TENSOR INTEGRAL AND EXTERIOR DERIVATIVE IN PARALLELIZABLE SPACES

BY

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The question of tensorial integration, defined as an inverse operation to covariant differentiation, was posed and studied for the first time in 1951 by A. Moór [6]. W. Fabian [2], H. Sasayama [8] and A. Moór [7] have subsequently contributed more to this problem.

Assuming an n-dimensional parallelizable space, we shall define in this paper the multiple tensor integral over a p-dimensional domain ($p \leqslant n$) by a generalization of (1.10) in [7]. We shall then find Stokes-type formulae which will enable us to define the exterior derivative of tensorial forms.

Introduction

Let E_n be an n-dimensional parallelizable space ([9], chap. III, § 4) of coordinates x^i ($i = 1, \ldots, n$) and having n fields ${}^\alpha a_i$ ($\alpha = 1, \ldots, n$) of linearly independent covariant vectors, i.e. for which

$$a = \det ({}^\alpha a_i) \neq 0. \tag{0.1}$$

By condition (0.1) the existence of the local (contravariant) frame ${}_\beta b^k$ is guaranted and we have

$${}^\alpha a_i \, {}_\alpha b^k = \delta^k_i \quad \text{and} \quad {}^\beta a_i \, {}_\alpha b^i = \delta^\beta_\alpha. \tag{0.2}$$

The coefficients Γ^i_{kl} of the connexion defined by the parallelism of the considered frames and of the torsion tensor $S^{i}_{.kl}$ of the connexion will be given by

$$\begin{aligned} \Gamma^{i}_{kl} &= {}_{\alpha}b^{i}\,\partial_{k}\,{}^{\alpha}a_{l} = -(\partial_{k}\,{}_{\alpha}b^{i})\,{}^{\alpha}a_{l} \\ S^{i}_{.kl} &= \Gamma^{i}_{[kl]} \quad \text{with} \quad \partial_{k} = \partial/\partial x^{k}, \end{aligned} \tag{0.3}$$

where the process of alternation is indicated by the square brackets ([9], chap. I, § 3).

If ∇_j is the covariant derivation operator, for the tensor $T^{\dots k_1 \dots k_r}_{i_1 \dots i_n} = T^{(k)}_{(i)}$ then :

$$\nabla_j T^{.(k)}_{(i)} = \partial_j T^{.(k)}_{(i)} - \sum_{r=1}^{n} \Gamma^{s_r}_{ji_r} T^{\dots\ \dots (k)}_{i_1 \dots s_r \dots i_n} + \sum_{r=1}^{r} \Gamma^{k_r}_{js_r} T^{.k_1 \dots s_r \dots k_r}_{(i)}. \tag{0.4}$$

where (i) is the notation for $i_1, i_2, \dots, i_n$. We can easily see that

$$\nabla_j\, {}^{\alpha}a_i = 0 \quad \text{and} \quad \Delta_j\, {}_{\beta}b^k = 0. \tag{0.5}$$

If $V_{j_1 \dots j_p}$ is a p-vector, then ([9], chap. III, § 2):

$$\nabla_{[l} V_{j_1 \dots j_p]} = \partial_{[l} V_{j_1 \dots j_p]} - p S^{h}_{.[lj_1|} V_{h|j_2 \dots j_p]}. \tag{0.6}$$

If $T^{.(k)}_{(i).j_1 \dots j_p}$ is a completely antisymmetric tensor of indices $j_1, \dots j_p$, we may define the operator ([9], chap. III, § 5)

$$\overset{p}{\nabla}_{[l} T^{.(k)}_{(i).|j_1 \dots j_p]} = \nabla_{[l} T^{.(k)}_{(i).|j_1 \dots j_p]} + p\, S^{h}_{.[lj_1|} T^{.(k)}_{(i).h|j_2 \dots j_p]}, \tag{0.7}$$

