STEADY EXTENSION OF COMPRESSIBLE SIMPLE FLUIDS

BY

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1. Introduction. The theory of incompressible simple fluids is based upon the constitutive equation

$$\mathbf{T} + p\,\mathbf{I} = \mathop{\mathfrak{H}}_{s=0}^{\infty}(\mathbf{F}(s)), \tag{1.1}$$

where $\mathbf{F}(s)$ is the history of the relative deformation gradient, $\mathfrak{H}$ is a functional, $\mathbf{T}$ is the stress tensor and p is an indeterminate hydrostatic pressure. The last years have seen considerable mathematical work done on the theory of incompressible simple fluids [1—7]. An elegant and complete treatment of steady viscometric flow is given by Coleman, Markovitz, and Noll [8]. Little has been done with general compressible simple fluids. Explicit exact solutions are obtained for a number of viscometric flows and for flows which are not viscometric, in the incompressible case. Here we shall discuss steady extension for a compressible simple fluid. The notation used is that of [5] and [8].

2. **Preliminaries.** A body is a three-dimensional differentiable manifold, the elements of which are called material points X. Let $\mathbf{x}$ be the position in Euclidean space $\mathcal{E}$ of the material point X at time t which we interpret as the present time. Suppose that at time τ, say $\tau \leqslant t$, this same material point X occupied the position ξ in $\mathcal{E}$. For the dependence of ξ on $\mathbf{x}$, t and τ, we write

$$\xi = \chi_t(\mathbf{x}, \tau). \tag{2.1}$$

We shall assume that a sufficient number of derivatives of χ_t exist and are continuous.

Assume that the velocity field $\mathbf{v}(\mathbf{x},t)$ in the spatial description is assigned. The relative deformation function (2.1) may then be obtained by solving the differential equations

$$\frac{d\xi}{d\tau} = \mathbf{v}(\xi, \tau), \tag{2.2}$$

with the initial conditions

$$\xi\big|_{\tau=t} = \mathbf{x}. \tag{2.3}$$

The gradient $\mathbf{F}_t(\tau)$ of χ_t with respect to $\mathbf{x}$

$$\mathbf{F}_t(\tau) = \nabla\chi_t(\mathbf{x}, \tau) \tag{2.4}$$

is a second order tensor field called the relative deformation gradient.

The tensor function defined by

$$\mathbf{F}(s) = \mathbf{F}_t(t - s) \quad (s \geq 0), \tag{2.5}$$

is called the history of the relative deformation gradient. It follows from (2.4) that

$$\mathbf{F}(0) = \mathbf{F}_t(t) = \mathbf{I}, \tag{2.6}$$

where $\mathbf{I}$ is the identity tensor.

The general constitutive equation of a compressible simple fluid may be written on the form

$$\mathbf{T} = -p(\rho)\,\mathbf{I} + \mathop{\mathfrak{F}}_{s=0}^{\infty}(\mathbf{F}(s); \rho), \tag{2.7}$$

where $p(\rho)$ is a scalar function of ρ. The response functional $\mathfrak{F}$ of (2.7) has the value $\mathbf{0}$ when $\mathbf{F}(s) = \mathbf{0}$. The principle of material objectivity requires that

$$\mathbf{R}(0) \mathop{\mathfrak{F}}_{s=0}^{\infty}(\mathbf{F}(s); \rho)\, \mathbf{R}(0)^T = \mathop{\mathfrak{F}}_{s=0}^{\infty}(\mathbf{R}(s)\,\mathbf{F}(s)\,\mathbf{R}(0)^T; \rho), \tag{2.8}$$

for every continuous function $\mathbf{R}(s)$ whose values are orthogonal tensors and every history $\mathbf{F}(s)$.

The equations of motion may be written in the form

$$\operatorname{div} \mathbf{T} + \rho\mathbf{b} = \rho\mathbf{a}, \tag{2.9}$$

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0, \tag{2.10}$$

where $\mathbf{a}$ is the acceleration and $\mathbf{b}$ is the density (per unit mass) of body force.

