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FORMULE DE CUBATURĂ OPTIMALE PENTRU ANUMITE CLASE DE FUNCŢII

Rezumat

În [1] şi [8] s-au dat formule de cuadratură optimale pentru anumite clase de funcţii. În această lucrare se extind unele din rezultatele amintite la formule de cubatură de tipul (2). Pentru aceasta se dă evaluarea exactă a restului (3), apoi se determină coeficienţii $A_{ij}^{(r,s)}$ şi nodurile $(x_i, y_j)$ astfel ca formula de cubatură să fie optimală pentru clasa de funcţii $W_{01_2}^{(r,s)}(M_{rs})$.

# DEGENERATE RIEMANNIAN AND DEGENERATE CONFORMAL CONNEXIONS

BY

V. OPROIU

**1. Introduction.** The Riemannian geometry deals with the study of the geometric properties of the Riemannian structure, i.e. the structure defined on a manifold $M$ by a nondegenerate quadratic form of definite sign. Since the fundamental tensor field of this structure is nondegenerate, there exists a tensor field of type (2,0), symmetric, which is given by the converse matrix of the fundamental tensor field. Then it can be established a canonical correspondence between the tangent and the cotangent bundle of the manifold $M$.

If the fundamental tensor field, denoted by $A$ is degenerate, then we cannot find a tensor field of type (2,0) with the properties of the tensor field constructed for the nondegenerate Riemannian case.

The structure defined by a degenerate symmetric tensor field $A$ of type (0,2), and rank constant $p$ on $M$ is called a degenerate Riemannian structure.

We shall study the integrability of the degenerate Riemannian structure determining the family of the linear connexions associated with this structure and calculating the obstructions to the integrability. These obstructions are cohomology classes in the Spencer cohomology associated with the Lie algebra of the structural group of the degenerate Riemannian structure. The same problems are solved for the degenerate conformal structures.

Let $M$ be a $C^\infty$-differentiable, $n$-dimensional manifold and on $M$ be defined a tensor field $A$ of type (0,2), symmetric and of rank constant $p$. We can associate with this tensor field a quadratic form and this form can be reduced in every point $x \in M$ to a sum (difference) of $p$ quadrates. If $P(M)$ is the principal frame bundle of $M$, we can construct the reduced principal sub-bundle $P_G(M)$ of $P(M)$, determined by the given tensor field.

Let $R^n$ be the real $n$-dimensional vector space with the natural basis $(e_\alpha)$, $\alpha = 1,2,\ldots,n$ and let us look at the fibre in $x \in M$ of the principal frame bundle $P(M)$ as the set of the linear mappings of $R^n$ into the tangent space $T_x(M)$ in the point $x \in M$, where $x$ is arbitrary in $M$. Then, if $(U, u^i)$ is a local chart defined in $x$ we obtain the following local chart in $P(M) : (\pi^{-1}(U), u^i\, u^i_\alpha)$ where $\pi : P(M) \to M$ is the natural projection and $(u^i_\alpha)$ is the matrix of the considered linear mapping defined by the given frame.

We suppose that the quadratic form associated with the tensor field $A$ can be reduced to a sum of $p$ quadrates and the other cases can be treated in a similar manner.

Let us consider the quadratic form $\varphi(y) = \varepsilon_{\alpha\beta}\, y^\alpha y^\beta$ defined on the vector space $R^n$ where $(y^1,\ldots,y^n)$ are the coordinates of the vector $y$ in the basis $(e_\alpha)$ and:

$$\varepsilon_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \leqslant p \\ 0 & \text{in the other cases.} \end{cases}$$

If the quadratic form associated with the tensor field $A$ is given in the local chart $(U, u^i)$ by $\psi = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$, then the reduced bundle $P_G(M)$ determined by the tensor field $A$ is defined by the following relation

$$P_G(M) = \{(u^i_\alpha) \in P(M) / g_{ij}\, u^i_\alpha u^j_\beta = \varepsilon_{\alpha\beta}\}.$$

Since the form $\psi$ can be written locally as a sum of quadrates, it follows easily that $P_G(M)$ is a principal sub-bundle of $P(M)$ and its structural group is:

$$G = \left\{ \begin{pmatrix} \lambda^q_r & 0 \\ \lambda^a_r & \lambda^a_b \end{pmatrix} \begin{matrix} q, r = 1,2,\ldots,p \\ a, b = p+1,2,\ldots,n \end{matrix} \quad (\lambda^q_r) \in O(p);\ (\lambda^a_b) \in GL(n-p, R) \right\},$$

where $O(p)$ is the orthogonal group in $p$ variables.

According to the general definition of the integrability of a $G$-structure, the degenerate Riemannian structure is integrable if the manifold $M$ can be covered by local charts so that the natural frames in these local charts belong to the reduced bundle $P_G(M)$. This implies that the quadratic form $\psi$ can be written in these local charts as:

$$\psi = \sum_{q=1}^{p} (\mathrm{d}u^q)^2. \tag{1.1}$$

The tensor field $A$ defines a mapping of the tangent bundle into the cotangent bundle and this mapping can be defined by its action on the sheaf of local sections in the tangent bundle. If $X$ is a local vector field giving a germ of vector field in $x \in M$ then the mapping determined by $A$ associates to the vector field $X$ the 1-form $A_X$ defined as follows:

$$(A_X)(Y) = A(X, Y). \tag{1.2}$$



The kernel of this mapping is an $n-p$-dimensional distribution $V$.

If the degenerate Riemannian structure is integrable, then the distribution $V$ is involutive and on the quotient manifold determined by this involutive distribution (the manifold of sheets of $V$) we have a local Euclidean, Riemannian metric.

In the sequell we shall denote by $\mathfrak{T}^s_t(M)$ the sheaf of germs of tensor fields of type $(s, t)$.

**2. The degenerate Riemannian connexions.** A degenerate Riemannian connexion is a connexion in the reduced principal bundle $P_G(M)$ defined by the degenerate Riemannian structure. A linear connexion on $P(M)$ determines by restriction a connexion on $P_G(M)$ if and only if the horizontal space in every point of $P_G(M)$ of the connexion in $P(M)$ is tangent to $P_G(M)$.

According to the usual methods of the theory of $G$-structures [3, 6] one shows that a linear connexion given by its covariant derivative $\nabla$ is a degenerate Riemannian connexion if and only if the covariant derivative of the fundamental tensor field $A$ vanishes, i.e.

$$\nabla_X A = 0, \qquad X \in \mathfrak{T}^1_0(M). \tag{2.1}$$

Let $V$ be the distribution defined in the section 1 of this paper and let us consider a complementary distribution $H$ to $V$. Such a distribution does always exist and it can be defined, for example as the orthogonal complement of $V$ with respect to an arbitrary Riemannian metric defined on $M$. We shall study the properties of the degenerate Riemannian structure with the help of the distribution $H$ and we shall show that many results do not depend on the choice of this distribution. Obviously, the dimension of the distribution $H$ is $p$.

