11. Rimer D. — *La géométrie différentielle des variétés uniparamétriques de cônes quadratiques dans l'espace équiaffine à trois dimensions*. Bull. Math. de la Soc. Sci Math. de la R. S. de Roumaine, t. IX (57), fasc. 3 (1965), pp. 211—225.
12. Rimer D. — *Geometria diferențială a varietăților uniparametrice de conuri pătratice în spațiul euclidian cu trei dimensiuni*. An. Univ. Timișoara, vol. V (1967), pp. 257—279.
13. Rimer D. — *Variétés biparamétriques de cônes quadratiques dans l'espace projectif et équiaffine à trois dimensions*. An. șt. Univ. Iași, s. I, Matematică t. XIII, fasc. 2 (1967), pp. 349—365.
14. Rimer D. — *Congruences de cônes quadratiques en E_3*. An. șt. Univ. Iași, s. I, Matematică, t. XV, fasc. 1, (1969), pp. 121—131.
15. Rimer D. — *La géométrie différentielle des complexes de cônes quadratiques dans l'espaces projectif à trois dimensions*. An. șt. Univ. Iași, s. I, Matematică t. XIV, (1968), pp. 137—147.
16. Рютин В. Р. — *Инвариантные образы комплекса коник в евклидовом пространстве*. Геом. Сборник, Томск, т. 191 (1967), стр. 140—142.

COMPLEXE DE CONURI PĂTRATICE ÎN SPAȚIUL ECHIAFIN ȘI EUCLIDIAN TRIDIMENSIONAL

Rezumat

Continuînd lucrările [10—15], se studiază geometria diferențială a varietăților cu trei parametri de conuri pătratice în A_3 și E_3. Se folosește, în acest scop, metoda reperului mobil, a lui Cartan, aplicată imaginii complexului într-un spațiu proiectiv cu 9 dimensiuni. Se obține, în ambele cazuri, reperul canonic și sistemul complet de invarianți. Se dau interpretări geometrice reperului și invarianților, utilizînd complexul, imaginea sa în S_9, precum și diferite figuri asociate complexului. Dintre cazurile de excepție se cercetează complexe de conuri de rotație în E_3.

EXTENSION AND BENDING OF ORTHOTROPIC NATURALLY TWISTED BARS

BY

C. I. BORȘ

We shall consider a bar limited by two planes $x_3 = 0$, $x_3 = h$ and a surface $\mathfrak{F}$ given by the equation

$$f(x_1 - kx_2x_3,\ x_2 + kx_1x_3) = 0, \tag{1}$$

where k is a small parameter, the square and higher powers of which can be neglected. The region occupied by the bar will be denoted by $\mathfrak{V}$.

We shall name such bars *slightly twisted bars* [1], [5]*).

We will suppose that the tractions applied at the ends of the bar are statically equivalent either to a longitudinal force or to one bending couple. The surface $\mathfrak{F}$ is free from tractions. Also we suppose that there are no body forces.

1. General equations

If we make the transformation:

$$\xi = x_1 - kx_2x_3, \qquad \eta = x_2 + kx_1x_3, \qquad \zeta = x_3, \tag{2}$$

then the surface (1) becomes

$$f(\xi, \eta) = 0 \tag{3}$$

and it is a cylindrical surface $\mathfrak{F}_1$ in the space ξ, η, ζ.

*) The work of Riz [5] is quoted after Khatiashvili [3]. I do not know if the surfaces (1) were introduced in [5] or not. Anyway, in the work [5] the isotropic case was discussed.

We will denote by S the region and also the area of a normal section of the cylindrical bar (3) and by Γ the boundary of the region S.

