call $a \in \mathfrak{A}^{\cdot}$ a left (right) quasi-regular, or quasi-invertible element with respect to the endomorphisms φ, ψ. *Lemma 6'*. D_a permutes with S_r, for any $a, r \in \mathfrak{A}^{\cdot}$, if and only if 1), 2), 3) (from Lemma 6' in the text) hold. *Lemma 7*. The additive subgroup $\mathfrak{A}^{\cdot}$, D_a is a left weak-ideal for any a ($\mathfrak{A}^{\cdot}$, S_r is a right weak-ideal for any r) if and only if (16') holds for any r (any a). *Theorem* 1. $\mathfrak{A}^{\cdot}$ is an associative weak-ring if and only if (16'), (23'), (24'), (25') hold. *Theorem* 2. In an associative weak-ring $\mathfrak{A}^{\cdot}$, the set of all its two-sided quasi-regular elements is a multiplicative subgroup G of the multiplicative semigroup of $\mathfrak{A}^{\cdot}$; the identity element of G is the identity element of $\mathfrak{A}^{\cdot}$; for any $a \in G$, D_a, S_a are automorphisms of $\mathfrak{A}^{\cdot}$.

SOME NUMBER THEORETIC APPLICATIONS OF CERTAIN POLYNOMIALS RELATED TO BERNOULLI POLYNOMIALS

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Let us introduce the notations

(1) $$t_r = r(r+1)/2,$$

(2) $$S_n(r) = \sum_{k=1}^{r} k^n.$$

As concerns the designation t_r, subscript r will be omitted whenever it is possible, hence $t = t_r = S_1(r)$. Also t will designate an indeterminate, if r need not be an integer.

By use of the known relations

$$S_{2m-1}(r) = \frac{1}{2m}\left[B_{2m}(r+1) - B_{2m}\right] = \frac{1}{2m}\sum_{s=0}^{2m-1}\binom{2m}{s}(r+1)^{2m-s} B_s,$$

where Bernoulli numbers appear in the even-suffix notation (except for B_1), we find

$$2S_{2m-1}(r) = S_{2m-1}(r) + S_{2m-1}(r-1) + r^{2m-1} =$$

$$= \frac{1}{2m}\sum_{s=0}^{2m-1}\binom{2m}{s}\left[(r+1)^{2m-s} + r^{2m-s}\right] B_s + r^{2m-1} =$$

$$= \frac{1}{2m}\sum_{s=0}^{2m-1}\binom{2m}{s}\left[(r+1)^{2m-s} + (1-2\delta_{1,s})\, r^{2m-s}\right] B_s.$$

Here $\delta_{i,k}$ is Kronecker's symbol. By introduction of the designation

$u^2 = -1/8t$ from the definition (1) we have $r = (-u \pm \sqrt{u^2-1})/2u$ and substituting it in the above sum we obtain

$$S_{2m-1}(r) = \frac{1}{4m} \sum_{s=0}^{2m-1} \binom{2m}{s} [((u \pm \sqrt{u^2-1})/2u)^{2m-s} +$$

$$+ (1-2\delta_{1,s})((-u \pm \sqrt{u^2-1})/2u)^{2m-s}] B_s =$$

$$= \frac{1}{4m} \sum_{s=0}^{2m-1} \binom{2m}{s} [(u \pm \sqrt{u^2-1})^{2m-s} + (u \mp \sqrt{u^2-1})^{2m-s}] (2u)^{s-2m} B_s .$$

The expression in the brackets is connected with the Tchebichef polynomials $T_n(x)$ in a known manner (cf. [1], p. 15), hence

$$S_{2m-1}(r) = \left(\frac{1}{2m}\right) \sum_{s=0}^{2m-1} \binom{2m}{s} (2u)^{s-2m} T_{2m-s}(u) B_s .$$

Substituting here the explicit representation of the Tchebichef polynomial one obtains

$$S_{2m-1}(r) = \sum_{s=0}^{2m-1} \sum_{k=0}^{\left[\frac{2m-s}{2}\right]} \binom{2m}{s} \frac{2m-s}{4m} \frac{(2m-s-k-1)!}{k!\,(2m-s-2k)!} (2t)^k B_s ,$$

where the order of summation will be changed

$$S_{2m-1}(r) = \sum_{k=1}^{m} t^k \frac{2^{k-1}}{k} \sum_{s=0}^{2m-2k} \binom{2m-1}{s} \binom{2m-s-k-1}{k-1} B_s .$$

For the polynomials on the right-hand side we shall introduce the following designations

(3) $$C_m(t) = \sum_{k=1}^{m} \varkappa(m,k)\, t^k , \qquad m = 1, 2, 3, \ldots ,$$

where

(4) $$\varkappa(m,k) = \frac{2^{k-1}}{k} \sum_{s=0}^{2m-2k} \binom{2m-1}{s} \binom{2m-s-k-1}{k-1} B_s , \qquad k = 1, 2, \ldots, m ,$$

and call them Jacobi polynomials for C.G.J. Jacobi was first who have found them in 1834 (cf. [4], p. 60). (It is to distinguish between $C_m(t)$ and Jacobi orthogonal polynomials!) The expression (4) being suitable for applications if k is close enough to m, gives also the following results: $\varkappa(1,1) = 1$, $\varkappa(m,1) = 0$ for $m = 2, 3, 4, \ldots$. Another expression for the coefficients $\varkappa(m,k)$ may be found, this time convenient for applications if k is close enough to 2.