**Theorem 2.1.** *Let $h$, $v$ be the projectors on $H$ and $V$, respectively. There exists a tensor field $\widetilde{A}$ of type $(2,0)$, symmetric, globally defined on $M$ such that:*

$$A_X \cdot \widetilde{A} = hX; \quad v\widetilde{A} = 0, \quad X \in \mathfrak{T}^1_0(M), \tag{2.2}$$

*where $A_X \cdot \widetilde{A}, v\widetilde{A}$ are defined in local coordinates in the following way: If $a_{ij}, a^{ij}, v^i_j, X^i$ are the expressions of $A, \widetilde{A}, v, X$ in local coordinates, respectively, then: $(A_X \cdot \widetilde{A})^k = a_{ij} X^i a^{kj}$; $(v\widetilde{A})^{ij} = v^i_k a^{kj}$.*

We prove the existence of the tensor field $\widetilde{A}$ in the following manner: The form $\psi$ can be written locally as:

$$\psi = \sum_{q=1}^{p} (\omega^q)^2,$$

and the distribution $H$ can be given locally by the system:

$$\omega^a = 0, \quad a = p+1, p+2, \dots, n,$$

where $\omega^q$, $\omega^a$ are $n$ linear independent Pfaffian forms defined on a suitable neighbourhood $U \subset M$. Let us consider the local dual basis of the cobasis $(\omega^p, \omega^a)$, given by the local vector fields $E_q, E_a$, defined on $U$ and satisfying the relations: $\omega^q(E_a) = 0$; $\omega^q(E_r) = \delta^q_r$; $\omega^b(E_a) = \delta^b_a$; $\omega^b(E_q) = 0$; $a, b = p+1, \dots, n$; $r, q = 1, 2, \dots, p$.

We can consider the following quadratic form:

$$\psi' = \sum_{q=1}^{p} (\omega^q)^2 + 2 \sum_{a=p+1}^{n} \omega^a \, dt^a,$$

defined on $U \times R^{n-p}$ where $R^{n-p}$ is the $n-p$ dimensional Euclidean space and $(R^{n-p}, t^a)$ is a chart in $R^{n-p}$.

Since the Pfaffian forms $\omega^q$, $\omega^a$ are linear independent it follows that the rank of $\psi'$ is $p + 2(n-p) = 2n - p$. The form $\psi'$ can be written in the neighbourhood $U \times R^{n-p}$ as:

$$\psi' = a_{ij} dx^i dx^j + 2 \sum_{a=p+1}^{n} \omega^a_i \, dx^i dt^a.$$

The matrix of the form $\psi'$ is:

$$B = \begin{pmatrix} a_{ij} & \omega^a_i \\ \omega^b_j & 0 \end{pmatrix},$$

and since the form $\psi'$ has the rank $2n-p$ it follows that $B$ is nondegenerate. Then there exists the converse

$$B^{-1} = \begin{pmatrix} a^{ij} & \xi^i_a \\ \xi^j_b & a^{ba} \end{pmatrix}$$

of the matrix $B$.

Using the Laplace method of calculation of determinants and the fact that rank $(a_{ij}) = p$, we can prove: $a^{ba} = 0$. Then we have:

(2.3) $$a_{ij} a^{kj} + \sum_{a=p+1}^{n} \omega^a_i \xi^k_a = \delta^k_i; \quad \omega^b_j a^{ij} = 0; \quad a_{ij} \omega^j_a = 0; \quad \omega^a_i \xi^i_b = \delta^a_b$$

$i, j, k = 1, 2, \dots, n$; $a, b = p+1, \dots, n$.

It follows that $\xi^i_a$ are the coordinates of the local vector field $E_a$ in the given chart and the relations;

(2.4) $$a_{ij} a^{kj} + \sum_{a=p+1}^{n} \omega^a_i \xi^k_a = \delta^k_i; \quad \omega^b_j a^{ij} = 0,$$

define uniquely the local tensor field $a^{ij}$.



If we choose in the same local chart another system defining the distribution $H$:

$$\tilde{\omega}^a = 0; \quad a = p+1, \dots, n$$

then $\tilde{\omega}^a$ must be linear combinations of the forms $\omega^a$ i. e.

$$\tilde{\omega}^a = \sum_{b=p+1}^{n} \lambda^a_b \omega^b, \quad \det(\lambda^a_b) \neq 0.$$

It follows that the dual basis of the cobasis $(\omega^q, \tilde{\omega}^a)$ is given by:

$$\tilde{E}_q = E_q; \quad \tilde{E}_a = \sum_{b=p+1}^{n} \mu^b_a E_b, \quad a, b = p+1, \dots, n,$$

where $(\mu^b_a)$ is the converse of the matrix $(\lambda^a_b)$.

The matrix of the quadratic form:

$$\tilde{\psi}' = a_{ij} dx^i dx^j + 2 \sum_{a=p+1}^{n} \tilde{\omega}^a_i \, dx^i dt^a,$$

is:

$$\tilde{B} = \begin{pmatrix} a_{ij} & \tilde{\omega}^a_i \\ \tilde{\omega}^b_j & 0 \end{pmatrix}$$

and its converse is:

$$\tilde{B}^{-1} = \begin{pmatrix} \tilde{a}^{ij} & \tilde{\xi}^i_a \\ \tilde{\xi}^j_b & 0 \end{pmatrix}$$

Since $\tilde{\xi}^i_a$ are the coordinates of the vector field $\tilde{E}_a$ we have:

$$\tilde{\xi}^i_a = \sum_{b=p+1}^{n} \mu^b_a \xi^i_b.$$

The tensor field $\tilde{a}^{ij}$ is uniquely determined by the relations:

(2.5) $$a_{ij} \tilde{a}^{kj} + \sum_{a=p+1}^{n} \tilde{\omega}^a_i \tilde{\xi}^k_a = \delta^k_i; \quad \tilde{\omega}^b_j \tilde{a}^{ij} = 0.$$

From the expressions of $\tilde{\omega}^a$, $\tilde{E}_a$ it follows that the system (2.5) is equivalent to the system (2.4) and since the tensor field $a^{ij}$ is uniquely determined by the system (2.4), it follows: $\tilde{a}^{ij} = a^{ij}$; We see that $\sum_{a=p+1}^{n} \omega^a_i \xi^k_a$ are the local coordinates of the projector $v$. Then $(a^{ij})$ depends only on the distribution $H$ and the tensor field $A$ which are globally defined on $M$. We can prove easily that $a^{ij}$ are the coordinates of a tensor field $\tilde{A}$ of type (2,0), symmetric and globally defined on $M$. From the relations (2.4) we obtain the relations (2.2).

