ECUAŢII CU DERIVATE FRÉCHET

Se studiază ecuaţiile (1.1) şi (1.3), unde funcţia f este derivabilă Fréchet numai în raport cu variabila y şi funcţiile f şi $\partial f/\partial y$ satisfac condiţia de completă integrabilitate (4.2). Existenţa şi unicitatea soluţiei ecuaţiei (1.1) şi dependenţa acestei soluţii de parametru se deduce după aceeaşi schemă, expusă în teorema 2.1. Pentru ecuaţia (3.1) se indică mulţimea pe care există o soluţie unică.

ON THE APPROXIMATE SOLUTION OF NON-LINEAR FUNCTIONAL EQUATIONS. II

BY

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In the paper [2] using the divided difference of a non-linear functional $F(x)$ defined in a Banach space two methods for solving the equation

$$F(x) = 0 \tag{1}$$

are given. The advantage of these methods in comparison with Sergejeff's method [4] for the solving of operational equations, consists in the fact that, in the methods of the paper [2], the existence and the boundedness of the inverse of the divided difference are not assumed. The convergence of the above mentioned methods is proved under the conditions of the existence and boundedness of the divided difference of $F(x)$.

The present paper shows: a) the existence of a divided difference for functionals defined in Banach spaces, and b) it proves a Theorem concerning the existence of the solution of the equation (1) under conditions, analogical to those of the theorem given in paper [4].

1. We shall denote with $F_{x',x''}$ [3] the divided difference of the functional $F(x)$ in the points $x', x'' \in X$.

Theorem 1. *For every $(x', x'') \in X \times X$, there exists a linear functional $F_{x',x''} \in X^*$ with the properties*

$$F_{x',x''}(x' - x'') = F(x') - F(x'') \tag{a}$$

$$\|F_{x',x''}\| = \frac{|F(x') - F(x'')|}{\|x' - x''\|}. \tag{b}$$

Proof. For fixed $(x', x'') \in X \times X$ and given F, let us define

$$f_{x',x''}[r(x' - x'')]' = rF(x') - rF(x''),$$

for every real number r. Hence we defined $f_{x',x''} \in L^*$, where

$$L = \{r(x' - x'') ; r \in R\}.$$

We have also

$$\|f_{x',x''}\| = \sup_{\|z\|=1} |f_{x',x''}(z)| = \left| f_{x',x''} \frac{x'-x''}{\|x'-x''\|} \right| = \frac{|F(x') - F(x'')|}{\|x'-x''\|}.$$

From the extension of the Theorem of Hahn-Banach it results the existence of the functional $F_{x',x''}$ with the properties:

$$F_{x',x''}(z) = f_{x',x''}(z)$$

for every $z \in L$, whence it results (a), and

$$\|F_{x',x''}\| = \|f_{x',x''}\|,$$

whence it results (b).

Theorem 2. *The functional $F(x)$ defined in X has a uniformly bounded divided difference $F_{x',x''}$ for every $(x', x'') \in X \times X$* (i.e. $\|F_{x',x''}\| \leq K$), *if and only if F satisfies the following Lipschitz condition:*

$$|F(x') - F(x'')| \leq K\|x' - x''\|. \tag{2}$$

Proof. Let us suppose that $F(x)$ satisfies the condition (2). Then from Theorem 1, it results the existence of a divided difference $F_{x',x''}$ with

$$\|F_{x',x''}\| = \frac{|F(x') - F(x'')|}{\|x' - x''\|}$$

and by the (2) we get $\|F_{x',x''}\| \leq K$.

Now let $F_{x',x''}$ be a uniformly bounded divided difference, i.e. $\|F_{x',x''}\| \leq K$. Then by the definition of the divided difference we get

$$|F(x') - F(x'')| = |F_{x',x''}(x' - x'')| \leq K\|x' - x''\|.$$

Remark. If the functional $F(x)$ satisfies a Lipschitz condition with the constant factor K', using the divided difference defined by Theorem 1, renouncing to the condition 3° and replacing the condition 4° by $B(K' + \|F_{x_0,x_0'}\|) < 1$ the conclusions of Theorem 1 of paper [2] are valid.

