(32) pentru care coeficienţii dezvoltării puterilor conduc la următorul rezultat: cel puţin unul din numerele Bernoulli $B_{p-n}, B_{p-2n}, \ldots, B_{(p+n+1)/2}$ nu este divizibil prin numărul prim $p = 4n - 1$, dacă $p \geq p_0(i)$, $i = 1, 2$ (teorema 9); cel puţin unul din numerele lui Bernoulli de mai sus nu este divizibil prin numărul prim $p = 4n + 1$, dacă $p \geq p_0(i)$, $i = 1, 2$ (teorema 10); cel puţin unul din numerele lui Bernoulli $B_{p-2q+1}, B_{p-2q-1}, \ldots, B_{(p+1)/2}$ nu sînt divizibile prin numărul prim impar p pentru orice q care satisface $2 \leq q \leq (p+3)/2$ (teorema 11).

APPROXIMATION TO CERTAIN CLASSES OF REAL IRRATIONAL NUMBERS BY RATIONAL FRACTIONS x/y WITH $(x, y) = 1$ AND $(x, m) = 1$ OR $(y, m) = 1$

BY

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I. Introduction

Let the continued fraction expansion of the irrational number θ be $\theta = [a_0, a_1, \ldots, a_n, \ldots]$; then the n-th convergent is $p_n/q_n = [a_0, a_1, \ldots, a_n]$ and the n-th denominator is a_n. We shall use the following known results:

A. If p_{n-2}/q_{n-2}, p_{n-1}/q_{n-1}, p_n/q_n are three consecutive convergents, then

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2} \text{ and } p_{n-1} q_n - p_n q_{n-1} = (-1)^n .$$

B. If $|\theta - p_n/q_n| = 1/(\lambda_n q_n^2)$, then λ_n is given by

$$\lambda_n = [a_{n+1}, a_{n+2}, \ldots] + [0, a_n, a_{n-1}, \ldots, a_1].$$

Also we can write $\lambda_n = \theta_n + 1/\Phi_n$, where $\theta_n = [a_{n+1}, a_{n+2}, \ldots]$ and $\Phi_n = [a_n, a_{n-1}, \ldots, a_1]$.

If θ is a given irrational number, then it is known that the rational fractions x/y which satisfy the inequality

$$(1) \qquad \left| \theta - \frac{x}{y} \right| < \frac{1}{\xi y^2}$$

are some convergents of θ. Precisely, we have the following three theorems for each irrational θ:

A_1. *Any convergent satisfies* (1) *with* $\xi = 1$.
A_2. *At least one of two consecutive convergents satisfies* (1) *with* $\xi = 2$.
A_3. *At least one of three consecutive convergents satisfies* (1) *with* $\xi = 5^{1/2}$.

These theorems are due to Dirichlet (as it is cited by F o r d [3]; probably before 1850), V a h l e n [14] in 1895 and B o r e l [1] in 1903.

The foregoing theorems have been extended by the following results:

A_1'. *p_{n-1}/q_{n-1} satisfies* (1) *with* $\xi = a_n$.

A_2'. *At least one of convergents p_{n-1}/q_{n-1} and p_n/q_n satisfies* (1) *with* $\xi = a_n + 1/a_n$.

A_3'. *At least one of convergents p_{n-2}/q_{n-2}, p_{n-1}/q_{n-1}, p_n/q_n satisfies* (1) *with* $\xi = (a_n^2 + 4)^{1/2}$.

The first theorem follows from B, and A_2' is a result of F u k a s a w a [4] in 1925, which can be also obtained directly.

Theorem A_3' is a result of M o r i m o t o [5] in 1926, found again by O b r e j k o v [8] (cited by E g g a n [2]), [9] and M. M ü l l e r [6] in 1949 and 1955 respectively. These results are cited in the book of G. S u d a n [12]. Theorem A_3' has been used by E g g a n [2] in 1961; it is also a particular case of our theorems B_1, B_2 [7] in 1961.

Considering the works of Raphael M. R o b i n s o n [11] in 1941 and Leonard T o r n h e i m [13] in 1955 on the approximation of real irrationals by classes of rational numbers, and θ being any irrational, we can state the theorems:

B_1. *At least one of two consecutive convergents satisfies* (1) *with* $\xi = 1$, *and* $(x, p^k) = 1$, *where p is a prime, $k \in N$ (N is the set of the natural numbers).*

B_2. *At least one of four consecutive convergents satisfies* (1) *with* $\xi = 2$, *and* $(x, p^k) = 1$, *where p is a prime* ≥ 2.