Hence we have the following relations:

$$h + v = I; \quad A_{hX} = A_X; \quad A_{vX} = 0; \quad vX + A_X \cdot \tilde{A} = X; \quad v\tilde{A} = 0. \tag{2.6}$$

where $I \in \mathcal{T}^1_1(M)$ is the identity tensor field.

Now, we look for the degenerate Riemannian connexions, i.e. the connexions which satisfy (2.1). If we write the covariant derivative $\nabla$ as:

$$\nabla = \dot{\nabla} + P, \tag{2.7}$$

where $\dot{\nabla}$ is the covariant derivative of a given initial connexion, then the tensor field $P$ of type (1,2) must satisfy the equation:

$$A(P(X, Y), Z) + A(Y, P(X, Z)) = (\dot{\nabla}_X A)(Y, Z), \quad X, Y, Z \in \mathcal{T}^1_0(M). \tag{2.8}$$

It follows easily that the equation (2.8) is equivalent to the equation:

$$hP(X, Y) + (A_Y P_X)\tilde{A} = (\dot{\nabla}_X A)_Y \tilde{A}, \tag{2.9}$$

where $A_Y P_X$ is the 1-form defined by $(A_Y P_X)(Z) = A(Y, P(X, Z))$.

We introduce the following tensor fields:

$$\begin{aligned} \Phi &= \frac{1}{2}(I \otimes I + I \otimes v - A \otimes \tilde{A}), \\ \Psi &= \frac{1}{2}(I \otimes I - I \otimes v + A \otimes \tilde{A}), \\ \Theta &= \frac{1}{2}(v \otimes h). \end{aligned} \tag{2.10}$$

These tensor fields are of type (2.2); we consider the operations $\Psi P_X$, $\Psi\Psi$, defined as follows: if $\Psi, \Phi, P_X$ are locally given by: $\Psi^{kl}_{ij}$, $\Phi^{kl}_{ij}$, $P^i_{jk}$, $X^j$ respectively, then:

$$\begin{aligned} (\Psi P_X)^i_j &= \Psi^{il}_{jk}(P^k_{hl} X^h), \\ (\Psi\Phi)^{kl}_{ij} &= \Psi^{ml}_{ih} \Phi^{kh}_{mj}, \qquad l, i, j, k, m = 1, 2, \ldots, n. \end{aligned}$$

Then the equation (2.9) can be written as:

$$\Psi_Y P_X = \frac{1}{2}(\dot{\nabla}_X A)_Y \tilde{A}, \tag{2.11}$$

where $\Psi_Y P_X = (\Psi P_X)(Y)$.

It can be verified by a straightforward calculation the following relations:



$$\begin{aligned} &\Psi + \Phi = I \otimes I; \quad \Psi\Psi = \Psi - \frac{1}{2}\Theta; \quad \Phi\Phi = \Phi - \frac{1}{2}\Theta, \\ &\Psi\Phi = \Phi\Psi = \Phi\Theta = \Theta\Phi = \Psi\Theta = \Theta\Psi = \Theta\Theta = \frac{1}{2}\Theta. \end{aligned} \tag{2.12}$$

Then the equation (2.11) is equivalent to the equation:

$$(\Psi + \Theta)_Y P_X = \frac{1}{2}(\dot{\nabla}_X A)_{Y+vY} \tilde{A}, \tag{2.13}$$

and from (2.12) we obtain:

$$\begin{aligned} &(\Phi - \Theta) + (\Psi + \Theta) = I \otimes I; \quad (\Phi - \Theta)(\Phi - \Theta) = \Phi - \Theta; \\ &(\Psi + \Theta)(\Psi + \Theta) = \Psi + \Theta, \quad (\Phi - \Theta)(\Psi + \Theta) = (\Psi + \Theta)(\Phi - \Theta) = 0. \end{aligned} \tag{2.14}$$

Hence $\Psi + \Theta$, $\Phi - \Theta$ are projectors and the equation (2.13) has a solution if and only if $\Phi - \Theta$, applied to the right hand side of the equation (2.13) gives zero. This can be verified by a straightforward calculation and then the general solution of the equation (2.13) is given by the formula:

$$P(X, Y) = \frac{1}{2}(\dot{\nabla}_X A)_{Y+vY} \tilde{A} + (\Phi - \Theta)_Y Q_X, \tag{2.15}$$

where $Q \in \mathcal{T}^1_2(M)$ is an arbitrary tensor field. Then the general family of degenerate Riemannian connexions is given by the formula:

$$\nabla_X Y = \dot{\nabla}_X Y + \frac{1}{2}(\dot{\nabla}_X A)_{Y+vY} \tilde{A} + (\Phi - \Theta)_Y Q_X. \tag{2.16}$$

We shall prove that this family does not depend on the distribution $H$. In fact if $H'$ is another distribution complementary to $V$ then the corresponding family obtained from (2.16) is:

$$\nabla'_X Y = \dot{\nabla}_X Y + \frac{1}{2}(\dot{\nabla}_X A)_{Y+v'Y} \tilde{A}' + (\Phi' - \Theta')_Y Q'_X, \tag{2.16'}$$

where $\Phi'$, $\Psi'$, $\Theta'$ are the tensor fields of type (2,2) obtained from (2.10) with the help of $h'$, $v'$, $\tilde{A}'$, corresponding to the distribution $H'$.

If the initial connexion $\dot{\nabla}$ is a degenerate Riemannian connexion, then a connexion from the family (2.16) is given by:

$$\nabla_X Y = \dot{\nabla}_X Y + (\Phi - \Theta)_Y Q_X, \tag{2.17}$$

and a connexion from the family (2.16') is given by:

$$(2.17') \qquad \nabla'_X Y = \dot{\nabla}_X Y + (\Phi' - \Theta')_Y Q'_X.$$

In order to show that the two families of connexions do coincide we must prove that every connexion of the family (2.17) belongs to the family (2.17) and vice-versa. From (2.17) and (2.17′) it follows that we must prove the following relations:

$$(2.18) \qquad \begin{aligned} (\Psi' + \Theta')(\Phi - \Theta)Q_X &= 0, \\ (\Psi + \Theta)(\Phi' - \Theta')Q'_X &= 0. \end{aligned}$$

As $h'$, $v'$ are projectors, we have:

$$(2.19) \qquad h' = h + l;\ v' = v - l;\ \widetilde{A}' = \widetilde{A} + B;\ l \in \mathfrak{T}^1_1(M);\ B \in \mathfrak{T}^2_0(M).$$

Since the distributions $H$, $H'$ are complementary to the same distribution $V$ it follows:

$$(2.20) \qquad A_X B = lX;\ B = lB;\ hB = l\widetilde{A};\ lv = 0;\ ll = 0;\ Al = 0.$$

