reale positivo $a > 0$, per la 3.24, esiste un $\varepsilon_a > 0$ tale che per ogni $t > \varepsilon_a$ si ha $g(t) < \mu_a$, e quindi $|\sigma(t)| < a$. Da cui si ha che

$$\lim_{t \to +\infty} \sigma(t) = 0, \tag{3.25}$$

ed il teorema è così dimostrato.


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ASUPRA COMPORTĂRII ASIMPTOTICE A UNEI ECUAŢII INTEGRALE NELINIARE

Rezumat

Se studiază ecuaţia (1.1) pe semiaxa $t \geq 0$, stabilindu-se faptul că $\lim \sigma(t) = 0$ pentru $t \to \infty$, unde $\sigma(t)$ este o soluţie mărginită iar $f(t), k(t)$ şi $\varphi(\sigma)$ satisfac condiţiile (1.2), (1.3), (1.5)—(1.7) din text.

Autorul foloseşte metoda lui V. M. Popov din teoria stabilităţii sistemelor de reglaj automat.


# MÖBIUS TRANSFORMATIONS AND A CLASS OF RICCATI EQUATIONS


BY

E. R. SURYANARAYAN and J. KANE


## 1. Introduction

In this paper we shall give an algebraic structure of a class of Riccati equations of the type

$$\frac{dy}{dx} = A + By + Cy^2, \tag{A}$$

where $A$, $B$, $C$ are functions of $x$. In order to study the structure, we shall first study some of the algebraic properties of the linear homogeneous second order equation

$$\frac{d^2y}{dx^2} + k^2 y = 0, \tag{B}$$

where $k$ is an integrable function of $x$ over the interval $[x_0, x]$. Over a partition $P$ of $[x_0, x]$ a sequence of step functions is defined over the subintervals of $P$ which converge uniformly to $k^2(x)$ as the norm of $P$ approaches zero. The solution of the differential equation over each subinterval with constant $k^2$ gives an infinitesimal Möbius transformation and the product of these transformations in limit leads to the solution of the Riccati equation corresponding to $B$, namely

$$\frac{dy}{dx} = -(y^2 + k^2).$$

We shall use this idea to generalize the result obtained by Davis ([1], p. 11).

## 2. Basic Equations

The second order linear differential equation

(1) $$\frac{d^2}{dt^2}V(t) + A(t)\frac{d}{dt}V(t) + B(t)V(t) = 0,$$

following a change of the unknown

(2) $$U(t) = V(t)\exp\left\{\frac{1}{2}\int^t A(\tau)\,d\tau\right\}$$

and an introduction of

(3) $$k^2(t) = B(t) - \frac{1}{2}\frac{d}{dt}A(t) - \frac{1}{4}A^2(t),$$

can be recast into the canonical form

(4) $$\frac{d^2u}{dx^2} + k^2(x)u = 0.$$

At some point $x = x_0$, we may suppose that we have a given constraint

(5) $$\frac{d}{dt}\ln u = \frac{1}{u}\frac{du}{dx} = g_0, \quad x = x_0,$$

which singles out one solution; another linearly independent solution can be specified by choosing a different value for the parameter $g_0$. Another change of the unknown

(6) $$\int_{x_0}^{x}\Gamma(\xi)d\xi = \ln u(x), \quad u(x) = e^{\int_{x_0}^{x}\Gamma(\xi)d\xi},$$

that is,

(7) $$\Gamma(x) = \frac{1}{u}\frac{du}{dx},$$

will be more useful in our work. The continuity of $\Gamma(x)$ is assured if we restrict our attention to continuous solutions of $u(x)$ with a continuous first derivative. If these functions are bounded, then we note that the poles and zeros of $\Gamma(x)$ correspond to the zeros and extrema respectively of $u(x)$. In terms of $\Gamma(x)$ equations (4) and (5) become

(8) $$\frac{d\Gamma}{dx} = -(k^2 + \Gamma^2), \quad x_0 \leq x \leq x_m,$$



(9) $$\Gamma = g_0, \qquad x = x_0,$$

wherein we have adopted the convention that the initial condition is given at $x = x_0$, and we wish to propagate the solution to the right of some arbitrary point $x_m$. Formula (8) describes $\Gamma(x)$ in terms of a Riccati equation.