B_3. *At least one of six consecutive convergents satisfies* (1) *with* $\xi = 5^{1/2}$, *and* $(x, p^k) = 1$, *where p is an odd prime.*

These theorems show that taking a double number of consecutive convergents in the results A_1, A_2, A_3 and imposing that condition $(x, p^k) = 1$ (p being some prime, $k \in N$) be verified, the approximation is not changed.

The principal purpose of this Note is to show that it is possible to generalise theorems $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$ in the sense in which theorems A_1', A_2', A_3' represent extensions of the results A_1, A_2, A_3.

The Note is concluded with results on approximations x/y which satisfy (1) and conditions $(x, y) = 1$ and $(x, p^k q^l) = 1$, where p and q are distinct primes and $k, l \in N$.

2. Intermediary results

Lemma $\bar{B}_1$. *If $(p_{n-2}, p) = p$, then p_{n-1}/q_{n-1} satisfies* (1) *with* $\xi = a_n$ *and* $(p_{n-1}, p^k) = 1$, *where p is a prime and* $k \in N$.

Proof. If $(p_{n-2}, p) = p$, because $p_{n-2}q_{n-1} - p_{n-1}q_{n-2} = (-1)^{n-1}$, we have $(p_{n-1}, p) = 1$ and p_{n-1}/q_{n-1} satisfies (1) with $\xi = a_n$, by virtue of statement B.

Lemma $\bar{B}_2$. *If $a_{n+1} \geq a_n$, $m = p^k$, $(p_{n-1}, p) = p$, where p is a prime $\geq a_n + 1$ and $k \in N$, then at least one of consecutive convergents p_n/q_n and p_{n+1}/q_{n+1} satisfies* (1) *with* $\xi = a_n + 1/a_n$ *and* $(x, m) = 1$.

Proof. According to $(p_{n-1}, p) = p$ and $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$, we have $(p_n, p) = 1$, because if p divides p_{n-1} and p_n, then p divides $p_{n-1}q_n - p_n q_{n-1}$, which has divisor 1 (contradiction since p is prime > 1).

If $a_{n+1} \geq a_n + 1$, p_n/q_n satisfies

$$\left|\frac{p_n}{q_n} - \theta\right| < \frac{1}{(a_n + 1)\, q_n^2} \leq \frac{1}{\left(a_n + \frac{1}{a_n}\right) q_n^2},$$

because $a_n + 1 \geq a_n + 1/a_n$ for $a_n \geq 1$ and taking in to account statement B.

If $a_{n+1} = a_n$, then $(p_{n+1}, p) = 1$, since p divides p_{n-1} and not $a_{n+1}p_n$. In this case $(a_{n+1} = a_n)$, according to A_2', at least one of two consecutive convergents p_n/q_n and p_{n+1}/q_{n+1} satisfies (1) with $\xi = a_n + 1/a_n$, $(p_n, p^k) = 1$ and $(p_{n+1}, p^k) = 1$.

Lemma $\bar{B}_3$. *If $a_{n+i} \geq a_n$ $(i = 1, 2)$, $m = p^k$, $(p_{n-1}, p) = p$, where p is a prime $> a_n^2 + 1$ and $k \in N$, then at least one of the n-th, $(n+1)$-th and $(n+2)$-th convergents satisfies* (1) *with* $\xi = (a_n^2 + 4)^{1/2}$ *and* $(x, m) = 1$.

Proof. Suppose that $(p_{n-1}, p) = p$. Therefore $(p_n, p) = 1$ since, if p divides p_n, p divides $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$.

If $a_{n+1} \geq a_n + 2$, then p_n/q_n satisfies

$$\left|\frac{p_n}{q_n} - \theta\right| < \frac{1}{(a_n + 2)q_n^2} < \frac{1}{q_n^2\,(a_n^2 + 4)^{1/2}} \quad \text{since}$$

$$\lambda_n = [a_{n+1}, a_{n+2}, \ldots] + [0, a_n, a_{n-1}, \ldots, a_1] > a_{n+1} \geq a_n + 2$$
$$\text{and } a_n + 2 > (a_n^2 + 4)^{1/2}.$$

In this way, inequality (1) with $\xi = (a_n^2 + 4)^{1/2}$ and $(p_n, p) = 1$ is satisfied.