3. Steady extension. We say that the fluid body is undergoing steady extension if there exists a fixed Cartesian coordinate system such that the components of $\mathbf{v}$ at x_1, x_2, x_3 are given by

$$v_i = a_i\, x_i, \tag{3.1}$$

where a_i are constants. Eqs. (2.2) are

$$\frac{d\xi_i}{d\tau} = a_i\, \xi_i \tag{3.2}$$

and, by using (2.3), the required solution is

$$\xi_i = x_i e^{-a_i s} \quad (s = t - \tau). \tag{3.3}$$

The history of deformation is given by

$$\lfloor \mathbf{F}(s) \rfloor = \begin{Vmatrix} e^{-a_1 s} & 0 & 0 \\ 0 & e^{-a_2 s} & 0 \\ 0 & 0 & e^{-a_3 s} \end{Vmatrix}, \tag{3.4}$$

where $\lfloor \mathbf{F}(s) \rfloor$ represents the matrix of the tensor $\mathbf{F}(s)$.

It follows from Eqs. (3.4), (2.7) and (2.8) that the stress tensor $\mathbf{T}$ must be given by an isotropic function of $\mathbf{M}$, where

$$\lfloor \mathbf{M} \rfloor = \begin{Vmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{Vmatrix}. \tag{3.5}$$

Using this fact, and the representation theorem for isotropic functions of symmetric matrices we find that

$$\mathbf{T} = (-p + \beta_0)\,\mathbf{I} + \beta_1 \mathbf{M} + \beta_2 \mathbf{M}^2, \tag{3.6}$$

where the coefficients β_0, β_1 and β_2 are functions of ρ and the three principal invariants of $\mathbf{M}$; their form depends on the fluid under consideration.

Using the matrix

$$\lfloor \mathbf{T} \rfloor = \begin{Vmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{Vmatrix}, \tag{3.7}$$

we obtain

$$T_1 - T_2 = (a_1 - a_2)\,\beta_1 + (a_1^2 - a_2^2)\,\beta_2\,, \tag{3.8}$$
$$T_1 - T_3 = (a_1 - a_3)\,\beta_1 + (a_1^2 - a_3^2)\,\beta_2\,.$$

If we assume that the body forces are zero, it follows from Eqs. (3.8) and (3.1) that the equations of motion (2.9) and (2.10) reduce to

$$\frac{\partial T_1}{\partial x_1} = \rho\, a_1^2 x_1\,, \quad \frac{\partial T_2}{\partial x_2} = \rho\, a_2^2 x_2\,, \quad \frac{\partial T_3}{\partial x_3} = \rho\, a_3^2 x_3\,, \tag{3.9}$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} a_i x_i + \rho \Sigma a_i = 0.$$

Suppose now that the rheologycal coefficients β_0, β_1 and β_2 depend only on the three principal invariants of **M**. By (3.8), we may write

$$\frac{\partial T_2}{\partial x_1} = \rho a_1^2 x_1\,, \quad \frac{\partial T_2}{\partial x_2} = \rho a_2^2 x_2\,, \quad \frac{\partial T_2}{\partial x_3} = \rho a_3^2 x_3\,. \tag{3.10}$$

It follows from (3.10) and (3.9) that

$$a_1^2 x_1 \frac{\partial \rho}{\partial x_2} - a_2 x_2 \frac{\partial \rho}{\partial x_1} = 0, \quad a_1^2 x_1 \frac{\partial \rho}{\partial x_3} - a_3^2 x_3 \frac{\partial \rho}{\partial x_1} = 0, \tag{3.11}$$
$$a_2^2 x_2 \frac{\partial \rho}{\partial x_3} - a_3^2 x_3 \frac{\partial \rho}{\partial x_2} = 0, \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} a_i x_i + \rho \Sigma a_i = 0.$$

Equations (3.11) show that

$$a_1^2 x_1 \frac{\partial V}{\partial x_2} - a_2^2 x_2 \frac{\partial V}{\partial x_1} = 0, \quad a_1^2 x_1 \frac{\partial V}{\partial x_3} - a_3^2 x_3 \frac{\partial V}{\partial x_1} = 0, \tag{3.12}$$
$$a_2^2 x_2 \frac{\partial V}{\partial x_3} - a_3^2 x_3 \frac{\partial V}{\partial x_2} = 0,$$