Let us multiply the Bernoulli polynomial

$$B_{2m-1}(x) = \sum_{s=0}^{2m-1} \binom{2m-1}{s} x^{2m-s-1} B_s$$

by x^{-k}, differentiate this equations $k-1$ times and take then $x = 1$. Since $B_n(1) = B_n$, we have

$$\sum_{s=0}^{2m-1} \binom{2m-1}{s} \binom{2m-k-s-1}{k-1} B_s = \sum_{s=0}^{k-1} (-1)^{k-s-1} \binom{2m-1}{s} \binom{2k-s-2}{k-1} B_{2m-s-1} .$$

In another manner one can show that

$$\sum_{s=0}^{2m-1} \binom{2m-1}{s} \binom{2m-k-s-1}{k-1} B_s = \sum_{s=0}^{2m-2k} \binom{2m-1}{s} \binom{2m-k-s-1}{k-1} B_s +$$

$$+ \sum_{s=0}^{k-1} (-1)^{k-1} \binom{2m-1}{s} \binom{2k-s-2}{k-1} B_{2m-s-1}$$

Comparing the last two equations with the relation (4) we obtain

(5) $$\varkappa(m,k) = (-1)^k \frac{2^{k-1}}{k} \sum_{s=0}^{k-1} (1-(-1)^s) \binom{2m-1}{s} \binom{2k-s-2}{k-1} B_{2m-s-1} ,$$

$m = 1, 2, 3, \ldots, k = 1, 2, \ldots, m$. If particularly $m > 1$ and $1 < k \leqslant m$, the formula (5) reduces to

(6) $$\varkappa(m,k) = (-1)^k \frac{2^k}{k} \sum_{s \geqq k/2}^{k-1} \binom{2m-1}{2k-2s-1} \binom{2s-1}{k-1} B_{2m-2k+2s} .$$

Now owing to the relations (4) and (6) one can calculate easily a few first and last coefficients of the Jacobi polynomials:

$\varkappa(m,m) = 2^{m-1}/m, \quad m \geqslant 1,$

$\varkappa(m,m-1) = -2^{m-2}(m-2)/3!, \quad m \geqslant 2,$

$\varkappa(m,m-2) = 2^{m-2}(m-1)(m-3)(7m-8)/3!5!, \quad m \geqslant 3,$

$\varkappa(m,m-3) = -2^{m-3}(m-1)(m-2)(m-4)(31m^2-89m+$
$+48)/3!7!, \quad m \geqslant 4,$

(7) $\varkappa(m,m-4) = 2^{m-5}(m-1)(m-2)(m-3)(m-5)(127m^3-691m^2+$
$+1038m-384)/5!7!, \quad m \geqslant 5,$

. . . .

$\varkappa(m,4) = 4\binom{2m-1}{3} B_{2m-4} + 40(2m-1)B_{2m-2}, \quad m \geqslant 4,$

$\varkappa(m,3) = -8(2m-1)B_{2m-2}, \quad m \geqslant 3,$

$\varkappa(m,2) = 2(2m-1)B_{2m-2}, \quad m \geqslant 2$

By means of the above formulas were calculated the following eight polynomials:

$$C_1(t) = t,\ C_2(t) = t^2,\ C_3(t) = \frac{1}{3}(4t^3 - t^2),\ C_4(t) = \frac{1}{3}(6t^4 - 4t^3 + t^2),$$

$$C_5(t) = \frac{1}{5}(16t^5 - 20t^4 + 12t^3 - 3t^2),\ C_6(t) = \frac{1}{3}(16t^6 - 32t^5 + 34t^4 - 20t^3 + 5t^2),$$

$$C_7(t) = \frac{1}{105}(960t^7 - 2800t^6 + 4592t^5 - 4720t^4 + 2764t^3 - 691t^2),$$

$$C_8(t) = \frac{1}{3}(48t^8 - 192t^7 + 448t^6 - 704t^5 + 718t^4 - 420t^3 + 105t^2).$$

A simple property of the Jacobi polynomials is a consequence of the relation

$$S_{2m-1}(r) = C_m(t).$$

Namely, if one put $r = 1$, it follows $t = 1$ and $C_m(1) = 1$, $m = 1, 2, 3, \ldots$.

Generally, the coefficients $\alpha(m, k)$ are fractions. The application of the congruence methods to these fractions will be possible if we know something about prime factors of their denominators in more detail. Without any effort one result is obtained by use of the von Staudt-Clausen theorem for Bernoulli numbers.

Lemma 1. *The denominator of the coefficient* $\alpha(m, k)$, $k = 2, 3, \ldots, m$, *is an odd number whose every prime factor is less than* $2m - 1$.

Proof. Factor before the sum in formula (6) is $2^k/k$. Let a be the greatest integer such that $2^a/k$. Then $k \geqslant 2^a > a$, hence $2^{k-a} \geqslant 2$. Therefore the denominator of the fraction $\alpha(m, k)$ is an odd number. Bernoulli number with the greatest subscript in formula (6) is B_{2m-2}, so that greatest primfactor of its denominator could be $2m-1$. But in the case $2m-1$ is a prime this Bernoulli number in formula (6) is multiplied by $2m - 1$. This proves Lemma 1.

To make further investigations on Jacobi polynomials successful, some generalizations are needed.

Theorem 1. *Let a and b are arbitrary numbers. Then*

$$\sum_{n=1}^{r} (an + b)^{2m-1} = a^{2m-1}\left[C_m(v/a + c) - C_m(c)\right], \tag{8}$$

where

$$v = at + br = \sum_{n=1}^{r}(an + b), \tag{9}$$

$$c = b(a + b)/2a^2. \tag{10}$$

Proof. Let for a moment x designate any non-negative integer. Then

$$\sum_{n=1}^{r}(n + x)^{2m-1} - C_m(t_{r+x}) + C_m(t_x) = 0.$$

Since the left-hand side can be considered as a polynomial in x whose degree is not greater than $2m$, and the equation is satisfied if x assumes any non-negative integral value, this equation must be identity in x. Hence the equation will be satisfied also for $x = b/a$. After some rearrangement the equation (8) follows.