If a_{n+1} is a_n or $a_n + 1$ and $a_{n+2} \geq a_n + 2$, then, since p divides p_{n-1} and not a_{n+1} and p_n, (p being a prime and $p > a_n^2 + 1$) and $p_{n+1} = a_{n+1}p_n + p_{n-1}$, we have inequality

$$\left|\frac{p_{n+1}}{q_{n+1}} - \theta\right| < \frac{1}{(a_n + 2)\, q_{n+1}^2} < \frac{1}{q_{n+1}^2\,(a_n^2 + 4)^{1/2}} \quad \text{with } (p_{n+1}, p) = 1.$$

If $a_{n+1} < a_n + 2$, $a_{n+2} < a_n + 2$, there are four possible cases:

Case 1: $a_{n+1} = a_n + 1$ and $a_{n+2} = a_n + 1$. In this case

$$\zeta_n = [a_n + 1, a_n + 1, \ldots] + [0, a_n, a_{n-1}, \ldots, a_1] > a_n + 1 + \frac{1}{a_n + 2} + \frac{1}{a_n + 1} > (a_n^2 + 4)^{1/2},$$

that is p_n/q_n satisfies (1) with $\xi = (a_n^2 + 4)^{1/2}$ and $(p_n, p) = 1$.

Case 2: $a_{n+1} = a_n + 1$, $a_{n+2} = a_n$, then

$$\zeta_n = [a_n + 1, a_n, \ldots] + [0, a_n, a_{n-1}, \ldots, a_1] > a_n + 1 + \frac{1}{a_n + 1} + \frac{1}{a_n + 1} = a_n + 1 + \frac{2}{a_n + 1} > (a_n^2 + 4)^{1/2},$$

that is p_n/q_n satisfies (1) with $\xi = (a_n^2 + 4)^{1/2}$ and $(p_n, p) = 1$.

Case 3: $a_{n+2} = a_n + 1$, $a_{n+1} = a_n$, then

$$\zeta_{n+1} = [a_{n+2}, a_{n+3}, \ldots] + [0, a_{n+1}, a_n, \ldots, a_1] > a_n + 1 + \frac{1}{a_n + \frac{1}{a_n}} > (a_n^2 + 4)^{1/2}.$$

In this case p_{n+1}/q_{n+1} satisfies (1) with $\xi = (a_n^2 + 4)^{1/2}$ and $(p_{n+1}, p) = 1$.

Case 4: $a_{n+1} = a_n$, $a_{n+2} = a_n$. Then

$$p_{n+1} q_{n+2} - p_{n+2} q_{n+1} = (a_{n+1} a_{n+2} + 1)(p_{n+1} q_n - q_{n+1} p_n) = (-1)^n (a_n^2 + 1).$$

If p divides p_{n+1} and p_{n+2}, then p divides $a_n^2 + 1$, which is a contradiction because p is a prime $> a_n^2 + 1$. Therefore $(p_{n+2}, p) = 1$.

In this case, in view of theorem A_3, at least one of convergents p_n/q_n, p_{n+1}/q_{n+1}, p_{n+2}/q_{n+2} satisfies (1) with $(p_n, p) = 1$, $(p_{n+1}, p) = 1$, $(p_{n+2}, p) = 1$.

We have shown that in the possible cases, at least one of three convergents p_n/q_n, p_{n+1}/q_{n+1}, p_{n+2}/q_{n+2} satisfies (1) with $(x, m) = 1$, where $m = p^k$.

Now we can state symmetric results for y:

Lemma B$_1$. *If $(q_{n-2}, q) = q$, $m = q^k$, where q is a prime and $k \in N$, then $(n-1)$-th convergent satisfies* (1) *with $\xi = a_n$ and $(q_{n-1}, q^k) = 1$.*

Lemma B$_2$. *If $a_{n+1} \geq a_n$, $m = q^k$, $(q_{n+1}, q) = q$, where q is a prime $\geq a_n + 1$ and $k \in N$, then at least one of two consecutive convergents p_n/q_n and p_{n+1}/q_{n+1} satisfies* (1) *with $\xi = a_n + 1/a_n$ and $(y, m) = 1$.*