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{3} a_i x_i \frac{\partial V}{\partial x_i} - \rho \sum_{i=1}^{3} a_i \frac{\partial V}{\partial \rho} = 0 \tag{3.13}$$

and

$$(a_2 - a_1)\left(a_1^2 x_1 \frac{\partial V}{\partial x_2} + a_2^2 x_2 \frac{\partial V}{\partial x_1}\right) = 0, \tag{3.14}$$
$$(a_3 - a_1)\left(a_1^2 x_1 \frac{\partial V}{\partial x_3} + a_3^2 x_3 \frac{\partial V}{\partial x_1}\right) = 0,$$
$$(a_3 - a_2)\left(a_2^2 x_2 \frac{\partial V}{\partial x_3} + a_3^2 x_3 \frac{\partial}{\partial x_2}\right) = 0,$$

where

$$V(\rho,\ x_1,\ x_2,\ x_3,\ t) = 0$$

is a solution of (3.11).

a) Assuming

$$a_1 \neq a_2 \neq a_3$$

we have, by virtue of (3.12) and (3.14)

$$V = V(\rho, t), \quad \rho = \rho(t),$$

and by the continuity equation (3.13),

$$\rho = c e^{-\Sigma a_i t}. \tag{3.15}$$

By integration of (3.10) and use of (3.8) we obtain

$$T_i = \frac{1}{2}\,\rho(t) \sum_{i=1}^{3} a_i^2 x_i^2 + a_i \beta_1 + a_i^2 \beta_2 + f(t), \tag{3.16}$$

where $f(t)$ is an arbitrary function of t.

b) Consider the case in which

$$a_1 = a_2 = a \neq a_3.$$

b$_1$) If a and a_3 are not zero, we get (3.16).

b$_2$) If $a_3 = 0$, we find that

$$\rho = \frac{1}{\alpha}\, g(\alpha e^{-2at})$$

and

$$T_i = \frac{a^2}{2}\int \rho(\alpha, t)\, d\alpha + a_i \beta_1 + a_i^2 \beta_2 + f(t),$$

where $\alpha = x_1^2 + x_2^2$, and $f(t)$ is an arbitrary function of t.

b$_3$) If $a = 0$, ρ and T_i can be written as

$$\rho = \frac{1}{x_3}\, g(x_3 e^{-a_3 t}),$$

$$T_1 = T_2 = a_3^2 \int g(x_3 e^{-a_3 t})\, dx_3 + f(t),$$

$$T_3 = T_1 + a_3 \beta_1 + a_3^2 \beta_2.$$

c) Assuming

$$a_1 = a_2 = a_3 = a,$$

we get

$$\rho = z^{-1/2} g(z e^{-2at}),$$

and

$$T_1 = T_2 = T_3 = \frac{a^2}{2}\int \rho \, dz + f(t),$$

where $z = x_1^2 + x_2^2 + x_3^2$ and $f(t)$ as before, is an arbitrary function of t.

4. An application. We assume that at some time the body has the shape of a box whose edges are parallel to the coordinate axes. It then follows from Eq. (3.1) that the fluid fills a box at all times. Let $b_i = b_i(t)$ be the length of the edge of the box parallel to the x_i axis. Equation (3.1) implies that

$$\frac{db_i}{dt} = a_i b_i, \quad a_i = \frac{d}{dt}(\log b_i). \tag{4.1}$$

If $a_1 \neq a_2 \neq a_3$, by (3.16) we obtain

$$\begin{aligned} T_1 - T_2 &= \frac{\rho}{4}(a_1^2 b_1^2 - a_2^2 b_2^2) + (a_1 - a_2)[\beta_1 + (a_1 + a_2)\beta_2], \\ T_1 - T_3 &= \frac{\rho}{4}(a_1^2 b_1^2 - a_3^2 b_3^2) + (a_1 - a_3)[\beta_1 + (a_1 + a_3)\beta_2], \end{aligned} \tag{4.2}$$

where T_i is the normal stress (per unit area or the present configuration) in the centre of the side perpendicular to the x_i axis. Eqs. (4.2) may be used to calculate the material functions β_1 and β_2.

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EXTENSIA STAȚIONARĂ A FLUIDELOR SIMPLE COMPRESIBILE

Rezumat

În lucrare se consideră fluidele pentru care tensiunea actuală este determinată, pînă la o presiune hidrostatică, de istoria gradientului de deformare relativă. Presupunînd că aceste fluide (2.7) admit cîmpul de viteze (3.1), se arată că tensiunile tangențiale sînt nule, iar diferențele tensiunilor normale satisfac relațiile (3.9). După integrarea ecuațiilor mișcării, putem determina natura fluidului cunoscînd tensiunile normale și forma volumului ocupat de fluid la un moment dat. Cazul cînd acest volum este un paralelipiped dreptunghic este considerat ca o aplicație.