves" of the three-dimensional matrix A, and $e_j\ (j = 1, \ldots, q)$ are certain symbols.

Using the product $e_i e_j = \delta_{ij}\, e_i$, where δ_{ij} is Kronecker's delta, we obtain the product (1) of two three-dimensional matrices A and B, namely $C = AB = (A_1 B_1) e_1 + \ldots + (A_q B_q) e_q$, where $A_j B_j\ (j = 1, \ldots, q)$ is the usual product of two bi-dimensional matrices. This product exists if $q = \bar{q}$, $r = \bar{p}$, where (p, q, r) and $(\bar{p}, \bar{q}, \bar{r})$ are the types of the matrices A and B respectively.

By this manner we have multiplied pairwise the bi-dimensional „leaves" A_j and B_j $(j = 1, \ldots, q)$, $A_j B_j$ being the „leaves" of the product matrix C. We observe that in this product, the „leaves" A_j, $B_l (j \neq l)$ are not related.

By using this very simple product, we obtained some results in the theory of functions of three-dimensional matrices [4, 1].

Now we shall introduce other products of three-dimensional matrices with three-dimensional product matrices, such that the relations between the „leaves" A_j, B_l will not be only pairwise for $j = l$.

Let $A = (a_{ijk})$ and $B = (b_{ijk})$ be two three-dimensional matrices, of types (p, q, r) and $(\bar{p}, \bar{q}, \bar{r})$ respectively, and $C = (c_{ijk})$ their product. Using scalar symbols z_{ijk} and preserving the places of all indices we may introduce four classes of products.

Class I. If the scalar symbol has three summation indices, we obtain eight kinds of products, according to the number of summation indices of A and B: two indices of A and one of B, one of A and two of B, three indices of A and none of B and conversely. The products and their types are

1. $c_{uvw} = z_{ijk}\, a_{ujk}\, b_{ivw}$ $\quad (p, q, r)$,
2. $z_{ijk}\, a_{ivk}\, b_{ujw}$ $\quad (\bar{p}, q, \bar{r})$,
3. $z_{ijk}\, a_{ijw}\, b_{uvk}$ $\quad (\bar{p}, \bar{q}, r)$,
4. $z_{ijk}\, a_{uvk}\, b_{ijw}$ $\quad (p, q, \bar{r})$,
5. $z_{ijk}\, a_{ujw}\, b_{ivk}$ $\quad (p, \bar{q}, r)$,
6. $z_{ijk}\, a_{ivw}\, b_{ujk}$ $\quad (\bar{p}, q, r)$,
7. $z_{ijk}\, a_{uvw}\, b_{ijk}$ $\quad (p, q, r)$,
8. $z_{ijk}\, a_{ijk}\, b_{uvw}$ $\quad (\bar{p}, \bar{q}, \bar{r})$.

Here the factor matrices may be of any type. All the products are associative, non-commutative, and distributive with respect to addition.

Class II. If the scalar symbol has two summation indices we obtain 12 kinds of products, as follows:

1. $c_{uvw} = z_{ujk}\, a_{uvk}\, b_{ujw}$ $\quad (p, q, r)$, this product exists if $p = \bar{p}$,
2. $z_{ujk}\, a_{ujw}\, b_{uvk}$ $\quad (p, \bar{q}, r)$, „ „ „ „ $p = \bar{p}$,
3. $z_{ivk}\, a_{uvk}\, b_{ivw}$ $\quad (p, q, \bar{r})$, „ „ „ „ $q = \bar{q}$,
4. $z_{ivk}\, a_{ivw}\, b_{uvk}$ $\quad (\bar{p}, q, r)$, „ „ „ „ $q = \bar{q}$,
5. $z_{ijw}\, a_{ujw}\, b_{ivw}$ $\quad (p, \bar{q}, r)$, „ „ „ „ $r = \bar{r}$,

6. $z_{ijw}\, a_{ivw}\, b_{ujw}$ $\quad (\bar{p}, q, r)$, this product exists if $r = \bar{r}$,
7. $z_{ujk}\, a_{ujw}\, b_{uvk}$ $\quad (p, \bar{q}, \bar{r})$, „ „ „ „ $p = \bar{p}$,
8. $z_{ivk}\, a_{ivk}\, b_{uvw}$ $\quad (\bar{p}, q, \bar{r})$, „ „ „ „ $q = \bar{q}$,
9. $z_{ijw}\, a_{ijw}\, b_{uvw}$ $\quad (\bar{p}, \bar{q}, r)$, „ „ „ „ $r = \bar{r}$,
10. $z_{ujk}\, a_{uvw}\, b_{ujk}$ $\quad (p, q, r)$, „ „ „ „ $p = \bar{p}$,
11. $z_{ivk}\, a_{uvw}\, b_{ivk}$ $\quad (p, q, r)$ „ „ „ „ $q = \bar{q}$,
12. $z_{ijw}\, a_{uvw}\, b_{ijw}$ $\quad (p, q, r)$, „ „ „ „ $r = \bar{r}$.

The above factor matrices have two arbitrary indices. All these products are associative, non-commutative, and distributive with respect to addition.