Lemma B$_3$. *If $a_{n+i} \geq a_n (i = 1, 2)$, $m = q^k$, $(q_{n+1}, q) = q$, where p is a prime $> a_n^2 + 1$ and $k \in N$, then at least one of three convergents p_n/q_n, p_{n+1}/q_{n+1}, p_{n+2}/q_{n+2} satisfies* (1) *with $\xi = (a_n^2 + 4)^{1/2}$ and $(y, m) = 1$.*

3. Principal results

Theorem B$_1$. *If $a_{n-1} \geq a_n$, $m = p^k$, where p is a prime and $k \in N$, then at least one of convergents p_{n-2}/q_{n-2} and p_{n-1}/q_{n-1} satisfies* (1) *with $\xi = a_n$ and $(x, m) = 1$.*

Proof. If $(p_{n-2}, p) = p$, then, according to lemma $\overline{B}_1$, p_{n-1}/q_{n-1} satisfies (1) with $\xi = a_n$ and $(p_{n-1}, p) = 1$.

If $(p_{n-2}, p) = 1$, then

$$\zeta_{n-2} = [a_{n-1}, a_n, \ldots] + [0, a_{n-2}, a_{n-3}, \ldots, a_1] > a_n.$$

Therefore, we have $\left| \frac{p_{n-2}}{q_{n-2}} - \theta \right| < \frac{1}{a_n q_{n-2}^2}$ with $(p_{n-2}, p) = 1$ or $(p_{n-2}, p^k) = 1$.

Theorem $\overline{B}_2$. *If $a_{n+i} \geq a_n (i = 1, 2)$, $m = p^k$, $p \geq a_n + 1$, $k \in N$ and p is a prime, then at least one of four consecutive convergents p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1}, p_{n+2}/q_{n+2} satisfies* (1) *with $\xi = a_n + 1/a_n$ and $(x, m) = 1$.*

Proof. There exist three possible cases:

either $(p_{n-1}, p) = p$, or $(p_n, p) = p$, or $(p_{n-1} p_n, p) = 1$.

In the first two cases, according to lemma B_2, at least one of three convergents p_n/q_n, p_{n+1}/q_{n+1}, p_{n+2}/q_{n+2} satisfies (1) with $\xi = a_n + 1/a_n$ and $(x, p) = 1$ or $(x, p^k) = 1$.

In the last case, $(p_{n-1} p_n, p) = 1$, according to A_2 at least one of two convergents p_{n-1}/q_{n-1}, p_n/q_n satisfies (1) with $\xi = a_n + 1/a_n$ and $(p_{n-1}, p) = 1$, $(p_n, p) = 1$.

In this way, the proof of theorem is complete.

Theorem $\overline{B}_3$. *If $a_{n+i} \geq a_n (i = 1, 2, 3, 4)$, $m = p^k$, $p > a_n^2 + 1$, $k \in N$ and p is a prime, then at least one of the six consecutive convergents p_{n-1}/q_{n-1}, p_n/q_n, $\ldots$, p_{n+4}/q_{n+4} satisfies* (1) *with $\xi = (a_n^2 + 4)^{1/2}$ and $(x, m) = 1$.*

Proof. We have four possibilities:

either $(p_{n-1}, p) = p$, or $(p_n, p) = p$, or $(p_{n+1}, p) = p$, or $(p_n p_{n-1} p_{n+2}, p) = 1$.

In the first three possibilities, by lemma B_1 at least one of the consecutive convergents $p_n/q_n, \ldots, p_{n+1}/q_{n+1}$ satisfies (1) with $\xi = (a_n^2 + 4)^{1/2}$ and $(x, p) = 1$.

In the last case, according to result A_3', at least one of convergents $p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1}$ satisfies (1) with $\xi = (a_n^2 + 4)^{1/2}$ and $(x, p) = 1$ or $(x, p^k) = 1$.

Obviously we can also state symmetric theorems for y:

Theorem $\mathbf{B}_1'$. *If $a_{n-1} \geq a_n$, $m = q^k$, where q is a prime and $k \in N$, then at least one of convergents $p_{n-2}/q_{n-2}, p_{n-1}/q_{n-1}$ satisfies (1) with $\xi = a_n$ and $(y, m) = 1$.*

Theorem $\mathbf{B}_1'$. *If $a_{n+i} \geq a_n$ $(i = 1, 2)$, $m = q^k$, $q \geq a_n + 1$, $k \in N$ and q is a prime, then at least one of four consecutive convergents $p_{n-1}/q_{n-1}, \ldots, p_{n+2}/q_{n+2}$ satisfies (1) with $\xi = a_n + 1/a_n$ and $(y, m) = 1$.*