Then the relations (2.18) can be verified by a straightforward calculation. Hence we have proved:

**Theorem 2.2.** *The general family of the degenerate Riemannian connexions is given by the formula* (2.16) *where* $\dot{\nabla}$ *is an arbitrary initial connexion and* $Q \in \mathfrak{T}^1_2(M)$ *is an arbitrary tensor field. This family is expressed with the help of the distribution* $H$ *but it does not depend on this distribution.*

From the expression of $\Phi - \Theta$, we see that it can be written as:

$$\Phi - \Theta = I \otimes v + \frac{1}{2}(h \otimes h - A \otimes \widetilde{A}),$$

and then the formula (2.15) can be written as:

$$(2.15') \qquad P(X, Y) = \frac{1}{2}(\dot{\nabla}_X A)_{Y+vY}\widetilde{A} + vQ(X, Y) + \frac{1}{2}(h \otimes h - A \otimes \widetilde{A})_Y Q_X.$$

**3. The torsion tensor field of the degenerate Riemannian connexions.** Let us suppose that the initial connexion $\dot{\nabla}$ is torsion-less and let us consider the general degenerate Riemannian connexion:

$$(3.1) \qquad \begin{aligned} \nabla_X Y = \dot{\nabla}_X Y + P(X, Y) = \dot{\nabla}_X Y + \frac{1}{2}(\dot{\nabla}_X A)_{Y+vY}\widetilde{A} + vQ(X, Y) + \\ + \frac{1}{2}(h \otimes h - A \otimes \widetilde{A})_Y Q_X. \end{aligned}$$

Since $\dot{\nabla}$ is torsion-less, the torsion tensor field of the connexion $\nabla$ is given by:



$$(3.2) \qquad T(X, Y) = \nabla_X Y + \nabla_Y X - [X, Y] = P(X, Y) - P(Y, X).$$

Since $\nabla$ is a degenerate Riemannian connexion then:

$$(3.3) \qquad A(P(X, Y), Z) + A(Y, P(X, Z)) = (\dot{\nabla}_X A)(Y, Z).$$

From this relation it follows in the same manner as in the classical case:

$$(3.4) \qquad \begin{aligned} A(P(X, Y) + P(Y, X), Z) + A(Y, T(X, Z)) + A(X, T(Y, Z)) = \\ = (\dot{\nabla}_X A)(Y, Z) + (\dot{\nabla}_Y A)(Z, X) - (\dot{\nabla}_Z A)(X, Y). \end{aligned}$$

We should be interested in the obtaining of a degenerate Riemannian. connexion with the torsion tensor field $T$ such that $hT(hX, hY) = 0$.

Then for such a connexion we must have:

$$(3.5) \qquad 2A(hP(hX, hY), hZ) = (\dot{\nabla}_{hX} A)(hY, hZ) + (\dot{\nabla}_{hY} A)(hX, hZ) - (\dot{\nabla}_{hZ} A)(hX, hY)$$

Hence:

$$(3.6) \qquad hP(hX, hY) = \frac{1}{2}(\dot{\nabla}_{hX} A)_{hY}\widetilde{A} + \frac{1}{2}(\dot{\nabla}_{hY} A)_{hX}\widetilde{A} - \frac{1}{2}(\dot{\nabla} A)(hX, hY)\widetilde{A}.$$

Since $hP(hX, hY)$ must be obtained from the formula (3.1) it results the following condition for $Q$:

$$(3.7) \qquad \frac{1}{2}(h \otimes h - A \otimes \widetilde{A})_Y Q_{hX} = \frac{1}{2}(\dot{\nabla}_{hY} A)_{hX}\widetilde{A} - \frac{1}{2}(\dot{\nabla} A)(hX, hY)\widetilde{A}.$$

If we introduce the tensor field $\frac{1}{2}(h \otimes h + A \otimes \widetilde{A})$ then we can verify the following relations:

$$(3.8) \qquad \begin{aligned} &\frac{1}{2}(h \otimes h - A \otimes \widetilde{A}) + \frac{1}{2}(h \otimes h + A \otimes \widetilde{A}) = h \otimes h, \\ &\frac{1}{2}(h \otimes h - A \otimes \widetilde{A})\frac{1}{2}(h \otimes h - A \otimes \widetilde{A}) = \frac{1}{2}(h \otimes h - A \otimes \widetilde{A}), \\ &\frac{1}{2}(h \otimes h + A \otimes \widetilde{A})\frac{1}{2}(h \otimes h + A \otimes \widetilde{A}) = \frac{1}{2}(h \otimes h + A \otimes \widetilde{A}) \\ &\frac{1}{2}(h \otimes h - A \otimes \widetilde{A})\frac{1}{2}(h \otimes h + A \otimes \widetilde{A}) = \\ &= \frac{1}{2}(h \otimes h + A \otimes \widetilde{A})\frac{1}{2}(h \otimes h - A \otimes \widetilde{A}) = 0. \end{aligned}$$

Then the general solution of the equation (3.7) is the tensor field $Q$, having the following property:

$$hQ(hX,hY)=\frac{1}{2}(\dot\nabla_{hY}A)_{hX}\widetilde{A}-\frac{1}{2}(\dot\nabla A)(hX,hY)\widetilde{A}+ \tag{3.9}$$
$$+\frac{1}{2}(h\otimes h+A\otimes\widetilde{A})_{hY}S_{hX},$$

where $S\in\mathcal{T}^1_2(M)$ is an arbitrary tensor field.

The tensor field $Q$ from the formula (3.1) can be written as follows:

$$Q_X=(Q_X-hQ_{hX}h)+hQ_{hX}h. \tag{3.10}$$

Using (3.9), we obtain:

$$Q(X,Y)=(Q(X,Y)-hQ(hX,hY))+\frac{1}{2}(\dot\nabla_{hY}A)_{hX}\widetilde{A}- \tag{3.11}$$
$$-\frac{1}{2}(\dot\nabla A)(hX,hY)\widetilde{A}+\frac{1}{2}(h\otimes h+A\otimes\widetilde{A})_{hY}S_{hX}$$

Substituting this expression in (3.1), we have:

$$\nabla_XY=\dot\nabla_XY+\frac{1}{2}(\dot\nabla_XA)_{Y+hY}\widetilde{A}+\frac{1}{2}(\dot\nabla_{hY}A)_{hX}\widetilde{A}-\frac{1}{2}(\dot\nabla A)(hX,hY)\widetilde{A}+ \tag{3.12}$$
$$+(\Phi-\Theta)_Y(Q-hQ_{hX}h).$$

Hence we have obtained.