The inverse transformation is given by:

$$x_1 = \xi + k\eta\zeta, \qquad x_2 = \eta - k\xi\zeta \qquad x_3 = \zeta. \tag{4}$$

It is easy to see that we have the following formulae

$$\begin{cases} \dfrac{\partial}{\partial x_1} = \dfrac{\partial}{\partial \xi} + k\zeta \dfrac{\partial}{\partial \eta}, \qquad \dfrac{\partial}{\partial x_2} = \dfrac{\partial}{\partial \eta} - k\zeta \dfrac{\partial}{\partial \xi}, \\ \dfrac{\partial}{\partial x_3} = \dfrac{\partial}{\partial \zeta} - k\left(\eta \dfrac{\partial}{\partial \xi} - \xi \dfrac{\partial}{\partial \eta}\right), \end{cases} \tag{5}$$

and

$$\begin{cases} n_1 = \cos\alpha + k\zeta\cos\beta, \quad n_2 = \cos\beta - k\zeta\cos\alpha, \\ n_3 = -k(\eta\cos\alpha - \xi\cos\beta), \end{cases} \tag{6}$$

where n_i are the direction cosines of the exterior normal to the surface (1) and $\cos\alpha$, $\cos\beta$ are the cosines of the exterior normal to the curve Γ.

We take the axis of ζ to be the central-line of the bar (3) and the axes of ξ and η to be the principal axes of inertia of the end $\zeta = 0$. In this case we have

$$\iint\limits_S \xi d\xi d\eta = 0, \qquad \iint\limits_S \eta d\xi d\eta = 0, \qquad \iint\limits_S \xi\eta d\xi d\eta = 0. \tag{7}$$

The stress components σ_{ij} must satisfy the equilibrium equations

$$\sigma_{ij,j} = 0 \quad \text{in } \mathcal{S} \tag{8}$$

and the boundary conditions

$$\sigma_{ij} n_j = 0 \quad \text{on } \mathcal{S}. \tag{9}$$

At the ends, the boundary conditions are dependent of the tractions applied at the ends.

The index j after comma, indicates partial differentiation with respect to x_j. We use also the summation convention over the repeated indices.

We shall suppose that the bar is made by one orthotropic material so that the Hooke's law is given in the form:

$$\begin{cases} \sigma_{11} = A\gamma_{11} + H\gamma_{22} + G\gamma_{33}, \\ \sigma_{22} = H\gamma_{11} + B\gamma_{22} + F\gamma_{33}, \\ \sigma_{33} = G\gamma_{11} + F\gamma_{22} + C\gamma_{33}; \end{cases} \tag{10 a}$$

$$\sigma_{23} = L\gamma_{23}, \quad \sigma_{31} = M\gamma_{31}, \quad \sigma_{12} = D\gamma_{12}, \tag{10 b}$$

where $A, B, \ldots, L, M, D$ are moduli of elasticity.

We shall write the inverse relations of (10 a) and (10 b) by the following formulae:

$$\begin{cases} \gamma_{11} = \dfrac{1}{E}(\nu_{11}\sigma_{11} + \nu_{12}\sigma_{22} - \sigma_1\sigma_{33}), \\ \gamma_{22} = \dfrac{1}{E}(\nu_{12}\sigma_{11} + \nu_{22}\sigma_{22} - \sigma_2\sigma_{33}), \\ \gamma_{33} = \dfrac{1}{E}(-\sigma_1\sigma_{11} - \sigma_2\sigma_{22} + \sigma_{33}); \end{cases} \tag{11 a}$$

$$\gamma_{23} = \frac{1}{L}\sigma_{23}, \quad \gamma_{31} = \frac{1}{M}\sigma_{31}, \quad \gamma_{12} = \frac{1}{D}\sigma_{12}. \tag{11 b}$$

The coefficients E, ν_{ij} and σ_i may be expressed in terms of moduli of elasticity as shown in [2].

In the formulae above we had denoted with γ_{ij} the technical components of strain given by

$$\gamma_{ii} = u_{i,i} \text{ (not summed)},$$

$$\gamma_{ij} = u_{i,j} + u_{j,i} \quad (i \neq j),$$

where u_i are the components of the displacement.