Replace in the equation (8) m by $m + 1$ and thus obtained equation differentiate with respect to b

$$\sum_{n=1}^{r}(an + b)^{2m} = \frac{a^{2m-1}}{2(2m+1)}\left[(2ar + a + 2b)\,C'_{m+1}(v/a + c) - (a + 2b)\,C'_{m+1}(c)\right].$$

In particular, if we take $a = 1$, $b = 0$, this relation is

$$S_{2m}(r) = \frac{2r + 1}{2(2m + 1)}\,C'_{m+1}(t). \tag{11}$$

Herefrom we conclude that $(S_{2m}(r))^2$ is always a polynomial in t. Another interesting application of the relation (11) leads to

Theorem 2. *Let p be an odd prime. Then*

$$\Sigma t_{i_1} t_{i_2} \ldots t_{i_k} \equiv -\frac{1}{k 2^{5k-1}}\binom{2k-2}{k-1} \pmod{p}, \tag{12}$$

where the summation is extended over all integral suffixes such that $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant (p - 1)/2$, $k = 1, 2, 3, \ldots, (p - 3)/2$, t_i *according to the definition* (1),

$$\Sigma (t_{i_1} t_{i_2} \ldots t_{i_k})^{-1} \equiv -\frac{2^k}{k + 1}\binom{2k}{k} \pmod{p} \tag{13}$$

where the summation is extended over all integral suffixes such that $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant (p - 1)/2$, $k = 1, 2, 3, \ldots, (p - 1)/2$, t_i *according to the definition* (1).

Proof. Let $p=2m+1$. In view of the Lemma 1 equation (11) shows that the congruence $C''_{m+1}(u) \equiv 0 \pmod{p}$ has solutions $u = t_0, t_1, t_2, \ldots, t_{(p-3)/2}$. Suppose any two of these m solutions are congruent, say $t_i \equiv t_j \pmod{p}$. Then either $i-j \equiv 0 \pmod{p}$ or $i+j+1 \equiv 0 \pmod{p}$, what is impossible. Therefore

$$(14) \qquad C''_{m+1}(u) \equiv d(u-t_0)(u-t_1)\ldots(u-t_{(p-3)/2}) \pmod{p},$$

where $d \equiv (m+1)\alpha(m+1,\, m+1) \equiv 2^m \equiv (-1)^{(p^2-1)/8} \pmod{p}$. On the other hand one can find by use of the relation (4) residues modulo p of the coefficients of the polynomials $C''_{m+1}(u)$

$$k\alpha(m+1,\, k) \equiv (-2)^{k-1}\binom{2k-2}{k-1} \pmod{p}.$$

Comparing the coefficients of the congruence (14), we obtain after some calculations

$$\Sigma\, t_{i_1} t_{i_2} \ldots t_{i_k} \equiv \frac{1}{2^{5k}}\binom{2k}{k} \pmod{p},\quad k \leqslant (p-3)/2,$$

where summation is extended over all integral suffixes such that $1 \leqslant i_1 < i_2 < \ldots i_k \leqslant (p-3)/2$. Designate the above sum by s_k. Then it is obvious that

$$\Sigma\, t_{i_1} t_{i_2} \ldots t_{i_k} = s_k + s_{k-1}\, t_{(p-1)/2},$$

the sum being extended over all integral suffixes such that $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant (p-1)/2$. The congruence (12) results.

If we divide corresponding sides of the congruence (12) and of the congruence

$$t_1 t_2 \ldots t_{(p-1)/2} \equiv \frac{1}{2}(-1)^{\frac{p^2-1}{8}+\frac{p+1}{2}} \pmod{p},$$

after some rearrangements one obtains the congruence (13). This proves the Theorem 2.

Theorem 3. *The Jacobi polynomials satisfy the differential-difference equation*

$$(15) \qquad \begin{gathered}\left(2x+\frac{1}{4}\right)y''_{m+1}(x) + y'_{m+1}(x) = 2m(2m-1)\,y_m(x) + (2m-1)B_{2m}, \\ m=1,2,3,\ldots, \textit{ where } y_m(x) = C_m(x).\end{gathered}$$

Proof. Differentiate the equation (8) two times with respect to h, put then $a=1$, $h=0$ and take into account that $C'_{m+1}(0)=0$ and $C''_{m+1}(0) = 2\alpha(m+1,\, 2) = 4(2m+1)B_{2m}$, $m=1,2,3,\ldots$, it follows

$$(16) \qquad \left(2t+\frac{1}{4}\right)C''_{m+1}(t) + C'_{m+1}(t) = 2m(2m-1)\,C_m(t) + (2m-1)B_{2m}.$$

If we now conceive t as an indeterminate, the above equation concerning polynomials of degree m must be satisfied for $t = t_1, t_2, t_3, \ldots$, hence it is identity in t. This proves Theorem 3.

Corollary 1. *Comparing the coefficients of the equal powers of t in the equation (16), we find a recurrence formula for the coefficients of Jacobi polynomials*

$$(17) \qquad \begin{gathered}\alpha(m+1,\, k+1) = 4\,\frac{2m(2m-1)}{k(k+1)}\,\alpha(m,\, k-1) - 4\,\frac{2k-1}{k+1}\,\alpha(m+1,\, k), \\ k \geqslant 2,\ m \geqslant 2.\end{gathered}$$

We shall mention here without proof two recurrence formulas for Jacobi polynomials:

$$(18) \qquad \begin{gathered}\frac{1}{m}\,C_{2m}(t) + 2\sum_{k=1}^{m-1}\binom{2m-1}{2k-1}\frac{B_{2k}}{2k}\,C_{2m-2k}(t) = (C_m(t))^2, \\ m=2,3,4,\ldots;\end{gathered}$$

the formula is correct also for $m=1$, if in this case the sum is omitted;

$$(19) \qquad \begin{gathered}\frac{(2m-1)^2}{m}\,C_{2m-1}(t) + 2\sum_{k=1}^{m-2}\binom{2m-1}{2k-1}\left[(2m-k-1)^2 + k(k-1)\right]\frac{B_{2k}}{2k}\,C_{2m-k-1}(t) = \\ = (2m-1)(2m-2)\,C_m(t)\,C_{m-1}(t) - (m-1)(2m-1)^2 B_{2m}\,C_m(t),\end{gathered}$$

$m=3,4,5,\ldots$; formula holds also for $m=2$, if in this case the sum is omitted.