**Theorem 3.1.** *The general family of the degenerate Riemannian connexions with $hTh.h=0$, is given by the formula (3.12) where $Q\in\mathcal{T}^1_2(M)$ is an arbitrary tensor field.*

*Remark.* We have denoted by $hTh.h$ the tensor field of type (1,2) defined as: $(hTh.h)(X,Y)=hT(hX,hY)$. In the sequel we shall use the following notations too: $(hTh)(X,Y)=hT(hX,Y)$; $(hT.h)(X,Y)=hT(X,hY)$.

Now we are interested in the finding of the conditions under which there exist torsion-less degenerate Riemannian connexions.

If in (3.4) we substitute the vector field $Z$ by $vZ$, we obtain:

$$A(Y,T(X,vZ))+A(X,T(Y,vZ))=(\dot\nabla_XA)(Y,vZ)+ \tag{3.13}$$
$$+(\dot\nabla_YA)(X,vZ)-(\dot\nabla_{vZ}A)(X,Y).$$

**Proposition 3.1.** *The tensor field $L(X,Y,Z)=(\dot\nabla_YA)(Z,vX)+(\dot\nabla_ZA)(Y,vX)-(\dot\nabla_{vX}A)(Y,Z)$ does not depend on the torsion-less connexion $\dot\nabla$ and $-L(X,Y,Z)$ is the Lie derivative of the tensor field $A$ with respect to the vector field $vX$, i.e.:*

$$L(X,Y,Z)=-(L_{vX}A)(Y,Z). \tag{3.14}$$



*Proof:* If we consider another torsion-less connexion $\nabla'=\dot\nabla+P$; $(P(X,Y)=P(Y,X))$ and the respective tensor field $L'$, then:

$$L'(Z,X,Y)=(\nabla'_XA)(Y,vZ)+(\nabla'_YA)(X,vZ)-(\nabla'_{vZ}A)(X,Y)=$$
$$=L(Z,X,Y)-(A(P(X,Y),vZ)-A(Y,P(X,vZ))-A(P(Y,X),vZ)-$$
$$-A(X,P(Y,vZ))+A(P(vZ,X),Y)+A(X,P(vZ,Y))=L(Z,X,Y).$$

The Lie derivative of the tensor field $A$ with respect to $vZ$ is given by the formula:

$$(L_{vZ}A)(X,Y)=vZ(A(X,Y))-A([vZ,X],Y)-A(X,[vZ,Y])=$$
$$=(\dot\nabla_{vZ}A)(X,Y)+A(\dot\nabla_{vZ}X,Y)+A(X,\dot\nabla_{vZ}Y)-A(\dot\nabla_{vZ}X,Y)+$$
$$+A(\dot\nabla_XvZ,Y)-A(X,\dot\nabla_{vZ}Y)+A(X,\dot\nabla_YvZ)=(\dot\nabla_{vZ}A)(X,Y)+$$
$$A(X,\dot\nabla_YvZ)+A(Y,\dot\nabla_XvZ)=(\dot\nabla_{vZ}A)(X,Y)-(\dot\nabla_YA)(X,vZ)-$$
$$-(\dot\nabla_XA)(Y,vZ)=-L(Z,X,Y),$$

where we have used the relation:

$$0=YA(X,vZ)=(\dot\nabla_YA)(X,vZ)+A(\dot\nabla_YX,vZ)+A(X,\dot\nabla_YvZ)=$$
$$=(\dot\nabla_YA)(X,vZ)+A(X,\dot\nabla_YvZ).$$

The relation (3.13) can be written as:

$$A(Y,T(X,vZ))-A(X,T(Y,vZ))=L(Z,X,Y). \tag{3.13'}$$

We have:

$$hT(Y,vX)-(A_YT_{vX})\widetilde{A}=L_{XY}\widetilde{A}, \tag{3.15}$$

$$hT(vY,vX)=L_{X,vY}\widetilde{A}. \tag{3.16}$$

From the expressions (2.10) of the tensor fields $\Phi,\Theta$ it follows:

$$A\otimes\widetilde{A}=2I\otimes v+h\otimes h-2(\Phi-\Theta), \tag{3.17}$$

and then we obtain:

$$hT(hX,vY)+(\Phi-\Theta)_XhT_{vY}=\frac{1}{2}L_{Y,hX}\widetilde{A}. \tag{3.18}$$

Let us consider the degenerate Riemannian connexion $\nabla$ with its torsion tensor field $T$, which satisfies: $hTh.h=0$.

The tensor field $T$ can be written as:

$$T(X,Y)=vT(X,Y)+hT(vX,vY)+hT(hX,vY)+hT(vX,hY). \tag{3.19}$$

We obtain:

$$(3.20)\qquad \begin{aligned} T(X, Y) &= \frac{1}{2} L_{Y,hX}\tilde{A} - \frac{1}{2} L_{X,hY}\tilde{A} + L_{Y,vX}\tilde{A} + \\ &(\Phi - \Theta)_Y\left(\frac{1}{2} vT_X + hT_{vX}\right) - (\Phi - \Theta)_X\left(\frac{1}{2} vT_Y + hT_{vY}\right). \end{aligned}$$

Let us consider the connexion:

$$(3.21)\qquad \tilde{\nabla}_X Y = \nabla_X Y - (\Phi - \Theta)_Y\left(\frac{1}{2} vT_X + hT_{vX}\right).$$

This is a degenerate Riemannian connexion and its torsion tensor field is:

$$(3.22)\qquad \tilde{T}(X, Y) = \frac{1}{2} L_{Y,hX}\tilde{A} - \frac{1}{2} L_{X,hY}\tilde{A} + L_{Y,vX}\tilde{A}.$$

For the tensor field $\frac{1}{2} vT + hTv$ which appear in (3.21), we have: $h\left(\frac{1}{2} vT + hTv\right)h.h = 0$. It follows that the torsion tensor field $\tilde{T}$ of the connexion $\tilde{\nabla}$ has the property $h\tilde{T}h.h = 0$, i.e. $\tilde{\nabla}$ is a connexion of the family (3.12).

**Proposition 3.2.** *The right hand side of the relation* (3.22) *vanishes if and only if* $L = 0$.

*Proof.* If $L = 0$ the assertion follows directly.

If $\frac{1}{2} L_{Y,hX}\tilde{A} - \frac{1}{2} L_{X,hY}\tilde{A} + L_{Y,vX}\tilde{A} = 0$, then:

$$L(Y, hX, hZ) - L(X, hY, hZ) + 2L(Y, vX, hZ) = 0 .$$

If $X \to hX$, we obtain: $L(Y, hX, hZ) = L(hX, hY, hZ)$. But $L(hX, hY, hZ) = (L_{vhX}A)(hY, hZ) = 0$, since $vhX = 0$. Then $L(Y, hX, hY) = 0$. It follows: $L(Y, vX, hZ) = 0$ and then $L(Y, X, hZ) = L(Y, vX, hZ) + L(Y, hX, hZ) = 0$. We have: $L(Y, hX, vZ) = -(L_{vY} A)(hX, vZ) = -(L_{vY} A)(vZ, hX) = L(Y, vZ, hX) = 0$.