Class III. If the scalar symbol has only one summation index, we obtain 6 kinds of products, of the form

1. $c_{uvw} = z_{uvk}\, a_{uvw}\, b_{uvk}$ $\quad (p, q, r)$, this product exists if $p = \bar{p}$, $q = \bar{q}$,
2. $z_{uvk}\, a_{uvk}\, b_{uvw}$ $\quad (p, q, \bar{r})$, „ „ „ „ $p = \bar{p}$, $q = \bar{q}$,
3. $z_{ujw}\, a_{uvw}\, b_{ujw}$ $\quad (p, q, r)$, „ „ „ „ $p = \bar{p}$, $r = \bar{r}$,
4. $z_{ujw}\, a_{ujw}\, b_{uvw}$ $\quad (p, \bar{q}, r)$, „ „ „ „ $p = \bar{p}$, $r = \bar{r}$,
5. $z_{ivw}\, a_{uvw}\, b_{ivw}$ $\quad (p, q, r)$, „ „ „ „ $q = \bar{q}$, $r = \bar{r}$,
6. $z_{ivw}\, a_{ivw}\, b_{uvw}$ $\quad (p, q, r)$, „ „ „ „ $q = \bar{q}$, $r = \bar{r}$.

Here the factor matrices have a single arbitrary index. All these products are associative, non-commutative, and distributive with respect to addition.

Class IV. If the scalar symbol has no summation index, we obtain eight kinds of products

1. $c_{uvw} = z_{uvw}\, a_{ijk}\, b_{ijk}$ $\quad (p, q, r)$, this product exists if $p = \bar{p}$, $q = \bar{q}$, $r = \bar{r}$,
2. $z_{uvw}\, a_{ujk}\, b_{ujk}$ „ „ „ „ $p = \bar{p}$, $q = \bar{q}$, $r = \bar{r}$,
3. $z_{uvw}\, a_{ivk}\, b_{ivk}$ „ „ „ „ $p = \bar{p}$, $q = \bar{q}$, $r = \bar{r}$,
4. $z_{uvw}\, a_{ijw}\, b_{ijw}$ „ „ „ „ $p = \bar{p}$, $q = \bar{q}$, $r = \bar{r}$,
5. $z_{uvw}\, a_{uvk}\, b_{uvk}$ „ „ „ „ $p = \bar{p}$, $q = \bar{q}$, $r = \bar{r}$,
6. $z_{uvw}\, a_{ujw}\, b_{ujw}$ „ „ „ „ $p = \bar{p}$, $q = \bar{q}$, $r = \bar{r}$,
7. $z_{uvw}\, a_{ivw}\, b_{ivw}$ „ „ „ „ $p = \bar{p}$, $q = \bar{q}$, $r = \bar{r}$,
8. $z_{uvw}\, a_{uvw}\, b_{uvw}$ „ „ „ „ $p = \bar{p}$, $q = \bar{q}$, $r = \bar{r}$.

In that class all the factor and product matrices must be of the same type. Thus we obtain eight kinds of linear algebras, the first seven nonassociative, commutative, and distributive and the last one associative, commutative, and distributive with respect to addition.

The above considered products are obtained by preserving the places of all indices, u, v, w, and i, j, k.

By keeping only the places of the indices u, v, w we obtain another classification containing as a very special case the Cayley-Scott's product. Namely, if we consider

$$c_{uvw} = \varepsilon_{ijk}\, a_{uik}\, b_{kjw}$$

with

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } i = j = k \\ 0 & \text{in the other cases,} \end{cases}$$

we get

$$c_{uvw} = a_{uvk}\, b_{kvw}.$$

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ASUPRA PRODUSULUI A DOUĂ MATRICE TRIDIMENSIONALE

Rezumat

Folosind simboli scalari ε_{ijk} şi păstrind locurile indicilor de sumare şi liberi s-au obţinut 4 clase distincte de produse pentru două matrice tridimensionale şi astfel ca matricea produs să fie tot tridimensională.

Dacă se păstrează numai locurile indicilor liberi se poate obţine o altă clasificare care conţine ca un caz particular produsul introdus de Cayley-Scott.

SPAŢIUL SPINORIAL

DE

AGNETA VESCAN

În lucrările anterioare am introdus noţiunea de spaţiu spinorial în felul următor: Fie R_n un spaţiu liniar euclidian complex n-dimensional, iar C_n algebra spinorilor din acest spaţiu. Această algebră este $m = 2^n$ dimensională şi poate fi privită ca un spaţiu liniar, formînd spaţiul spinorial S_{2^n}. Vectorii din acest spaţiu sînt spinori, iar tensorii spin-tensori. (Vezi [5], [7]). Orice spinor poate fi scris sub forma

$$\varphi = \varphi^0 + \sum_{i_1} \varphi^{i_1} \vec{e}_{i_1} + \sum_{i_1<i_2} \varphi^{i_1 i_2} \underset{(2)}{\vec{e}_{i_1 i_2}} + \cdots + \varphi^{12\ldots n} \underset{(n)}{\vec{e}_{12\ldots n}},$$

descompus după multivectorii de bază. Să mai amintim că înmulţirea multivectorilor de bază se face după regula

$$\underset{(k)}{\vec{e}_{i_1 i_2 \ldots i_k}}\, \underset{(l)}{\vec{e}_{j_1 j_2 \ldots j_l}} = (-1)^N \underset{(s)}{\vec{e}_{q_1 q_2 \ldots q_s}},$$

unde indicii $q_1, q_2, \ldots, q_s$ sînt în ordine crescătoare şi sînt egali cu indicii $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l$, după ce s-au eliminat indicii care se intîlnesc de două ori, iar N este numărul inversiunilor prezentat de şirul de indici $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l$.

În spaţiul spinorial putem introduce şi produsul scalar cu ajutorul axiomelor cunoscute:

1. $(\varphi, \psi) = \overline{(\psi, \varphi)}$
2. $(\lambda\varphi, \psi) = \lambda(\varphi, \psi)$
3. $(\varphi_1 + \varphi_2, \psi) = (\varphi_1, \psi) + (\varphi_2, \psi)$
4. $(\varphi, \varphi) \geq 0$, egalitatea avînd loc atunci şi numai atunci cînd $\varphi = 0$,