so that

$$\begin{aligned} \overset{p+q}{\nabla}_{[l} (T^{.(k)}_{(i).|j_1 \dots j_p} U^{.(t)}_{(s).|h_1 \dots h_q]}) &= \overset{p}{\nabla}_{[l} T^{.(k)}_{(i).|j_1 \dots j_p|}) U^{.(t)}_{(s).|h_1 \dots h_q]} + \\ &+ T^{.(k)}_{(i).[j_1 \dots j_p} \nabla_{l|} U^{.(t)}_{(s).|h_1 \dots h_q]} \end{aligned} \tag{0.8}$$

and

$$\overset{p+1}{\nabla}_{[h} \overset{p}{\nabla}_{l|} T^{.(k)}_{(i).|j_1 \dots j_p]} = 0, \tag{0.9}$$

as the curvature tensor of space E_n vanishes.

Let

$$dS^{i_1 \dots i_p} = dx^{i_1} \wedge \dots \wedge dx^{i_p} \tag{0.10}$$

be the tensor of the element of a p-dimensional hypersurface. The Stokes theorem for the p-vector $V_{j_1 \dots j_p}$ becomes ([9], chap. II, § 8)

$$\int_{(S_p)} V_{j_1 \dots j_p}\, dS^{j_1 \dots j_p} = \int_{(S_{p+1})} \partial_{[l} V_{j_1 \dots j_p]}\, dS^{lj_1 \dots j_p}, \tag{0.11}$$

where (S_p) is the boundary of the domain (S_{p+1}).

§ 1 The definition of the Tensor-Integral

Let V^i be a vector field. If the components $V^i(Q)$ are subjected to a parallel displacement in P, with $P, Q \in E_n$, then

$$V^i(Q, P) = {}_{\alpha}b^i(P)\, {}^{\alpha}a_k(Q)\, V^k(Q). \tag{1.1}$$

Similarly, for a field of covariant vectors V_i, we have

$$V_i(Q, P) = {}^{\alpha}a_i(P)\, {}_{\alpha}b^k(Q)\, V_k(Q). \tag{1.2}$$

In the case of a parallel displacement of a tensor $T^{.(k)}_{(i)}$, we have

$$T^{.(k)}_{(i)}(Q, P) = {}^{(\alpha)}_{(\beta)}h^{.(k)}_{(i)}(P)\, {}^{(\beta)}_{(\alpha)}h^{(j)}_{.(l)}(Q)\, T^{.(l)}_{(j)}(Q), \tag{1.3}$$

where

$$\left\{ \begin{aligned} {}^{(\alpha)}_{(\beta)}h^{.(k)}_{(i)} &= {}^{(\alpha)}a_{(i)}\, {}_{(\beta)}b^{(k)} \quad \text{and} \\ {}^{(\alpha)}a_{(i)} &= {}^{\alpha_1}a_{i_1} \dots {}^{\alpha_n}a_{i_n}; \\ {}_{(\beta)}b^{(k)} &= {}_{\beta_1}b^{k_1} \dots {}_{\beta_r}b^{k_r}. \end{aligned} \right. \tag{1.4}$$

The definition of the *tensor-integral* of the tensor $T^{.(k)}_{(i).j_1 \dots j_p}$ which is completely antisymmetric in indices $j_1, \dots, j_p$ *over the p-dimensional domain* (S_p) described by the point Q, will consist in the following relation [3], [4]

$$\underset{(S_p)}{\mathrm{I}}\, T^{.(k)}_{(i).j_1 \dots j_p}\, dS^{j_1 \dots j_p} = \int_{(S_p)} (T^{.(k)}_{(i).j_1 \dots j_p}\, dS^{j_1 \dots j_p})(Q, P), \tag{1.5}$$

and, thus, by (1.3)

$$\begin{aligned} &\underset{(S_p)}{\mathrm{I}}\, T^{.(k)}_{(i).j_1 \dots j_p}\, dS^{j_1 \dots j_p} = \\ &= {}^{(\alpha)}_{(\beta)}h^{.(k)}_{(i)}(P) \int_{(S_p)} {}^{(\beta)}_{(\alpha)}h^{(s)}_{.(l)}\, T^{.(l)}_{(s).j_1 \dots j_p}\, dS^{j_1 \dots j_p}. \end{aligned} \tag{1.6}$$

It is now obvious that the tensor-integral is defined over all points P of E_n, and is, of course, function of P. As in (1.6) there are scalar factors under the integral and tensorial ones before it, it follows that the tensor-integral of a tensor has a tensorial character as well.