By the help of the above relations we shall prove

Theorem 4. *The coefficients of the Jacobi polynomials have alternating signs. More precisely*

$$(20) \qquad \alpha(m,\, k) = (-1)^{m+k}\,|\alpha(m,\, k)|.$$

Proof. As base of this proof by induction, we may state an already found result: $\alpha(2,2) = (-1)^{2+2}|\alpha(2,2)|$. Suppose that $\alpha(n,k) = (-1)^{n+k}|\alpha(n,k)|$ for $k=2,3,\ldots,n$, $n=2,3,\ldots,2m-2$. By comparing the coefficients in the equation (19) one can find for $r=2,3,\ldots,m+1$

$$\frac{(2m-1)^2}{m}\varkappa(2m-1,r)+2\sum_{k=1}^{m-2}\binom{2m-1}{2k-1}\Big[(2m-k-1)^2+$$

$$+k(k-1)\Big]\frac{B_{2k}}{2k}\varkappa(2m-k-1,r)=(2m-1)(2m-2)\sum_{i+j=r}\varkappa(m,i)\,\varkappa(m-1,j)-$$

$$-(m-1)(2m-1)^2B_{2m-2}\,\varkappa(m,r)\,,$$

where the sum on the right-hand side is to be taken over all partitions of the integer r in integers i, j, $2 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m-1$, and by definition $\varkappa(m,r)=0$ for $r>m$. Formula holds for $m=3,4,\ldots,$ and also for $m=2$ if in this case the sum on the left-hand side is omitted. For $r=m+2,\ m+3, \ldots, 2m-2$ one finds

$$\frac{(2m-1)^2}{m}\varkappa(2m-1,r)+2\sum_{k=1}^{2m-r-1}\binom{2m-1}{2k-1}\Big[(2m-k-1)^2+$$

$$+k(k-1)\Big]\frac{B_{2k}}{2k}\varkappa(2m-k-1,r)=$$

$$=(2m-1)(2m-2)\sum_{i+j=r}\varkappa(m,i)\varkappa(m-1,j)\,,$$

with the same explanations as above. The formula holds also for $r=2m-1$ if then the sum on the left-hand side is omitted. By virtue of the induction hypothesis and the fact that $B_{2k}=(-1)^{k-1}|B_{2k}|$ for both above cases we have

$$\varkappa(2m-1,r)=\frac{m(2m-2)}{2m-1}(-1)^{r-1}\sum_{i+j=r}|\varkappa(m,i)|\,|\varkappa(m-1,j)|+$$

$$+(m-1)m(-1)^{r-1}|B_{2m-2}|\,|\varkappa(m,r)|+$$

$$+\frac{2m}{(2m-1)^2}(-1)^{r-1}\sum_{k=1}\binom{2m-1}{2k-1}\Big[(2m-k-1)^2+$$

$$+k(k-1)\Big]\frac{|B_{2k}|}{2k}|\varkappa(2m-k-1,r)|\,,$$

hence

$$\varkappa(2m-1,r)=(-1)^{2m-1+r}|\varkappa(2m-1,r)|,\quad r=2,3,\ldots,2m-1\,.$$

Suppose now that $\varkappa(n,k)=(-1)^{n+k}|\varkappa(n,k)|$ for $k=2,3,\ldots,n$, $n=2,3,\ldots,2m-1$. Comparing the coefficients in the equation (18) one can find for $r=2,3,\ldots,m+1$

$$\frac{1}{m}\varkappa(2m,r)+2\sum_{k=1}^{m-1}\binom{2m-1}{2k-1}\frac{B_{2k}}{2k}\varkappa(2m-k,r)=\sum_{i+j=r}\varkappa(m,i)\,\varkappa(m,j)\,,$$

where the sum on the right-hand side is to be taken over all partitions of the integer r in integers i, j, $2 \leqslant i, j \leqslant m$. For $r=m+2,\ m+3,\ldots,2m-1$ one finds

$$\frac{1}{m}\varkappa(2m,r)+2\sum_{k=1}^{2m-r}\binom{2m-1}{2k-1}\frac{B_{2k}}{2k}\varkappa(2m-k,r)=\sum_{i+j=r}\varkappa(m,i)\,\varkappa(m,j)\,,$$

with the same explanations as above. The formula is true also for $r=2m$ if then the sum on the left-hand side is omitted. Owing to the induction hypothesis in both cases we have

$$\varkappa(2m,r)=(-1)^r m\sum_{i+j=r}|\varkappa(m,i)|\,|\varkappa(m,j)|+$$

$$+(-1)^r\,2m\sum_{k=1}\binom{2m-1}{2k-1}\frac{|B_{2k}|}{2k}|\varkappa(2m-k,r)|\,;$$

hence

$$\varkappa(2m,r)=(-1)^{2m+r}|\varkappa(2m,r)|,\qquad r=2,3,\ldots,2m\,.$$

This proves the theorem.

It might be of general interest to establish the following relations between Jacobi and Bernoulli polynomials

$$C_m(t)=\frac{1}{2m}\left[B_{2m}\left(\frac{1}{2}\left(1\pm\sqrt{1+8t}\right)\right)-B_{2m}\right], \tag{21}$$

$$C'_m(t)=\pm 2(1+8t)^{-1/2}B_{2m-1}\left(\frac{1}{2}\left(1\pm\sqrt{1+8t}\right)\right),\quad m=1,2,3,\ldots \tag{22}$$

From the last two relations by use of the Theorem 4 one arrives immediately at the following results:

if we assume $-1/8<t<0$, then result reads

$(-1)^m\,(B_{2m}(z)-B_{2m})>0$ in the interval $0<z<1$,

$(-1)^m\,B_{2m-1}(z)>0$ in the interval $0<z<1/2$,

$(-1)^m\,B_{2m-1}(z)<0$ in the interval $1/2<z<1$.