## 3. Infinitesimal Transformation and Solution

In this section we shall obtain an expression for $\Gamma(x)$ in terms of a product of $n$ $2\times 2$ unimodular matrices for an $n$-th order of quantization. In the limit as $n \to \infty$, each matrix operator becomes an infinitesimal transformation. In order to accomplish this, we shall approximate the function $k^2(x)$ by step functions and solve the differential equation (4) over each subinterval on which $k(x)$ is constant and express $\Gamma$ as product of infinitesimal transformations.

We assume that $g_0$ and $k^2(x)$ are real so that we confine our attention to real solutions of (4). Once the real solutions are known, complex solutions can easily be constructed by the superposition principle.

Let $P = \{x_0, x_1, \ldots, x_m\}$ be a partition which divides a portion of the $x$-axis say the interval

$$-\infty < x_0 \leq x \leq x_m + \infty.$$

Let

$$\Delta x_i = x_i - x_{i-1}.$$

Define the sequence of step function

(10) $$\Phi_r(x) = k_r^2 = k^2(t_r) \qquad x_{r-1} \leq t_r \leq x_r$$

and

$$\lim_{n\to\infty}\Phi_n(x) = k^2(x),$$

where $n \to \infty$ such that the norm of the set $\{\Delta x_k\}$ approaches zero.

The differential equation (4) in the interval $[x_{i-1}, x_i]$, replacing $k^2(x)$ by $k_i^2$, now becomes

(11) $$\frac{d^2u_i}{dx^2} + k_i^2u_i = 0, \qquad x_{i-1} \leq x \leq x_i$$

and its solution is

$$u_i = A_i\left[\Gamma_i k_i^{-1}\sin k_i(x - x_{i-1}) + \cos k_i(x - x_{i-1})\right]$$

wherein the discrete parameters $A_i$ and $\Gamma_i$ are

(12) $$A_i = u_i, \quad \Gamma_i = \frac{du_i/dx}{u_i}, \qquad x = x_{i-1}.$$

$A_i$ and $\Gamma_i$ can be identified as the amplitude and logarithmic derivative of the solutions at the left end point. Even though $k_i$ is a jump function this initial data can be continued as a smooth function for all $x$, provided we choose

$$\Gamma_{i+1} = \frac{\Gamma_i \cos k_i \Delta x_i - k_i \sin k_i \Delta x_i}{\Gamma_i k_i^{-1} \sin k_i \Delta x_i + \cos k_i \Delta x_i} \tag{13}$$

and

$$A_{i+1} = A_i [\Gamma_i k_i^{-1} \sin k_i \Delta x_i + \cos k_i \Delta x_i]. \tag{14}$$

In this manner we obtain a function which satisfies (4) everywhere except that $k^2(x)$ must be replaced by $k_i^2$. Furthermore, the iterations required to extend $u_i(x)$ are very easily performed, since (13) has the form of Möbius transformation whose coefficient matrix

$$M_i = \begin{bmatrix} \cos k_i \Delta x_i & -k_i \sin \Delta x_i \\ k_i^{-1} \sin k_i \Delta x_i & \cos k_i \Delta x_i \end{bmatrix} \tag{15}$$

has its determinant equal to unity. As a result the coefficient matrix $T_m$ required to transform $T_0$ into some final value $\Gamma_m$ at the arbitrary point $x = x_m$ is the product of $m$ factors

$$T_m = M_m M_{m-1} \ldots M_1 \tag{16}$$

and is likewise unimodular.

It is now a simple calculation to show that as $m \to \infty$ in a manner that the norm of the set $\{\Delta x_m\}$ approaches 0, then the process converges to the differential equation. Since $\Gamma(x)$ defines $u(x)$ uniquely except for a multiplicative constant, it will be sufficient to show that the continuous limit of $\Gamma_i$'s defined in (13) satisfies the Riccati equation (8) on any interval $\Delta x_i$. We find from (13) that

$$\frac{\Gamma_{i+1} - \Gamma_i}{\Delta x_i} = \frac{k_i \sin k_i \Delta x_i - (\Gamma_i^2 k_i^{-1}) \sin k_i \Delta x_i}{\Delta x_i [(\Gamma_i k_i^{-1}) \sin k_i \Delta x_i + \cos k_i \Delta x_i]}; \tag{17}$$

as the norm of $P$ approaches zero, we obtain

$$\frac{d\Gamma}{dx} = -(k^2 + \Gamma^2), \tag{18}$$

which is precisely the equation (10). Also we find from (14) that

$$\lim_{\Delta x_i \to 0} \frac{A_{i+1} - A_i}{\Delta x_i} = \frac{du}{dx} = u\Gamma \tag{19}$$

which checks with (12).