2. Let us suppose the existence of a sequence of points $y_n \in X$ satisfying the following two conditions:

$$\|y_n\| = 1, \ |F_{x_n,x_{n-1}}(y_n)| = \|F_{x_n,x_{n-1}}\| \quad (n = 0, 1, 2, \ldots).$$

We remark that in the case of the divided difference defined by Theorem 1, the sequence (y_n) may be given by the formula

$$y_n = \frac{x_n - x_{n-1}}{\|x_n - x_{n-1}\|}.$$

The approximate method, used for the solving of the equation (1) is given by the formula

$$x_{n+1} = x_n - \frac{F(x_n)}{F_{x_n,x_{n-1}}(y_n)} y_n \quad (n = 0, 1, \ldots), \tag{3}$$

where x_{-1}, x_0 are two initial approximations.

Further on we assume the functional $F(x)$ as being continuous in the sphere $S(x_0, r)$ specified later on.

Theorem 3. *We suppose that the following conditions are satisfied:*

1° *For the divided difference of the functional $F(x)$ in the points x_0, x_{-1} we have*

$$\frac{1}{\|F_{x_0,x_{-1}}\|} \leq B_0;$$

2° *For the approximations x_0, x_{-1} the estimations*

$$\|x_1 - x_0\| \leq \frac{|F(x_0)|}{\|F_{x_0,x_{-1}}\|} \leq \eta_0, \ \|x_0 - x_{-1}\| \leq \eta_0' \ (\eta_0 \leq \eta_0')$$

are known;

3° *The divided difference of the second order of the functional $F(x)$ is uniformly bounded in the sphere $S(x_0, r)$, where* $r = \max\{\eta_0', (3/2)\eta_0\}$:

$$\|F_{x',x'',x'''}\| \leq K;$$

4° *The constant numbers B_0, η_0, η_0', K satisfy the inequality*

$$h_0 = B_0 K(\eta_0 + \eta_0') < \frac{1}{4}.$$

Then the equation (1) has at least a solution $x^* \in S(x_0, r)$ which is the limit of the sequence of the approximations (x_n), and the rapidity of the convergence is characterized by the inequality

$$\|x^* - x_n\| \leq \frac{1}{2^{s_n-1}} q^{s_n-1} (4h_0)^{s_n} \eta_0.$$

Proof. According to formula (3) it results immediately that

$$F_{x_n,x_{n-1}}(x_n - x_{n+1}) = F(x_n), \tag{4}$$

whence we obtain the identity

$$F_{x_n,x_{n-1}}(x_n - x_{n+1}) = F(x_n) - F(x_{n-1}) - F_{x_{n-1},x_{n-2}}(x_n - x_{n-1}). \tag{5}$$

Using the relations (4) and (5) by formula (3) we have

$$\|x_{n+1} - x_n\| \leqslant \frac{1}{\|F_{x_n, x_{n-1}}\|} \, |F(x_n) - F(x_{n-1}) - F_{x_{n-1}, x_{n-2}}(x_n - x_{n-1})|$$

from where, applying Newton's formula and using condition 3° we obtain

$$\|x_{n+1} - x_n\| \leqslant \frac{K}{\|F_{x_n, x_{n-1}}\|} \|x_n - x_{n-1}\| \, \|x_n - x_{n-2}\|. \tag{6}$$