On the other hand:

$$\begin{aligned} -L(Y, vX, vZ) &= (L_{vY}A)(vX, vZ) = vY(A(vX, vZ)) - \\ &- A([vY, vX], vZ) - A(vX, [vY, vZ]) = 0. \end{aligned}$$

Then $L(X, Y, vZ) = L(Y, vX, vZ) + L(Y, hX, vZ) = 0$.

Hence: $L(X, Y, Z) = L(Y, X, vZ) + L(Y, X, hZ) = 0$.

From the proposition (3.1), (3.2) and the formulas (3.13′), (3.22) we obtain the following theorem:



**Theorem 3.2.** *There exists a torsion-less degenerate Riemannian connexion if and only if the Lie derivative of the tensor field* $A$ *with respect to every vector field of the distribution* $V$, *vanishes.*

We know that the Chern-Bernard invariant of a $G$-structure vanishes if and only if there exists a torsion-less connexion of this $G$-structure. Hence the tensor field $\tilde{T}$ of formula (3.22) is a representative of the Chern-Bernard invariant of the degenerate Riemannian structure in the Spencer cohomology attached to the Lie algebra of the structural group $G$ of the degenerate Riemannian structure and it vanishes if and only if the Chern-Bernard invariant of the degenerate Riemannian structure vanishes.

Let us suppose that the Chern-Bernard invariant of the degenerate Riemannian structure vanishes, i.e. the tensor field $L$ vanishes. If $(U, u^i)$ is a local chart on $M$ and in this chart $A, \tilde{A}$ are given by $a_{ij}, a^{ij}$ respectively then we can verify easily that:

$$(3.23)\qquad \Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial a_{ih}}{\partial u^j} + \frac{\partial a_{jh}}{\partial u^i} - \frac{\partial a_{ij}}{\partial u^h}\right) a^{hk},$$

are the connexion coefficients of a torsion-less degenerate Riemannian connexion. Hence if $L = 0$ we have a „Levi Civita-connexion" for the degenerate Riemannian structure, but this connexion is not uniquely determined since $\Gamma^k_{ij}$ depends on $a^{hk}$ which depend on the distribution $H$. If we consider another distribution $H$ then the corresponding connexion is obtained from $\Gamma^k_{ij}$, adding a tensor field of type $vS$.

**Proposition 3.3.** *If* $L = 0$, *the distribution* $V$ *is involutive.*

*Proof.* Let $X, Y, \in V$ be two arbitrary vector fields in $V$. We must show that $[X, Y] \in V$, i.e. $A_{[X, Y]} = 0$.

If in the formula:

$$0 = -L(X, Y, Z) = (L_{vX}A)(Y, Z) - vX(A(Y, Z)) - A([vX, Y], Z) - A(Y, [vX, Z]),$$

we take $X = vX$: $Y = vY$ then it follows $A([X, Y], Z) = 0$, i.e. $[X, Y] \in V$. Hence, if $L = 0$, the distribution $V$ is involutive and we can choose local charts $(U, u^i)$ such that the distribution $V$ is generated on $U$ by the local vector fields $\frac{\partial}{\partial u^{p+1}}, \ldots, \frac{\partial}{\partial u^n}$. Expressing the fact that the Lie derivatives of the tensor field $A$, with respect to $\frac{\partial}{\partial u^{p+1}}, \ldots, \frac{\partial}{\partial u^n}$ vanish, it follows that $a_{ij}$ does not depend on $u^{p+1}, \ldots, u^n$. Hence $(U, u^i)$ can be written: $(U, u^i) = (U', u^1, \ldots, u^p) \times (U'', u^{p+1}, \ldots, u^n)$ and on $U'$ we have a Riemannian metric. It remains to see if we can choose $(U', u^1, \ldots, u^p)$ such that $a_{qr}(u) = \delta^q_r$ (the Kronecker symbol); $q, r = 1, \ldots, p$.

**4. The curvature tensor field of the degenerate Riemannian connexions.** The degenerate Riemannian connexion given by the formula (3.21) has the torsion tensor field $\widetilde{T}$ given by the formula (3.22). We intend to express the family of degenerate Riemannian connexions having the same torsion tensor field. Obviously, this family must be given by the formula:

$$(4.1)\qquad \begin{aligned}\nabla_X Y &= \widetilde{\nabla}_X Y + (\Phi - \Theta)_Y(Q_X - hQ_{hX}h) = \widetilde{\nabla}_X Y + \\ &+ vQ(X, Y) + \frac{1}{2} hQ(vX, hY) - \frac{1}{2} A_Y Q_{vX} \widetilde{A},\end{aligned}$$

where $\widetilde{\nabla}$ is the connexion (3.21) and $Q$ a tensor field of type (1,2) which must be determined. Since $\nabla$ given by (4.1) and $\widetilde{\nabla}$ have the same torsion tensor fields it follows:

$$(4.2)\qquad (\Phi - \Theta)_Y(Q_X - hQ_{hX}h) - (\Phi - \Theta)_X(Q_Y - hQ_{hY}h) = 0.$$

This relation gives:

$$(4.3)\qquad \begin{aligned}vQ(X, Y) - vQ(Y, X) + \frac{1}{2} hQ(vX, hY) - \frac{1}{2} hQ(vY, hX) - \frac{1}{2} A_Y Q_{vX} \widetilde{A} + \\ + \frac{1}{2} A_X Q_{vY} \widetilde{A} = 0.\end{aligned}$$

It follows:

$$(4.4)\qquad vQ(X, Y) = vQ(Y, X).$$

If $X \to vX$, then:

$$hQ(vX, hY) - A_Y Q_{vX} \widetilde{A} = 0.$$

Hence we have proved:

**Theorem 4.1.** *The general degenerate Riemannian connexion with the torsion tensor field $\widetilde{T}$ is given by the formula:*

$$\nabla_X Y = \widetilde{\nabla}_X Y + vQ(X, Y),$$

*where $Q$ is a tensor field of type (1,2), symmetric in the lower indices. This family does not depend on the distribution $H$.*

The last assertion follows easily from the formulas (2.19), (2.20).

We shall study the curvature tensor field of a connexion of this family since this tensor field furnishes an obstruction to the formal integrability of the degenerate Riemannian structure.