The preceding analysis furnishes a simple illustration of the algebraic structure of a particular Riccati equation: one example of the general theory of such equations which shows that its solution can be characterized by Matrix operators. In the limit, each Matrix $M_i$ in the product (16) approaches an infinitesimal transformation, and the total transformation $T_m$ giving $\Gamma(x_m)$ can be expressed either as a Matrizant [2] or as a product integration [3]: the finite result of compounding sequence of infinitesimal transformations.

By direct multiplication, it is easily checked that each $M_k$ can be reexpressed as the product of three matrices

$$M_i = C_i^{-1} R_i C_i, \tag{20}$$

where

$$C_i = \begin{pmatrix} k_i^{1/2} & 0 \\ 0 & k_i^{-1/2} \end{pmatrix}, \quad R_i = \begin{pmatrix} \cos k_i \Delta x_i & -\sin k_i \Delta x_i \\ \sin k_i \Delta x_i & \cos k_i \Delta x_i \end{pmatrix}. \tag{21}$$

The geometric meaning of $M_i$ is discussed in [4].

## 4. Solutions of a Class of Riccati Equations

In this section we shall obtain explicit solutions of a class of Riccati equations by use of the analysis of the preceding section. From the results (13) and (15) we find that for any function $\mu(x) \neq 0$ in $[x_0, x]$ the matrices $M_i$ and $\mu_i M_i$ represent the same Möbius transformation. And in this context we shall write

$$\mu_i M_i = M_i. \tag{22}$$

Let us consider the infinitesimal transformation

$$\Gamma_{i+1} = \frac{\Gamma_i \cos k_i \Delta x_i - \sin k_i \Delta x_i}{\Gamma_i \sin k_i \Delta x_i + \cos k_i \Delta x_i}$$

for an integrable function $k(x)$ in $[x_0, x]$. Forming

$$\lim_{\Delta x_i \to 0} \{(\Gamma_{i+1} - \Gamma_i)/\Delta x_i\},$$

we find that (23) simplifies to the differential equation

$$\frac{d\Gamma}{dx} = -k(x)[1 + \Gamma^2]. \tag{24}$$

And the corresponding matrix of the Möbius transformation is the rotation matrix

$$M_i = \begin{pmatrix} \cos k_i \Delta x_i & -\sin k_i \Delta x_i \\ \sin k_i \Delta x_i & \cos k_i \Delta x_i \end{pmatrix}.$$

The limit of the successive transformations is given by the limit of the product matrices

$$M = \lim_{m\to\infty} \prod_{i=1}^{m-1} M_{m-1} = \begin{bmatrix} \cos I & -\sin I \\ \sin I & \cos I \end{bmatrix}$$

where $I = \int_{x_0}^{x} k(t)dt$. Hence the solution of (24) is

$$\Gamma(x) = \frac{\Gamma_0 \cos I - \sin I}{\Gamma_0 \sin I + \cos I}.$$

The above method is capable of generalization. For this, let us consider the functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and $\delta(x)$ continuous on $[x_0, x]$ and differentiable in $(x_0, x)$. Let us assume that the determinant

$$\begin{vmatrix} \alpha(x) & \gamma(x) \\ \beta(x) & \delta(x) \end{vmatrix} \neq 0 \tag{25}$$

over $[x_0, x]$. For an integrable function $k(x)$ on $[x_0, x]$, we consider the matrix

$$M_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} \cos k_i \Delta x_i & -\sin k_i \Delta x_i \\ \sin k_i \Delta x_i & \cos k_i \Delta x_i \end{pmatrix} \begin{pmatrix} \delta_{i-1} & -\beta_{i-1} \\ -\gamma_{i-1} & \alpha_{i-1} \end{pmatrix}. \tag{26}$$