(which is of course already known, cf. for instance [5], p. 22) and thereupon for $t<-1/8$ the result can be stated as

Theorem 5. *The polynomials $B_{2m}(z) - B_{2m}$ and $B_{2m-1}(z)$ have no zero on the straight line $z = 1/2 + iy$, $y \neq 0$.*

It may be interesting to present a possibility of extension of the Jacobi polynomials. Here the main reason for this lies in the desire to examine the properties of the coefficients $\alpha(m, k)$ given by the formula (6), which retains a meaning even if $m \geq 1$, $k \geq 2$, $m < k$. It is worthy to mention that we could start our investigations by the differential-difference equation (15) with a properly chosen $y_1(x)$. After this remark, the exposition can be given as follows. Our goal is to find an expression in the closed form for the sequence of functions

$$c_m(x) = \sum_{k=1}^{\infty} \alpha(m, k)\, x^k, \tag{23}$$

where $m = 1, 2, 3, \ldots$, $\alpha(1, 1) = 1$, $\alpha(m, 1) = 0$ for $m = 2, 3, \ldots$, and the remaining coefficients $\alpha(m, k)$ according to formula (6).

A simple method for the summation of series may be used to obtain

$$c_1(x) = \frac{1}{4} + 2x - \frac{1}{4}(1 + 8x)^{1/2},$$

$$c_2(x) = \frac{1}{4} + \frac{3}{2}x + x^2 - \frac{3}{16}(1 + 8x)^{1/2} - \frac{1}{16}(1 + 8x)^{3/2}. \tag{24}$$

In order to find similar expressions for the remaining terms of the sequence $\{c_m(x)\}$, the extended coefficients $\alpha(m, k)$ should be examined.

Theorem 6. *The coefficients $\alpha(m, k)$ defined by the formula (6) for every $k \geq 2$ and $m \geq 2$, satisfy the difference equation (17).*

Instead of the proof we shall mention only that the direct checking of the above theorem is very simple. In fact, if the coefficients $\alpha(m, k)$, given by the formula (6), are substituted in the expression formed as a difference of the left-hand and the right-hand side of the equation (17), and this expression arranged as a linear form in Bernoulli numbers, the coefficients of Bernoulli numbers are all equal to zero.

Theorem 7.

$$\alpha(m, k) = 0, \tag{25}$$

for $k = m + 1, m + 2, m + 3, \ldots, 2m - 2$, $m \geq 3$.

Proof. By direct checking, one finds that $\alpha(3, 4) = 0$. Suppose now that $\alpha(m, k) = 0$ for $k = m + 1, m + 2, \ldots, 2m - 2$, $m > 3$. With regard to the relation (17) we have

$$\alpha(m + 1, m + 2) = 4\,\frac{2m(2m + 1)}{(m + 1)(m + 2)}\,\alpha(m, m) - 4\,\frac{2m + 1}{m + 2}\,\alpha(m + 1, m + 1) = 0,$$

because of $\alpha(m, m) = 2^{m-1}/m$, and then

$$\alpha(m + 1, k) = 4\,\frac{2m(2m + 1)}{k(k - 1)}\,\alpha(m, k - 2) - 4\,\frac{2k - 3}{k}\,\alpha(m + 1, k - 1),$$

whence by the method of finite induction we find successively $\alpha(m + 1, k) = 0$ for $k = m + 3, m + 4, \ldots, 2m$. This proves the theorem.

Corollary 2. *From the relations (6) and (25) it results that*

$$\sum_{s \geq m - k/2}^{m-1} \binom{2s + 2k - 2m - 1}{k - 1} \binom{2m - 1}{2s} B_{2s} = 0, \tag{26}$$

for $k = m + 1, m + 2, \ldots, 2m - 2$, $m \geq 3$.

The next unexplored coefficient is $\alpha(m, 2m - 1)$, $m \geq 3$. We shall show that

$$\alpha(m, 2m - 1) = -2^{2m-2}, \quad m \geq 3. \tag{27}$$

Substituting in the formula (6) $m = 3$, $k = 5$ one can find $\alpha(3, 5) = -16$. Suppose the formula (27) holds for $m = n$. Then by equation (17) and Theorem 7 it follows that $\alpha(n + 1, 2n + 1) = -2^{2n}$.

Theorem 8.

$$\alpha(m, k) = (2m - 1)\,\frac{(-1)^k\, 2^{k-1}}{k} \binom{2k - 2m}{k - 2m + 1}, \tag{28}$$

for $m = 2$, $k \geq 4$ and $m \geq 3$, $k \geq 2m - 1$.

Proof. We designate the expression on the right-hand side of the relation (28), for the moment by $\bar{\alpha}(m, k)$. The direct checking shows that $\bar{\alpha}(m, k)$ satisfies the difference equation (17) for $k \geq 2m$, $m \geq 2$. On the other hand putting $m = 2$ in the formula (6), one finds that $\bar{\alpha}(2, k) = \alpha(2, k)$, $k \geq 4$, whilst the relation (27) shows that $\bar{\alpha}(m, 2m - 1) = \alpha(m, 2m - 1)$, $m \geq 3$. Thus we conclude that $\bar{\alpha}(m, k) = \alpha(m, k)$, $m = 2$, $k \geq 4$ and $m \geq 3$, $k \geq 2m - 1$ what is essentially the content of the Theorem 8.