Theorem $\mathbf{B}_3'$. *If $a_{n+i} \geq a_n$ $(i = 1, 2, 3, 4)$, $m = q^k$, $q > a_n^2 + 1$, $k \in N$ and q is a prime, then at least one of the six consecutive convergents $p_{n-1}/q_{n-1}, p_n/q_n, \ldots, p_{n+4}/q_{n+4}$ satisfies (1) with $\xi = (a_n^2 + 4)^{1/2}$ and $(y, m) = 1$.*

4. Consequences of the theorems $\bar{\mathbf{B}}'$ and $\mathbf{B}'$.

We shall now statesome consequences of the theorems $\bar{\mathbf{B}}_1'$, $\mathbf{B}_2'$, $\bar{\mathbf{B}}_3'$, which will be designed by C_1, $\bar{C}_2$, $\bar{C}_3$ respectively.

C_1'. If the continued fraction expansion of the irrational number θ has infinitely many blocks (b, a) of denominators with $b \geq a$ and $m = p^k$, where p is a prime and $k \in N$, then there exist infinitely many convergents of θ which satisfy (1) with $\xi = a$ and $(x, m) = 1$.

C_2. If the continued fraction expansion of the irrational number θ has infinitely many blocks (b, a, a_1, a_2) of denominators with $a_i \geq a$ $(i = 1, 2)$ and $m = p^k$, where p is a prime $\geq a + 1$ and $k \in N$, then there exist infinitely many convergents of θ which satisfy (1) with $\xi = a + 1/a$ and $(x, m) = 1$.

$\bar{C}_3$. If in the continued fraction expansion of the irrational number θ there exist infinitely many blocks $(b, a, a_1, a_2, a_3, a_4)$ of denominators with $a_i \geq a$ $(i = 1, 2, 3, 4)$ and $m = p^k$, where p is a prime $> a^2 + 1$ and $k \in N$, then there exist infinitely many convergents of θ which satisfy (1) with $\xi = (a^2 + 4)^{1/2}$ and $(x, m) = 1$.

We can also state symmetric consequences C_1, C_2, C_3 in the same words with obvious modifications.

We shall state only one of other obvious consequences:

C_3''. If the continued fraction expansion of the irrational θ has $a_n \geq k$ for $n \geq n_0$ and there exist infinitely many $a_n = k$ and $m = p^l$, where p is a prime $> k^2 + 1$ and $l \in N$, then there exist infinitely many convergents of θ which satisfy (1) with $\xi = (k^2 + 4)^{1/2}$ and $(x, m) = 1$.

In case $k = 1$, we obtain the results of Leonard Tornheim [13].

5. A theorem of Vahlen — type for the interconvergents of θ

Theorem D. *At least one of the interconvergents $(p_n - p_{n-1})/(q_n - q_{n-1})$ and $(p_n + p_{n-1})/(q_n + q_{n-1})$ satisfies (1) with $\xi = [(a_n + 1)^2 + 1]/[(a_n + 1)^2 - 1]$.*

Proof. Let n be an odd number, then

$$\frac{p_n - p_{n-1}}{q_n - q_{n-1}} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} = \frac{2(p_n q_{n-1} - q_n p_{n-1})}{(q_n - q_{n-1})(q_n + q_{n-1})} = \frac{2}{(q_n - q_{n-1})(q_n + q_{n-1})},$$

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \theta = \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{\theta_n p_n + p_{n-1}}{\theta_n q_n + q_{n-1}} = \frac{1 - \theta_n}{(q_n + q_{n-1})(\theta_n q_n + q_{n-1})} < 0$$

$$\text{and} \quad \frac{p_n - p_{n-1}}{q_n - q_{n-1}} - \theta = \frac{1 + \theta_n}{(q_n - q_{n-1})(\theta_n q_n - q_{n-1})} > 0.$$

It follows

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} < \theta < \frac{p_n - p_{n-1}}{q_n - q_{n-1}}.$$

Suppose that

$$(2) \quad \begin{cases} \theta - \dfrac{p_n + p_{n-1}}{q_n + q_{n-1}} < \dfrac{1}{\xi (q_n + q_{n-1})^2}, \\[2ex] \dfrac{p_n - p_{n-1}}{q_n - q_{n-1}} - \theta < \dfrac{1}{\xi (q_n - q_{n-1})^2}. \end{cases}$$

By adding these inequalities we have

$$\frac{2}{(q_n - q_{n-1})(q_n + q_{n-1})} < \frac{1}{\xi}\left[\frac{1}{(q_n + q_{n-1})^2} + \frac{1}{(q_n - q_{n-1})^2}\right],$$

that is $\xi (q_n^2 - q_{n-1}^2)^2 \geq q_n^2 + q_{n-1}^2$ or $q_n/q_{n-1} \geq \sqrt{(\xi + 1)/(\xi - 1)}$.