The infinitesimal fractional transformation corresponding to the matrix (26) is

$$y_{i+1} = \frac{L_1 y_i + L_2}{L_3 y_i + L_4}, \tag{27}$$

where

$$L_1 = (\alpha_i \delta_{i-1} - \beta_i \delta_{i-1}) \cos k_i \Delta x_i + (\beta_i \delta_{i-1} + \alpha_i \gamma_{i-1}) \sin k_i \Delta x_i,$$
$$L_2 = (\beta_i \alpha_{i-1} - \alpha_i \beta_{i-1}) \cos k_i \Delta x_i - (\alpha_i \alpha_{i-1} + \beta_i \beta_{i-1}) \sin k_i \Delta x_i,$$
$$L_3 = (\gamma_i \delta_{i-1} - \delta_i \gamma_{i-1}) \cos k_i \Delta x_i + (\delta_i \delta_{i-1} + \gamma_i \gamma_{i-1}) \sin k_i \Delta x_i,$$
$$L_4 = (\delta_i \alpha_{i-1} - \gamma_i \beta_{i-1}) \cos k_i \Delta x_i - (\delta_i \beta_{i-1} + \gamma_i \alpha_{i-1}) \sin k_i \Delta x_i.$$

Forming the limit of $(y_{i+1} - y_i)/\Delta x_i$ as $\Delta x_i$ approaches zero, we find, after some simplification, that

$$y' = A + By + Cy^2, \tag{28}$$

where

$$(\alpha\delta - \gamma\beta)A = \alpha\beta' - \beta\alpha' - k(\alpha^2 + \beta^2),$$
$$(\alpha\delta - \gamma\beta)B = \beta\gamma' - \gamma\beta' - \alpha\delta' + \delta\alpha' + 2k(\alpha\gamma + \beta\delta),$$
$$(\alpha\delta - \gamma\beta)C = \delta\gamma' - \gamma\delta' - k(\gamma^2 + \delta^2).$$



We find also that the corresponding limit of product matrices, by use of (22) is

$$\lim_{m\to\infty} \prod_{i=1}^{m} M_{m-i} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \cos I & -\sin I \\ \sin I & \cos I \end{pmatrix} \begin{pmatrix} \delta_0 & -\beta_0 \\ -\gamma_0 & \alpha_0 \end{pmatrix},$$

and the Möbius transformation is

$$y(x) = \tag{29}$$

$$= \frac{\{(\delta_0\alpha - \gamma_0\beta)\cos I + (\delta_0\beta + \alpha_0\gamma)\sin I\} y_0 + (\alpha_0\beta - \beta_0\alpha)\cos I - (\beta_0\beta + \alpha_0\alpha)\sin I}{\{(\delta_0\gamma - \gamma_0\delta)\cos I + (\delta_0\delta + \gamma_0\gamma)\sin I\} y_0 + (\alpha_0\delta - \beta_0\gamma)\cos I - (\alpha_0\gamma + \beta_0\delta)\sin I}.$$

(29) is the solution of the equation (28). This is a generalization of the result by Davis ([1], p. 11).

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## TRANSFORMĂRI MÖBIUS ŞI O CLASĂ DE ECUAŢII RICCATI

### Rezumat


În această Notă se prezintă un procedeu de construcţie a soluţiei pentru o clasă de ecuaţii Riccati cu ajutorul produsului de transformări Möbius.

Considerînd ecuaţia diferenţială liniară de ordinul al doilea (1) şi forma sa canonică (4), acesteia i se pune în corespondenţă în mod obişnuit o ecuaţie de tip Riccati (8). Se construiesc soluţiile pentru ecuaţiile diferenţiale de ordinul al doilea (11) în care $\Gamma_i$ se consideră constante pe intervalul $[x_{i-1}, x_i]$, punîndu-se astfel în evidenţă transformările infinitezimale Möbius (15) al căror produs $T_m$ din (16) permite determinarea printr-un proces de trecere la limită a soluţiei ecuaţiei Riccati (8). În final rezultatul se extinde pentru ecuaţia Riccati (28) punîndu-se în evidenţă soluţia sub forma (29).