Corollary 3. *From the relations (6) and (28) it results that*

$$\sum_{s=1}^{m-1} \binom{2s + 2m - 3}{2m - 2} \binom{2m - 1}{2s} B_{2s} = (2m - 1)/2, \quad m \geq 3. \tag{29}$$

and

(30) $$\sum_{s=0}^{m-1}\binom{2s+2k-2m-1}{k-1}\binom{2m-1}{2s}B_{2s}=\frac{2m-1}{2}\binom{2k-2m}{k-1},\quad k\geq 2m,\ m\geq 2.$$

We are now in a position to find suitable expressions for functions $c_m(x)$ defined by the relation (23). Let us start by the function

$$g_m(x)=\sum_{k=m+1}^{\infty} z(m,k)\,x^k,\qquad m\geq 3.$$

The coefficients $z(m,k)$ given by the relation (28) should be transformed in a simple manner to obtain

$$g_m(x)=-\frac{1}{2^{2m}}\sum_{k=m+1}^{\infty}(8x)^k\,\frac{(-1)^{k+1}(2m-1)!\,(2k-2m)!}{2^{2k-1}(k-1)!\,k!}\,2^{2m-2}\binom{k-1}{2m-2}.$$

By use of the formula (cf. [3], (32), p. 253)

$$2^{2m-2}\binom{k-1}{2m-2}=\sum_{i=0}^{m-1}(-1)^i\binom{k-1}{i}\binom{2(k-1)-2i}{2(m-1)-2i},$$

the expression for $g_m(x)$ may be modified again

$$g_m(x)=-\frac{1}{2^{2m}}\sum_{k=m+1}^{\infty}(8x)^k\sum_{i=1}^{m}\binom{2m-1}{2i-1}\binom{i-1/2}{k}.$$

Let us now define the polynomials $D_m(x)$ thus

(31) $$D_m(x)=C_m(x)+\frac{1}{2^{2m}}\sum_{k=0}^{m}(8x)^k\sum_{i=1}^{m}\binom{2m-1}{2i-1}\binom{i-1/2}{k},\qquad m\geq 2.$$

Hence we can write

$$c_m(x)=C_m(x)+g_m(x)=D_m(x)-\frac{1}{2^{2m}}\sum_{k=0}^{\infty}(8x)^k\sum_{i=1}^{m}\binom{2m-1}{2i-1}\binom{i-1/2}{k},$$

where the order of summation may be reversed

$$c_m(x)=D_m(x)-\frac{1}{2^{2m}}\sum_{i=1}^{m}\binom{2m-1}{2i-1}(1+8x)^{(2i-1)/2},$$

so that finally we arrive at the proposed relation

(32) $$c_m(x)=D_m(x)-\frac{1}{2^{2m+1}}\left[(1+(1+8x)^{1/2})^{2m-1}-(1-(1+8x)^{1/2})^{2m-1}\right]$$

valid for $m\geq 3$. This relation remains true also for $m=2$, what is checked by means of the relation (24).

If the definition of the function $c_1(x)$ is modified,

$$\bar{c}_1(x)=c_1(x)-x=\frac{1}{4}-x-\frac{1}{4}(1+8x)^{1/2},$$

the sequence of functions $\bar{c}_1(x)$, $c_2(x)$, $c_3(x)$, ... should satisfy the differential-difference equation (15), if there $y_1(x)=\bar{c}_1(x)$, $y_m(x)=c_m(x)$, $m=2,3,4,\dots$. For the present it is a principal property of the sequence of functions (23).

Consider now the relation (26) and take $2m-1=p$, where the prime $p\geq 5$. Introduce the notation

(33) $$\frac{1}{p}\binom{p}{2s}B_{2s}=x_{2s},$$

and the substitutions $s+u=m$, $k=m+r$ to obtain

(34) $$\sum_{u=1}^{h}\binom{p+2r-2u}{m+r-1}x_{2m-2u}=0,$$

where

$$h=\left[\frac{m+r}{2}\right],$$

and $r=1,2,3,\dots,m-2$. Introduce also the notation

(35) $$a_{r,u}=\binom{p+2r-2u}{m+r-1},$$

$r=1,2,3,\dots,m-2$, $u=1,2,3,\dots,h$. In the case $u\leq r$ it is possible to verify that $0\leq 2r-2u<2r+1\leq m+r-1$, $2r+1\leq m+r-1\leq 2m-3=p-2$, so one may apply a result of L u c a s (cf. [2], p. 52), to show that

$$a_{r,u}\equiv\binom{1}{0}\binom{2r-2u}{m+r-1}\equiv 0\pmod{p}$$

for $u\leq r$. In particular it is necessary to examine the residue of $a_{r,1}/p$ modulo p. By use of Wilson's theorem it results

(36) $$a_{r,1}/p\equiv(-1)^{\frac{r-1}{2}}\,2^{p-2}\,\frac{r!\,(r-1)!}{(2r)!}\pmod{p}.$$

Now, after these preliminary expositions we are able to rewrite the equation (34) as the congruence

$$\sum_{u=r+1}^{h} a_{r,u}\, x_{2m-2u} \equiv a_{r,1}\, p \pmod{p}, \tag{37}$$

$r = 1, 2, 3, \dots, m-2$. Henceforth we shall distinguish the two cases: a) $m = 2n$ and b) $m = 2n+1$.