If we denote by $R$, $\widetilde{R}$ the curvature tensor fields of the connexions $\nabla$, $\widetilde{\nabla}$ respectively then we obtain by an usual calculation:

$$(4.6)\qquad R(X, Y) = \widetilde{R}(X, Y) + (\widetilde{\nabla}_X vS)_Y - (\widetilde{\nabla}_Y vS)_X + vS_X vS_Y - vS_Y vS_X.$$



Since $\widetilde{\nabla}$ is a degenerate Riemannian connexion we can verify that $h((\widetilde{\nabla}_X vS)_Y - (\widetilde{\nabla}_Y vS)_X + vS_X vS_Y - vS_Y vS_X) = 0$.

Then:

$$(4.7)\qquad hR = h\widetilde{R}.$$

Hence:

**Theorem 4.2.** *The tensor field $hR$, where $R$ is the curvature tensor field of a connexion of the family (4.5) does not depend on the connexion $\nabla$.*

Now let us consider the integrability problem for the degenerate Riemannian structure. If there exists a torsion-less degenerate Riemannian connexion, then the distribution $V$ is involutive and we can consider local charts on $M$ so that if $(U, u^i)$ is such a local chart then: $(U, u^i) = (U', u^1,\dots, u^p) \times (U'', u^{p+1},\dots, u^n)$; $a_{ai} = 0$ and $a_{qr}$ depend only on $x^1,\dots, x^p$; $(q, r = 1, 2,\dots, p;\ a, b = p+1,\dots, n)$. In the neighbourhood $U$ we consider the local distribution $H$ given by $du^a = 0$, $a = p+1,\dots, n$. In local coordinates, the tensor field $\widetilde{A}$ is given by: $a^{ai} = 0$; $a = p+1,\dots,n$; $i = 1, 2,\dots, n$ $a^{qr}$ depends only on $u^1,\dots, u^p$.

We have:

$$a_{qr}\, a^{rs} = \delta^s_q;\ q, r, s = 1, 2,\dots, p.$$

It this coordinate neighbourhood the „Levi Civita-connexion" is given by the formula (3.22) and we have:

$$(4.8)\qquad \begin{aligned}\Gamma^q_{rs} &= \frac{1}{2}\left(\frac{\partial a_{rt}}{\partial u^s} + \frac{\partial a_{st}}{\partial u^r} - \frac{\partial a_{rs}}{\partial u^t}\right) a^{tq};\ q, r, s, t = 1, 2,\dots, p. \\ \Gamma^a_{ij} &= \Gamma^i_{aj} = 0;\ a = p+1,\dots, n;\ i, j = 1, 2,\dots, n.\end{aligned}$$

We see that this connexion is the Levi-Civita connexion associated to the Riemannian structure determined by the degenerate Riemannian structure on $(U', u^1,\dots, u^p)$.

According to a classical result the quadratic form $a_{qr}\, du^q\, du^r$ can be reduced to the canonical form; $(du^1)^2 + \dots (du^p)^2$ if and only if the curvature tensor field of the Levi-Civita connexion on $U'$ vanishes. If the tensor field $hR$ defined by (4.7) vanishes, then we see that the curvature tensor field of the connexion (4.8) for which $hR = R$, vanishes, too. Hence the Riemannian structure induced on $(U', u^1,\dots, u^p)$ is integrable, i.e. the degenerate Riemannian structure is integrable.

If the degenerate Riemannian structure is integrable then obviously $hR = 0$.

Hence:

**Theorem 4.3.** *The degenerate Riemannian structure is integrable if and only if there exists a degenerate Riemannian connexion with vanishing torsion and curvature tensor fields. In this case: $L = 0$, $hR = 0$.*

**5. Degenerate conformal structures.** The degenerate conformal structure is defined by a symmetric tensor field of type $A$ of type (0,2) of rank constant $p$ on the $C^\infty$-differentiable, $n$-dimensional manifold $M$.

The reduced bundle corresponding to the degenerate conformal structure has the structural group:

$$G' = \left\{ \begin{pmatrix} \lambda_r^q & 0 \\ \lambda_r^a & \lambda_b^a \end{pmatrix} ; \begin{matrix} q, r = 1, 2, \dots, p \\ a, b = p+1, \dots, n. \end{matrix} \quad (\lambda_r^q) \in CO(p);\ (\lambda_b^a) \in GL(n-p, R) \right\},$$

where $CO(p)$ is the conformal group in $p$-variables.

The degenerate conformal structure is integrable if and only if $M$ can be covered by local charts such that in this charts the quadratic form $\psi$ associated with the tensor field $A$ can be is written as:

$$\psi = f(u) \sum_{q=1}^{p} (\mathrm{d}u_q)^2,$$

where $f(u)$ is a function defined on the respective chart.

In the same manner as in the case of the degenerate Riemannian structures we shall study the conformal degenerate structure with the help of a distribution $H$ complementary to the distribution $V$ determined by the tensor field $A$.

A linear connexion given by its covariant derivative $\nabla$ is a degenerate conformal connexion if:

$$\nabla_X A = \lambda(X) A, \tag{5.1}$$

where $\lambda \in \mathcal{T}_1^0(M)$ is the factor of the connexion $\nabla$.

Using the procedure of the section 2 of this paper we obtain the formula for the general family of the degenerate conformal connexions:

$$\nabla_X Y = \dot{\nabla}_X Y - \frac{1}{2} \lambda(X) hY + (\Phi - \Theta)_Y Q_X, \tag{5.2}$$

where $\dot{\nabla}$ is an arbitrary degenerate Riemannian connexion, $h, \Phi, \Theta$ are the tensor fields determined in the section 2 of this paper and $Q$ is an arbitrary tensor field of type (1,2).

We have seen that the tensor field $L$ introduced in the section 3 of this paper is closely related to the integrability of the degenerate Riemannian structure. If $A' = e^{\varphi} A$ is the tensor field of type (0,2) obtained by the multiplication of $A$ with a function $e^{\varphi}$, then:

$$L'(X, Y, Z) = e^{\varphi} (L(X, Y, Z) - A(Y, Z) \mathrm{d}\varphi(vX)), \tag{5.3}$$

where $L'$ is the tensor field corresponding to $A'$.

The tensor field of type (2,0) corresponding to $\tilde{A}$ is given by:



$$\tilde{A}' = e^{-\varphi} \tilde{A},$$

and we obtain:

$$L'(X, \tilde{A}') = L(X, \tilde{A}) - p\mathrm{d}\varphi(vX), \tag{5.4}$$

where $L(X, \tilde{A})$ is the value of $L$ calculated for the vector field $X$ and the tensor field of type (2,0) $\tilde{A}$.