For $\xi = \dfrac{(a_n + 1)^2 + 1}{(a_n + 1)^2 - 1}$ we have $\sqrt{\dfrac{\xi + 1}{\xi - 1}} = 1 + a_n$ or $\dfrac{q_n}{q_{n-1}} \geq a_n + 1$.

This is a contradiction since $q_n/q_{n-1} = [a_n, a_{n-1}, \ldots, a_1] < a_n + 1$. This contradiction proves completly the theorem for n odd because the inequalities (2) are not true simultaneously. In the same way, we can prove the theorem for n even.

Remarks. Alexander Oppenheim [10] in 1941 proved the theorem D only for $\xi = 1$. Here, we have a new precision of Oppenheim's result, because ξ is a function of a_n. Oppenheim's result gives an universal constant for all a_n.

This new result, in the cases

$$a_n = 1, \quad \xi = \frac{4+1}{4-1} = \frac{5}{3} > 1,$$

$$a_n = 2, \quad \xi = \frac{5}{4} > 1, \quad \text{etc.}$$

gives us some improvements of Oppenheim's result.

6. Approximations x/y with the conditions $(x, y) = 1$ and $(x, p^k q^l) = 1$

Theorem D'. *If $a_{n+1} \geq a_n$, $m = p^k q^l$ (p and q distinct primes, $k, l \in N$) and either $a_n \geq 2$ or $a_n = 1$ and a_{n+1}, a_{n-1} are 1 or 2, then at least one of the fractions p_{n-1}/q_{n-1}, p_n/q_n, $(p_n + p_{n-1})/(q_n + q_{n-1})$, $(p_n - p_{n-1})/(q_n - q_{n-1})$ satisfies (1) with $\xi = [(a_n + 1)^2 + 1]/[(a_n + 1)^2 - 1]$ and $(x, m) = 1$.*

Proof. In the case $a_n \geq 2$, if one of the convergents p_{n-1}/q_{n-1}, p_n/q_n has the numerator prime with m, then

$$\lambda_{n-1} = [a_n, a_{n+1}, \ldots] + [0, a_{n-1}, \ldots, a_1] > a_n \geq 2 > \frac{(a_n+1)^2+1}{(a_n+1)^2-1},$$

$$\lambda_n = [a_{n+1}, a_{n+2}, \ldots] + [0, a_n, a_{n-1}, \ldots, a_1] > a_n \geq 2 > \frac{(a_n+1)^2+1}{(a_n+1)^2-1} \quad \text{and}$$

that convergent satisfies (1) with $\xi = \dfrac{(a_n+1)^2+1}{(a_n+1)^2-1}$ and $(x, m) = 1$.

Otherwise p_{n-1} is divisible by p or q. Suppose that p_{n-1} is divisible by p, then, because $(p_{n-1}, p_n) = 1$ according to A, it follows that p_n is divisible by q. In this case the interconvergents $(p_n + p_{n-1})/(q_n + q_{n-1})$ and $(p_n - p_{n-1})/(q_n - q_{n-1})$ have numerators prime with m since $(p_{n-1}, m) = p$ and $(p_n, m) = q$. In view of theorem D, at least one of the interconvergents $(p_n + p_{n-1})/(q_n + q_{n-1})$ and $(p_n - p_{n-1})/(q_n - q_{n-1})$ satisfies (1) with $\xi = [(a_n + 1)^2 + 1]/[(a_n + 1)^2 - 1]$ and $(x, m) = 1$.

In the case $a_n = 1$, if $(p_{n-1}, m) = 1$, then

either $\lambda_{n-1} = [1, 1, \ldots] + [0, 1, \ldots] > 1 + \frac{1}{3} + \frac{1}{2}$,

or $\lambda_{n-1} = [1, 1, \ldots] + [0, 2, \ldots] > 1 + \frac{1}{2} + \frac{1}{3} = 1\frac{5}{6}$,

or $\lambda_{n-1} = [1, 2, \ldots] + [0, 1, \ldots] > 1 + \frac{1}{3} + \frac{1}{2} = 1\frac{5}{6}$,

or $\lambda_{n-1} = [1, 2, \ldots] + [0, 2, \ldots] > 1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$.