By the definition of the divided differences and from the conditions of this Theorem, we have

$$\|F_{x_1, x_0}\| \geqslant \frac{1}{\|F_{x_0, x_{-1}}\|}\left(1 - \frac{\|F_{x_1, x_0} - F_{x_0, x_{-1}}\|}{\|F_{x_0, x_{-1}}\|}\right) \geqslant \frac{1 - h_0}{B_0},$$

whence it results

$$\frac{1}{\|F_{x_1, x_0}\|} \leqslant \frac{B}{1 - h_0} = B_1.$$

In (6), for $n = 1$ we obtain

$$\|x_2 - x_1\| \leqslant \frac{B_0 K}{1 - h_0} (\eta_0 + \eta_0') \, \eta_0 = \frac{h_0 \eta_0}{1 - h_0} = \eta_1 < \frac{\eta_0}{3}.$$

We have

$$h_1 = B_1 K (\eta_1 + \eta_0) \leqslant \frac{h_0}{2(1 - h_0)}\left(1 + \frac{h_0}{1 - h_0}\right) \leqslant \frac{8}{9} h_0 = q h_0 < \frac{1}{4}.$$

Hence the conditions of the Theorem are satisfied for the points x_1, x_0

By induction we obtain

$$\frac{1}{\|F_{x_n, x_{n-1}}\|} \leqslant B = \frac{B_{n-1}}{1 - h_{n-1}}, \tag{7}$$

$$\|x_{n+1} - x_n\| \leqslant \frac{h_{n-1} \eta_{n-1}}{1 - h_{n-1}} = \eta_n < \frac{\eta_0}{3^n}, \tag{8}$$

$$h_n = B_n K (\eta_n + \eta_{n-1}) \leqslant \frac{h_{n-1} h_{n-2}}{(1 - h_{n-1})^2} < \frac{1}{4}. \tag{9}$$

From the relations (8) and (9) it results

$$h_n \leqslant \frac{1}{2} q^{u_n} (2h_0)^{u_{n+1}}, \qquad \eta_n \leqslant \left(\frac{2}{3}\right)^n q^{s_{n-1}} (2h_0)^{s_n} \eta_0, \tag{10}$$

where the sequence (u_n) is defined by the formula $u_{n+1} = u_n + u_{n-1}$, $u_1 = 1$, $u_2 = 1$, and $s_n = \sum_{i=1}^{m} u_i$.

On the basis of the formula (10) we obtain

$$\|x_{n+p} - x_n\| \leqslant \frac{1}{2^{s_n - 1}} q^{s_n - 1} (4h_0)^{s_n} \eta_0,$$

from where using the completeness of the space X it results that the sequence (x_n) has a limit $x^* \in X$. From (10) we also obtain the inequality characterizing the rapidity of the convergence.

By (8) it results that the terms of the sequence (x_n) as well as x^* belong to the sphere $S(x_0, r)$. Indeed we have

$$\|x_n - x_0\| \leqslant \|x_0 - x_1\| + \|x_1 - x_2\| + \ldots + \|x_{n-1} - x_n\| \leqslant$$

$$\leqslant \eta_0 + \frac{\eta_0}{3} + \ldots + \frac{\eta_0}{3^{n-1}} < \frac{3}{2} \eta_0.$$

The fact that the point x^* is the solution of the equation (1) is shown by (4) for $n \to \infty$, using the boundedness of the operator $F_{x_n, x_{n-1}}$ [2].

Remark. If the functional $F(x)$ has derivatives of the first and second order, a Theorem analogical with Theorem 3 was proved by M. Altman [1].

The authors wish to express their acknowledgement to their colleague H. Wiesler for some useful discussions on the subject.

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DESPRE SOLUŢII APROXIMATIVE ALE UNOR ECUAŢII FUNCŢIONALE

Rezumat

După ce se examinează şi se rezolvă problema existenţei şi mărginirii uniforme a diferenţelor divizate în cazul funcţionalelor, lucrarea prezintă o metodă aproximativă (3) de rezolvare a ecuaţiilor (1). Condiţiile în care se studiază convergenţa metodei şi existenţa soluţiei sînt analoge cu cele date în [4].