If we denote: $\tilde{L}(X) = \dfrac{1}{p} L(X, \tilde{A})$, then:

**Proposition 5.1.** *The tensor field*

$$C(X, Y, Z) = L(X, Y, Z) - \tilde{L}(X) A(Y, Z), \tag{12.5}$$

*is an invariant of the degenerate conformal structure.*

Using the method described in the section 3 of this paper we can prove the following theorem:

**Theorem 5.1.** *There exists a degenerate conformal connexion $\nabla$ with the torsion tensor field:*

$$T(X, Y) = \frac{1}{2} C_{Y, hX} \tilde{A} - \frac{1}{2} C_{X, hY} \tilde{A} + C_{Y, vX} \tilde{A}, \tag{5.6}$$

*and the vanishing of this tensor field is equivalent to the vanishing of $C$.*

*The general family of degenerate conformal connexions admitting $T$ as the torsion tensor field, is given by the formula:*

$$\begin{aligned} \tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} \lambda(hX) hY - \frac{1}{2} \lambda(hY) hX + \\ + \frac{1}{2} A(X, Y) \lambda \tilde{A} + vQ(X, Y), \end{aligned} \tag{5.7}$$

where $\lambda \in \mathcal{T}_1^0(M)$ is an arbitrary Pfaffian form and $Q \in \mathcal{T}_2^1(M)$ is an arbitrary tensor field symmetric in the lower indices.

We look for an invariant of the degenerate conformal structure given by the curvature tensor field of a connexion of the family (5.7).

We denote $\frac{1}{2} \lambda h = \sigma$ and then $\sigma h = \sigma$; $\sigma v = 0$.

We have:

$$\tilde{\nabla}_X Y = \nabla_X Y - \sigma(X) hY - \sigma(Y) hX + A(X, Y) \sigma \tilde{A} + vQ(X, Y). \tag{5.7'}$$

Calculating the curvature tensor field $\tilde{R}$ of the connexion (5.7') we obtain:

$$(5.8)\qquad \begin{aligned} h\widetilde{R}(X, Y) = hR(X, Y) + K_X\widetilde{A}\otimes A_Y - K_Y\widetilde{A}\otimes A_X + \\ + K_Y \otimes hX - K_X \otimes hY + K(Y, X)h - K(X, Y)h, \end{aligned}$$

where $K\in\mathcal{T}^0_2(M)$ is defined by the formula:

$$(5.9)\qquad K(X, Y) = (\nabla_X\sigma)(Y) + \sigma(X)\sigma(Y) - \frac{1}{2}\widetilde{A}(\sigma, \sigma)A(X, Y).$$

$\widetilde{A}(\sigma, \sigma)$ is the value of $\widetilde{A}$, calculated for the form $\sigma$.

We can obtain, by some calculations similar to those used in the classical case, the „Weyl tensor field" of the degenerate conformal structure:

$$(5.10)\qquad \begin{aligned} W(X, Y)Z = hR(X, Y)Z + \Big(\frac{E(X, Y)}{p-2} + \\ + \frac{B(X, Y)}{p(p-2)}\Big)hZ - \Big(\frac{E(Y,X)}{p-2} + \frac{B(Y,X)}{p(p-2)}\Big)hZ + \Big(\frac{E(X,Z)}{p-2} + \frac{B(X,Z)}{p(p-2)}\Big)hY - \\ - \Big[\frac{E(Y,Z)}{p-2} + \frac{B(Y,Z)}{p(p-2)}\Big]hX - A(Y,Z)\Big[\frac{E_X}{p-2} + \frac{B_X}{p(p-2)}\Big]\widetilde{A} + \\ + A(X,Z)\Big[\frac{E_Y}{p-2} + \frac{B_Y}{p(p-2)}\Big]\widetilde{A}, \end{aligned}$$

where $E\in\mathcal{T}^0_2(M)$ is given by the formula:

$$(5.11)\qquad E(X, Y) = D(Y, X) + \frac{B(Y, hX) + B(X, hY)}{p} - \frac{\bar{R}}{2(p-1)}A(X,Y),$$

and $B, D$ are obtained from $hR\in\mathcal{T}^1_3(M)$ by the contractions like in the case of the Ricci tensor fields, $(B(X, Y) = -B(Y, X))$. $R$ is the scalar Ricci curvature of the tensor field $hR$.

The Schouten-Weyl tensor field of the degenerate conformal structure is given by the formula:

$$(5.12)\qquad W(X,Y,Z) = \nabla_X\Big(\frac{E}{p-2} + \frac{B}{p(p-2)}\Big)(Y,Z) - \nabla_Y\Big(\frac{E}{p-2} + \frac{B}{p(p-2)}\Big)(X,Z).$$

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## CONEXIUNI RIEMANNIENE DEGENERATE ŞI CONFORME DEGENERATE

Rezumat

Se studiază problema integrabilităţii structurilor riemanniene degenerate cu ajutorul familiei de conexiuni compatibile cu aceste structuri. Se obţine un rezultat analog cu rezultatul clasic pentru structuri riemanniene. O structură riemanniană degenerată este integrabilă dacă şi numai dacă există o conexiune compatibilă cu această structură avînd tensorii de curbură şi torsiune nuli.

Pentru cazul structurilor conforme singulare se introduc nişte invarianţi care amintesc de tensorii Weyl ai structurilor conforme.

# AUTOMORFISME INFINITEZIMALE ALE UNEI $r$-$\pi$-STRUCTURI

DE

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Fie $V_n$ o varietate diferenţiabilă dotată cu o $r$-$\pi$-structură şi $\varphi$ tensorul fundamental al acestei structuri, deci $\overset{r}{\varphi} = \lambda^r I$ [2]. Vom numi automorfism infinitezimal al unei r-$\pi$-structuri un cîmp de vectori $X \in H(V_n)$ în raport cu care

$$L_X \varphi = 0, \tag{1}$$

unde $H(V_n)$ este algebra Lie a cîmourilor de vectori pe $V_n$, iar $L_X$ simbolul derivatei Lie în raport cu cîmpul $X$ *).

În cele ce urmează ne propunem să studiem cîteva proprietăţi ale acestor automorfisme.

**Teorema 1.** *Un cîmp de vectori $X \in H(V_n)$ este un automorfism infinitezimal al unei r-$\pi$-structuri dacă şi numai dacă*

$$[X, \varphi Y] = \varphi([X, Y]) \textit{ pentru orice } Y \in H(V_n). \tag{2}$$

([.] reprezintă croşetul a două cîmpuri de vectori).

*Demonstraţie.* Fie $X, Y \in H(V_n)$. Avem:

$$[X, \varphi Y] = L_X(\varphi Y) = (L_X \varphi) Y + \varphi([X, Y]). \tag{3}$$

Din (3) rezultă că $L_X \varphi = 0$ dacă şi numai dacă $[X, \varphi Y] = \varphi([X, Y])$, c. c. t. d.

*Observaţie.* Din proprietăţile derivatei Lie rezultă că $L_X \varphi = 0$ atrage şi

$$L_X \overset{r-1}{\varphi} = 0, \tag{4}$$

*) Pentru detalii vezi [1] şi [3].