The components γ_{ij} must satisfy the compatibility conditions of Saint-Venant

$$\gamma_{11,22} + \gamma_{22,11} = \gamma_{12,12}, \tag{12 a}$$

. . . .

$$(-\gamma_{23,1} + \gamma_{31,2} + \gamma_{12,3})_{,1} = 2\gamma_{11,23}, \tag{12 b}$$

. . . .

2. Extension by a longitudinal force

Let the tractions applied at the end $x_3 = h$ be statically equivalent to a longitudinal force F_3, acting in the centroid of this end.

We will see that the problem of extension has an elementary solution. We try to find the solution in the form

$$u_1 = -\gamma\sigma_1\xi + ku_1^0, \quad u_2 = -\gamma\sigma_2\eta + ku_2^0, \quad u_3 = \gamma\zeta + ku_3^0, \tag{13}$$

where u_i^0 are complementary displacements to be determined and γ is a constant.

The stress components corresponding to the displacements (13) are given by

$$
(14)\quad \begin{cases} \sigma_{11} = k\tau_{11}, \quad \sigma_{22} = k\tau_{22}, \quad \sigma_{33} = \gamma E + k\tau_{33}, \\ \sigma_{23} = -k\gamma L\sigma_2\xi + k\tau_{23}, \\ \sigma_{31} = k\gamma M\sigma_1\eta + k\tau_{31}, \\ \sigma_{12} = k\gamma D(\sigma_1 - \sigma_2)\zeta + k\tau_{12}, \end{cases}
$$

where τ_{ij} are the stresses corresponding to the displacements u_i^0.

If we substitute σ_{ij} from (14) into (9), the boundary conditions take the form

$$
(15)\quad \begin{cases} \tau_{11}\cos\alpha + \tau_{12}\cos\beta = -\gamma D(\sigma_1 - \sigma_2)\zeta\cos\beta, \\ \tau_{12}\cos\alpha + \tau_{22}\cos\beta = -\gamma D(\sigma_1 - \sigma_2)\zeta\cos\alpha, \\ \tau_{31}\cos\alpha + \tau_{23}\cos\beta = -\gamma M\sigma_1\eta\cos\alpha + \gamma L\sigma_2\xi\cos\beta + \\ \qquad \gamma E(\eta\cos\alpha - \xi\cos\beta) \quad \text{on } \mathcal{F}_1. \end{cases}
$$

The equilibrium equations (8) require that

$$
(16)\quad \tau_{ij,j} = 0.
$$

Here the indices 1, 2, 3 after comma indicate partial differentiation with respect to ξ, η, ζ.

The equilibrium equations (16), the boundary conditions (15) and the compatibility conditions (12) will be satisfied if we take

$$
(17)\quad \begin{cases} \tau_{11} = 0, \quad \tau_{22} = 0, \quad \tau_{33} = 0, \quad \tau_{23} = \gamma(L\sigma_2 + E)\xi, \quad \tau_{31} = -\gamma(M\sigma_1 - E)\eta, \\ \qquad \tau_{12} = -\gamma D(\sigma_1 - \sigma_2)\zeta. \end{cases}
$$

Taking into account (14) and (17), we find that

$$
(18)\quad \begin{cases} \sigma_{11} = 0, \quad \sigma_{22} = 0, \quad \sigma_{33} = \gamma E, \\ \sigma_{23} = k\gamma E\xi, \quad \sigma_{31} = -k\gamma E\eta, \quad \sigma_{12} = 0. \end{cases}
$$

The end conditions at $x_3 = h$, require that

$$
\gamma = \frac{F}{SE}
$$

and to add at (18) the stresses corresponding to the torsion problem of cylindrical bar (3) by the torque

$$
\mathfrak{M}_3 = -k\iint_S E(\xi^2 + \eta^2)\,d\xi\,d\eta.
$$

It follows that the extension is accompanied by a twisting effect.

3. Bending due to couples

Let the bar be subjected to two equal and opposite couples $\mathfrak{M}_2$ acting is the plane of the ends.