Case a). $m = 2n$, $p = 4n-1$. From the system of congruences (37) where $r = 1, 2, 3, \dots, 2i+1$ one may eliminate $2i$ numbers: $x_{m+2i-2}, x_{m+2i-4}, \dots, x_m, x_{m-2i+2}, x_{m-2i}$. To that end one could multiply the congruences (37) by properly choosen determinants and add the corresponding sides of these obtained congruences. On the left-hand side remains the numbers $x_{2m-4}, x_{2m-6}, \dots, x_{m+2i}$ whereas on the right-hand side it appears a determinant of the $2i+1$-th order, say D_{2i+1}. Hence, the result of this process of elimination is the congruence

$$\sum_{j=1}^{n-2} b_j\, x_{2n+2j} \equiv D_{2i+1} \pmod{p}. \tag{38}$$

As concerns the elements of the determinant D_{2i+1}, they are given by the formulas (35) and (36). If one requires that the condition $u > r$ for the elements $a_{r,u}$ must be satisfied, this implies a condition on i. It is easy to show that the mentioned condition reads

$$p > 12i+3. \tag{39}$$

Expression (35) for the elements $a_{r,u}$ is used to find the following form of their residues modulo p:

$$a_{r,u} \equiv \binom{2\nu+2r-2[r/2]-1/2}{2\nu+r-2[r/2]} \pmod{p}, \tag{40}$$

where $\nu = h-u = n-u+[r/2]$, $u = r+1, r+2, \dots, h$.

With regard to the expressions (36) and (40) we infer that the determinant D_{2i+1} does not depend on p if i has a fixed value. Hence, there exists a prime $p_0(i)$ such that for every prime $p \geqslant p_0(i)$ we have $D_{2i+1} \not\equiv 0 \pmod{p}$, and the condition (39) is fulfilled. On the other hand it follows from the definition (33) that $x_{2s} \equiv 0 \pmod{p}$ if and only if $B_{2s} \equiv 0 \pmod{p}$ provided that $2s < p-1$. Owing to the congruence (38) we can state

Theorem 9. *Let i be a fixed natural number. There exists a prime $p_0(i)$ such that for every prime p of the form $4n-1$, $p \geqslant p_0(i)$, at least one of the numerators of the Bernoulli numbers $B_{p-3}, B_{p-5}, \dots, B_{(p-4i-1)/2}$ is not divisible by p.*

For the first two particular cases, $i = 1, 2$, one finds

$$D_3 = \frac{13}{4 \cdot 5}, \quad D_5 = -\frac{5^2 \cdot 733}{2^9 \cdot 3^2},$$

and because of the condition (39), $p_0(1) = 19$, $p_0(2) = 31$.

Case b). $m = 2n+1$, $p = 4n+1$. Similarly as in the case a), from the system of congruences (37), $r = 1, 2, 3, \dots, 2i$, it is possible to eliminate $2i-1$ numbres $x_{m+2i-3}, x_{m+2i-5}, \dots, x_{m-1}, \dots, x_{m-2i+3}, x_{m-2i+1}$. On the left-hand side it remains numbers $x_{2m-4}, x_{2m-6}, \dots, x_{m+2i-1}$, and on the right-hand side it appears a determinant D_{2i} of the $2i$-th order. The result of this elimination is therefore

$$\sum_{j=1}^{n-1} c_j\, x_{2n+2j} \equiv D_{2i} \pmod{p}. \tag{41}$$

The elements of the determinant D_{2i} are given by formulas (35) and (36). But in this case from the formula (35) it follows that

$$a_{r,u} \equiv \binom{2\nu+2r-2[(r+1)/2]+1/2}{2\nu+r-2[(r+1)/2]+1} \pmod{p}, \tag{42}$$

where $\nu = [(m-r)/2] - u = h-u = n-u+[(r+1)/2]$, $u = r+1, r+2, \dots, h$. To satisfy the condition $u > r$ for elements $a_{r,u}$ of the determinant D_{2i}, the condition

$$p > 12i-3 \tag{43}$$

should be fulfilled.

Again, since the elements of the determinant D_{2i} are given by formulas (36) and (42), we conclude that D_{2i} does not depend on p. Hence it is possible to find a prime $p_0(i)$ such that for every prime $p \geqslant p_0(i)$ the condition (43) is satisfied and $D_{2i} \not\equiv 0 \pmod{p}$, i being a fixed natural number. On the base of the congruence (41) and the definition (33) we are able to state

Theorem 10. *Let i be a fixed natural number. There exists a prime $p_0(i)$ such that for every prime p of the form $4n+1$, $p \geqslant p_0(i)$, at least one of the numerators of the Bernoulli numbers $B_{p-3}, B_{p-5}, \dots, B_{(p+4i-1)/2}$ is not divisible by p.*

For the first three cases, $i = 1, 2, 3$, one can find

$$D_2 = -\frac{1}{3}, \quad D_4 = -\frac{3 \cdot 47}{2^5 \cdot 7}, \quad D_6 = \frac{691339}{2^{14} \cdot 7 \cdot 11},$$

and by virtue of the condition (43) $p_0(1) = 13$, $p_0(2) = 29$, $p_0(3) = 37$.

We shall start the further investigation by the formula (30) rewritten in a slightly different form:

$$\sum_{s=0}^{2m-2}\binom{s+2k-2m-1}{k-1}\binom{2m-1}{s}B_s=0,\quad k<2m,\ m\geq 2.$$

Introduce the symbol

$$(a)_n=a(a-1)\cdots(a-n+1),\ (a)_0=1,$$

and replace k by $2m-r-2$ in the above formula to obtain

$$\sum_{s=0}^{2m-2}(s-r-1)_{2m-r-3}\binom{2m-1}{s}B_s=0,$$

$r=2,3,\ldots,\ m=2,3,\ldots$. It is easy to verify that the formula

$$\sum_{s=0}^{2m-2}(s-r-1)_{2m-r-3}(2m-s)_{2q-2}\binom{2m+2q-3}{s}B_s=0,$$

$q=1,2,3,\ldots$, is equivalent to the preceding one. Make here another substitution $m=n-q+1$ to obtain finally

$$\text{(44)}\qquad \sum_{s=0}^{2n-2q}(s-r-1)_{2n-2q-r-1}(2n-2q-s+2)_{2q-2}\binom{2n-1}{s}B_s=0,$$

$r=2,3,\ldots,\ n=2,3,\ldots,\ q=1,2,3,\ldots$. Now we shall make the assumptions $2n-1=p$, the prime $p\geq 3$, $r=2q+(p-1)/2$, $2\leq q\leq(p+3)/4$, and introduce the designations

$$a_s=(s-r-1)_{2n-2q-r-1},\quad b_s=(2n-2q-s+2)_{2q-2}.$$

Ascertain first that b_s is divisible by p only for $s=0$, and secondly that:

a_s is divisible by p but not by p^2 for $s=p-2q+1,\ p-2q,\ \ldots,\ (p-4q+5)/2$,

a_s is divisible by p^2 for $s=(p-4q+3)/2,\ (p-4q+1)/2,\ \ldots,\ 1,0$.