§ 2. Stokes-type formulae

If (S_p) is a closed hypersurface, we shall obtain individually by (1.6), (0.11), (0.6) and (0.5)

$$\mathop{\mathbf{I}}_{(S_p)} T^{\cdot(k)}_{(i)\cdot j_1\ldots j_p}\, dS^{j_1\ldots j_p} = {}^{(\alpha)}_{(\beta)}h^{\cdot(k)}_{(i)}\,(P)\int\limits_{(S_{p+1})} \partial_{[l}\, \left({}^{(\beta)}_{(\alpha)}h^{(s)}_{\cdot(t)}\, T^{\cdot(t)}_{(s)\cdot|j_1\ldots j_p]}\right) dS^{lj_1\ldots j_p} =$$

$$= {}^{(\alpha)}_{(\beta)}h^{\cdot(k)}_{(i)}\,(P)\int\limits_{(S_{p+1})} \{\Delta_{[l}\,({}^{(\beta)}_{(\alpha)}h^{(s)}_{\cdot(t)}\, T^{\cdot(t)}_{(s)\cdot|j_1\ldots j_p]}) +$$

$$+ p\, S^{k}_{\cdot[lj_1|}\, {}^{(\beta)}_{(\alpha)}h^{(s)}_{\cdot(t)}\, T^{\cdot(t)}_{(s)\cdot k|j_2\ldots j_p]}\}\, dS^{lj_1\ldots j_p} =$$

$$= {}^{(\alpha)}_{(\beta)}h^{\cdot(k)}_{(i)}\,(P)\int\limits_{(S_{p+1})} {}^{(\beta)}_{(\alpha)}h^{(s)}_{\cdot(t)}\, \{\nabla_{[l}\, T^{\cdot(t)}_{(s)\cdot|j_1\ldots j_p]} + p\, S^{k}_{\cdot[lj_1|}\, T^{\cdot(t)}_{(s)\cdot k\,j_2\ldots j_p]}\}\, dS^{lj_1\ldots j_p},$$

and, from (0.7) and (1.6)

$$\mathop{\mathbf{I}}_{(S_p)} T^{\cdot(k)}_{(i)\cdot j_1\ldots j_p}\, dS^{j_1\ldots j_p} = \mathop{\mathbf{I}}_{(S_{p+1})} \overset{p}{\nabla}_{[l}\, T^{\cdot(k)}_{(i)\cdot j_1\ldots j_p]}\, dS^{lj_1\ldots j_p}, \tag{2.1}$$

where (S_p) is the boundary of (S_{p+1}). (2.1) *is the generalization of the Stokes theorem for the tensor-integral.*

§ 3. Tensorial Forms and Exterior Derivative

We shal call a *tensorial form of order p* (or shortly tensorial p-form) [5] the expression

$$\overset{p}{\omega}{}^{\cdot(k)}_{(i)} = T^{\cdot(k)}_{(i)\cdot j_1\ldots j_p}\, dS^{j_1\ldots j_p}, \tag{3.1}$$

where T is a tensor whose components are functions of necessary class.

We can now introduce an exterior derivative operator for tensorial forms, which will enable us to find a simple expression for Stokes-type formulae.

We can derive this operator by interpreting (0.7) as an operator upon tensorial forms by the relation

$$d\overset{p}{\omega}{}^{\cdot(k)}_{(i)} = \overset{p}{\nabla}_{[l}\, T^{\cdot(k)}_{(i)\cdot|j_1\ldots j_p]}\, dS^{lj_1\ldots j_p}. \tag{3.2}$$

So that, from (2.1), Stokes-type formulae become

$$\mathop{\mathbf{I}}_{(S_p)} \omega = \mathop{\mathbf{I}}_{(S_{p+1})} d\omega, \tag{3.3}$$

where (S_p) is the boundary of domain (S_{p+1}).