In this case p_{n-1}/q_{n-1} satisfies (1) with $\xi = 5/3$.

The proof can be continued as above.

This theorem can be stated also for rational fractions x/y which satisfy the condition $(y, m) = 1$.

One can also state consequences of type C' and C''.

Remarks. 1° The condition $(x, p^k q^l) = 1$ is equivalent with the two conditions $(x, p^k) = 1$ and $(x, q^l) = 1$.

2° All the aproximations x/y being convergents of the number θ, it follows that condition $(x, y) = 1$ is also satisfied.

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APROXIMAREA UNOR CLASE DE NUMERE IRAȚIONALE REALE PRIN FRACȚII RAȚIONALE x/y CU $(x, y) = 1$ ȘI $(x, n) = 1$ SAU $(x, m) = 1$

Rezumat

În paragrafele 2–4 se modifică teoremele $\overline{\mathbf{B}}_1$, $\overline{\mathbf{B}}_2$, $\overline{\mathbf{B}}_3$, enunțate în introducere, în sensul ca acestea să poată fi enunțate pentru anumite clase de numere iraționale reale cărora li se impun unele condiții pentru cîturile parțiale a_n din dezvoltarea lor în fracție continuă regulată. Teoremele principale obținute aici, sînt $\overline{\mathbf{B}}_1$, $\mathbf{B}_2$, $\overline{\mathbf{B}}_3'$ și $\underline{\mathbf{B}}_1$, $\underline{\mathbf{B}}_2'$, $\underline{\mathbf{B}}_3'$. Ele constituie rezultatele asupra aproximării unor clase de iraționale prin fracții raționale, satisfăcînd condiții de forma $(x, y) = 1$ și $(x, p^k) = 1$, unde p este un număr prim și k un număr natural. În cazul cînd $a_n \geqslant 1$ (condiție satisfăcută de orice irațional real) obținem rezultatele lui Leonard Tornheim din 1955.

În paragraful 5 se obține o teoremă de tip Vahlen pentru aproximarea numerelor iraționale reale cu întreredușe, teoremă ce constituie o îmbunătățire a unui rezultat cunoscut al lui Alexander Oppenheim. În ultimul paragraf se aplică această teoremă la studiul aproximării numerelor iraționale reale prin fracții raționale x/y satisfăcînd condițiile $(x, y) = 1$ și $(x, p^h q^l) = 1$, unde p și q sînt numere prime distincte și k și l numere naturale.

SUR QUELQUES PROBLÈMES DE QUASI-COMPACITÉ

PAR

OLGA COSTINESCU

Dans le cadre de l'étude des espaces quasi-compacts, on peut poser différents problèmes qui réclament, à bon droit, l'attention des mathématiciens (voir par ex. [2]).

Dans la Note présente, nous allons nous occuper de certains problèmes auxquels nous avons fait allusion plus haut.

La première partie contient un théorème de quasi-compactification d'un espace topologique quelconque (c'est le résultat principal), tandis que dans la seconde partie on établit certaines conditions suffisantes de quasi-compacité.

§ 1. Afin de formuler les résultats annoncés, il est nécessaire de préciser d'abord le sens des notions qu'on va utiliser :

Définition 1. *Un espace topologique (X, τ), s'appelle quasi-compact si tout filtre $\mathcal{F}$ sur X admet un point adhérent (c'est-à-dire $\bigcap_{M \in \mathcal{F}} M^{\tau} \neq \varnothing$, M^{τ} signifie l'adhérence de M par rapport à la topologie τ).*

On dit qu'une partie A de (X, τ) este un ensemble quasi-compact si l'espace (A, τ_A) est quasi-compact.

Définition 2. *Un espace (X, τ) est un T_2-espace si tout filtre sur X peut avoir au plus un point limite.*

Définition 3. *Un espace topologique (X, τ) est localement quasi-compact si (X, τ) est T_2-espace, et tout point $x \in X$ admet un voisinage quasi-compact.*

Définition 4. *Un espace topologique (X, τ) est compact s'il est quasi-compact et T_2-espace.*