Hence, with the notation $s_0=(p-4q+5)/2$, from the equation (44) it results

$$\text{(45)}\qquad \sum_{s=s_0}^{2n-2q}\frac{a_s}{p}\,b_s\,\frac{1}{p}\binom{p}{s}B_s\equiv 0\pmod p.$$

Put here $q=2$ to conclude on the base of the theorems 9 and 10 for $i=1$ and the congruence (45) that at least two of the numerators of

the Bernoulli numbers $B_{p-3}, B_{p-5}, \ldots, B_{(p-3)/2}$ are not divisible by the prime p, $p\geq 19$. Take further $q=3$ to conclude on the base of the foregoing result and the congruence (45) that at least two of the numerators of the Bernoulli numbers $B_{p-5}, B_{p-7}, \ldots, B_{(p-7)/2}$ are not divisible by the prime p, $p\geq 19$. If we continue this chain of conclusions we prove

Theorem 11. *At least two of the numerators of the Bernoulli numbers* $B_{p-2q+1}, B_{p-2q-1}, \ldots, B_{(p-4q+5)/2}$ *are not divisible by the prime* p, $p\geq 19$, *for every integer* q *from the interval* $2\leq q\leq(p+3)/4$.

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CÎTEVA APLICAȚII ÎN TEORIA NUMERELOR ALE UNOR POLINOAME LEGATE DE POLINOAMELE LUI BERNOULLI

Rezumat

Lucrarea este consacrată examinării unor proprietăți ale polinoamelor numite aici ale lui Jacobi, pentru că Jacobi a fost cel care le-a descoperit, dar care trebuie deosebite de polinoamele ortogonale ale lui Jacobi. Sînt găsite două formule diferite (4) și (6), pentru coeficienții polinoamelor Jacobi. Se pare că nu sînt cunoscute pînă acum. Cu ajutorul acestor coeficienți și al unor proprietăți ale polinoamelor Jacobi sînt găsite reziduurile funcțiilor simetrice elementare ale numerelor triangulare după un modul prim (teorema 2). Sînt găsite formule de recurență pentru coeficienții (17) și pentru polinoamele Jacobi (18) și (19). Ultimele două relații au dat posibilitatea să se demonstreze că coeficienții polinoamelor Jacobi au semne alternante (teorema 4). În sfîrșit se generalizează polinoamele Jacobi (23) și se demonstrează că aceste generalizări sînt funcții algebrice

(32) pentru care coeficienții dezvoltării puterilor conduc la următorul rezultat: cel puțin unul din numerele Bernoulli $B_{p-3}, B_{p-5}, \dots, B_{(p+1)/2}$ nu este divizibil prin numărul prim $p = 4n - 1$, dacă $p \geq p_0(i)$, $i = 1, 2, \dots$ (teorema 9); cel puțin unul din numerele lui Bernoulli de mai sus nu este divizibil prin numărul prim $p = 4n + 1$, dacă $p \geq p_0(i)$, $i = 1, 2, \dots$ (teorema 10); cel puțin unul din numerele lui Bernoulli B_{p-2q+1}, $B_{p-2q-1}, \dots, B_{(p-4q+5)/2}$ nu sînt divizibile prin numărul prim impar p pentru orice q care satisface $2 \leq q \leq [(p+3)/2]$ (teorema 11).

APPROXIMATION TO CERTAIN CLASSES OF REAL IRRATIONAL NUMBERS BY RATIONAL FRACTIONS x/y WITH $(x, y) = 1$ AND $(x, m) = 1$ OR $(y, m) = 1$

BY

N. NEGOESCU

I. Introduction

Let the continued fraction expansion of the irrational number θ be $\theta = [a_0, a_1, \dots, a_n, \dots]$; then the n-th convergent is $p_n/q_n = [a_0, a_1, \dots, a_n]$ and the n-th denominator is q_n. We shall use the following known results:

A. If p_{n-2}/q_{n-2}, p_{n-1}/q_{n-1}, p_n/q_n are three consecutive convergents, then

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2} \text{ and } p_{n-1} q_n - p_n q_{n-1} = (-1)^n .$$

B. If $|\theta - p_n/q_n| = 1/(\lambda_n q_n^2)$, then λ_n is given by

$$\lambda_n = [a_{n+1}, a_{n+2}, \dots] + [0, a_n, a_{n-1}, \dots, a_1] .$$

Also we can write $\lambda_n = \theta_n + 1/\Phi_n$, where $\theta_n = [a_{n+1}, a_{n+2}, \dots]$ and $\Phi_n = [a_n, a_{n-1}, \dots, a_1]$.

If θ is a given irrational number, then it is known that the rational fractions x/y which satisfy the inequality

$$(1) \qquad \left| \theta - \frac{x}{y} \right| < \frac{1}{\xi y^2}$$

are some convergents of θ. Precisely, we have the following three theorems for each irrational θ:

A_1. *Any convergent satisfies* (1) *with* $\xi = 1$.

A_2. *At least one of two consecutive convergents satisfies* (1) *with* $\xi = 2$.

A_3. *At least one of three consecutive convergents satisfies* (1) *with* $\xi = 5^{1/2}$.