The operator (3.2) may be called an exterior derivative ([1], chap. IV, § 6) because it satisfies the following properties:

1) if $p = 0$, $d\overset{0}{\omega}$ agrees with the definition of the absolute differential of $\overset{0}{\omega}$;

2) domain of $\overset{p}{\omega}$ = domain of $d\overset{p}{\omega}$;

3) d is R-liniar (R = real numbers)

4) $d(\overset{p}{\omega} \wedge \overset{q}{\omega}) = (d\overset{p}{\omega}) \wedge \overset{q}{\omega} + (-1)^p \overset{p}{\omega} \wedge d\overset{q}{\omega}$;

5) $d(d\overset{p}{\omega}) = 0$ for all $\overset{p}{\omega}$.

From (0.7) it follows indeed that $\overset{0}{\nabla}_l = \nabla_l$, so that property No. 1 is satisfied. Properties 2) and 3) are easily verifiable.

If now

$$\overset{q}{\omega}{}^{\cdot(t)}_{(s)} = U^{\cdot(t)}_{(s)\cdot h_1\ldots h_q}\, dS^{h_1\ldots h_q}$$

is a tensorial q-form, we have

$$\overset{p}{\omega}{}^{\cdot(k)}_{(i)} \wedge \overset{q}{\omega}{}^{\cdot(t)}_{(s)} =$$

$$= T^{\cdot(k)}_{(i)\cdot j_1\ldots j_p}\, U^{\cdot(t)}_{(s)\cdot h_1\ldots h_q}\, dx^{j_1} \wedge \ldots \wedge dx^{j_p} \wedge dx^{h_1} \wedge \ldots \wedge dx^{h_q}$$

so that

$$d\,(\overset{p}{\omega}{}^{\cdot(k)}_{(i)} \wedge \overset{q}{\omega}{}^{\cdot(t)}_{(s)}) = \overset{p+q}{\nabla}_{[l}\, T^{\cdot(k)}_{(i)\cdot|j_1\ldots j_p|}\, U^{\cdot(t)}_{(s)\cdot|h_1\ldots h_q]}\, dS^{lj_1\ldots j_p h_1\ldots h_q}$$

and by (0.8)

$$d\,(\overset{p}{\omega}{}^{\cdot(k)}_{(i)} \wedge \overset{q}{\omega}{}^{\cdot(t)}_{(s)}) = \{(\overset{p}{\nabla}_{[l}\, T^{\cdot(k)}_{(i)\cdot|j_1\ldots j_p|})\, U^{\cdot(t)}_{(s)\cdot|h_1\ldots h_q]} +$$

$$T^{\cdot(k)}_{(i)\cdot[j_1\ldots j_p}\nabla_{l]}\, U^{\cdot(t)}_{(s)\cdot|h_1\ldots h_q]}\}\, dS^{lj_1\ldots j_p h_1\ldots h_q} =$$

$$= \{\overset{p}{\nabla}_{[l}\, T^{\cdot(k)}_{(i)\cdot|j_1\ldots j_p]}\, dx^l \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_p}\} \wedge$$

$$\wedge \{U^{\cdot(t)}_{(s)\cdot|h_1\ldots h_q]}\, dx^{h_1} \wedge \ldots \wedge dx^{h_q}\} + (-1)^p \{T^{\cdot(k)}_{(i)\cdot[j_1\ldots j_p]}\, dx^{j_1} \wedge \ldots \wedge dx^{j_p}\} \wedge$$

$$\wedge \{\overset{q}{\nabla}_{[l}\, U^{\cdot(t)}_{(s)\cdot|h_1\ldots h_q]}\, dx^l \wedge dx^{h_1} \wedge \ldots \wedge dx^{h_q}\}.$$

consequently

$$d(\overset{p}{\omega}{}^{\cdot(k)}_{(i)} \wedge \overset{q}{\omega}{}^{\cdot(t)}_{(s)}) = (d\overset{p}{\omega}{}^{\cdot(k)}_{(i)}) \wedge \overset{q}{\omega}{}^{\cdot(t)}_{(s)} + (-1)^p\, \overset{p}{\omega}{}^{\cdot(k)}_{(i)} \wedge d\overset{q}{\omega}{}^{\cdot(t)}_{(s)}, \tag{3.4}$$

and property 4) is verified.

Finally the exterior derivative of (3.2) will be

$$d(d\overset{p}{\omega}{}_{(i)}^{\cdot(k)}) = \overset{p+1}{\nabla}_{[s}\overset{p}{\nabla}_{[l|} T_{(i).|j_1\ldots j_p]]}^{\cdot(k)}\, dS^{slj_1\ldots j_p} = \overset{p+1}{\nabla}_{[s}\overset{p}{\nabla}_{l]} T_{(i).|j_1\ldots j_p]}^{\cdot(k)}\, dS^{slj_1\ldots j_p},$$

so that by (0.9)

$$d(d\overset{p}{\omega}{}_{(i)}^{\cdot(k)}) = 0, \tag{3.5}$$

which is the property No. 5 — Poincaré's theorem.

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TENSOR-INTEGRALA ȘI DIFERENȚIALA EXTERIOARĂ ÎN SPAȚII PARALELIZABILE

Rezumat

În prezenta Notă autorii definesc tensor-integrala într-un spațiu paralelizabil cu ajutorul relației (1.5). Tensor-integrala unui tensor are de asemenea caracter tensorial. Se obține generalizarea formulei lui Stokes pentru tensor-integrală dată de (2.1). Definind formele tensoriale de grad p prin relația (3.1) precum și diferențiala exterioară a acestora prin (3.2), formulele lui Stokes capătă expresia simplă (3.3). Diferențiala exterioară astfel definită satisface proprietățile 1)—5) din § 3.

DEFORMAREA PLANĂ TERMOELASTICĂ A MEDIILOR NEOMOGENE IZOTROPE

DE

D. IEȘAN

Considerăm un mediu elastic izotrop care ocupă o regiune finită, simplu conexă S care constă din reunirea domeniilor $S_j (j = 0, 1, 2, \ldots, m)$, domeniul S_0 fiind multiplu conex, mărginit de curbele închise L_j $(j = 1, 2, \ldots, m+1)$ suficient de netede și fără puncte comune, dintre care L_{m+1} conține în interiorul său pe celelalte. Domeniile $S_j (j = 1, 2, \ldots, m)$ sînt simplu conexe, disjuncte, mărginite de curbele $L_j (j = 1, 2, \ldots, m)$. Fie L frontiera lui S_0. Presupunem că mediul elastic ce ocupă domeniul $S_j (j = 0, 1, 2, \ldots, m)$ este omogen dar trecînd de la un domeniu la altul proprietățile elastice și termice se schimbă. Notăm cu E_j și σ_j modulul lui Young și coeficientul lui Poisson pentru materialul din domeniul $S_j (j = 0, 1, 2, \ldots, m)$. Fie k_j coeficientul de conductivitate termică corespunzător materialului din domeniul S_j, iar α_j coeficientul de dilatare termică liniară. Considerăm cazul în care temperatura este staționară și nu avem surse termice. Presupunem corpul liber de tensiuni exterioare.

Alegem un reper cartezian rectangular xOy cu originea în S_0. Vom nota cu u_j, v_j componentele vectorului deplasare în domeniul S_j.

1. Determinarea variației de temperatură

Fie T_j variația temperaturii mediului din S_j față de temperatura din starea nedeformată. Dacă temperatura este prescrisă pe frontieră, problema revine la determinarea funcțiilor T_j care satisfac [1]:

$$\Delta T_j = 0 \text{ în } S_j, \quad (j = 0, 1, \ldots, m), \tag{1.1}$$

$$T_0 = T_j \text{ pe } L_j, \tag{1.2}$$