Let us find the solution of the problem in the following form

$$
(19)\quad \begin{cases} u_1 = \frac{1}{2}a(\sigma_1\xi^2 - \sigma_2\eta^2 + \zeta^2) + kau_1^0, \\ u_2 = a\sigma_2\xi\eta + kau_2^0, \\ u_3 = -a\xi\zeta + kau_3^0, \end{cases}
$$

where u_i^0 are unknown functions of ξ, η, ζ and a is a constant which will be determined from the conditions at the end $x_3 = h$.

The stress components corresponding to displacements (19) are given by

$$
(20)\quad \begin{cases} \sigma_{11} = ka[G - \sigma_2(A + H)]\eta\zeta + ka\tau_{11}, \\ \sigma_{22} = ka[F - \sigma_2(H + B)]\eta\zeta + ka\tau_{22}, \\ \sigma_{33} = -aE\xi + ka[E + G(\sigma_1 - \sigma_2)]\eta\zeta + ka\tau_{33}, \\ \sigma_{23} = ka[L\sigma_2(\xi^2 - \eta^2) + L\zeta^2] + ka\tau_{23}, \\ \sigma_{31} = -kaM(\sigma_1 + \sigma_2)\xi\eta + ka\tau_{31}, \\ \sigma_{12} = -kaD(\sigma_1 - \sigma_2)\xi\zeta + ka\tau_{12}, \end{cases}
$$

where τ_{ij} are the stress components corresponding to u_i^0.

If we take into account (20), (9) and (8) we find that the components τ_{ij} must satisfy

$$
(21)\quad \begin{cases} \tau_{11}\cos\alpha + \tau_{12}\cos\beta = -[G - \sigma_2(A + H)]\eta\zeta\cos\alpha + D(\sigma_1 - \sigma_2)\xi\zeta\cos\beta, \\ \tau_{12}\cos\alpha + \tau_{22}\cos\beta = -[F - \sigma_2(H + B)]\eta\zeta\cos\beta + D(\sigma_1 - \sigma_2)\xi\zeta\cos\alpha, \\ \tau_{31}\cos\alpha + \tau_{23}\cos\beta = M(\sigma_1 + \sigma_2)\xi\eta\cos\alpha - L\sigma_2(\xi^2 - \eta^2)\cos\beta - \\ \qquad - L\zeta^2\cos\beta - E(\xi\eta\cos\alpha - \xi^2\cos\beta) \quad \text{on } \mathcal{F}_1 \end{cases}
$$

and the equations

$$
(22)\quad \begin{cases}
\dfrac{\partial \tau_{11}}{\partial \xi} + \dfrac{\partial \tau_{12}}{\partial \eta} + \dfrac{\partial \tau_{31}}{\partial \zeta} = 0, \\[2mm]
\dfrac{\partial \tau_{12}}{\partial \xi} + \dfrac{\partial \tau_{22}}{\partial \eta} + \dfrac{\partial \tau_{23}}{\partial \zeta} + [F - \sigma_2(H + B) - D(\sigma_1 - \sigma_2) + 2L]\,\zeta = 0, \\[2mm]
\dfrac{\partial \tau_{31}}{\partial \xi} + \dfrac{\partial \tau_{23}}{\partial \eta} + \dfrac{\partial \tau_{33}}{\partial \zeta} + [2E - 2L\sigma_2 - M(\sigma_1 + \sigma_2) + G(\sigma_1 - \sigma_2)]\,\eta = 0,
\end{cases}
$$

in the region occupied by the bar (3).

It is easy to see that if we take

$$
(23)\quad \begin{cases}
\tau_{11} = -[G - \sigma_2(A + H)]\,\eta\zeta, \\
\tau_{22} = -[F - \sigma_2(H + B)]\,\eta\zeta, \\
\tau_{33} = -[E + G(\sigma_1 - \sigma_2)]\,\eta\zeta, \\
\tau_{23} = L\dfrac{\partial \psi}{\partial \eta} - L\sigma_2(\xi^2 - \eta^2) - L\xi^2 + L\sigma_1\xi^2, \\
\tau_{31} = M\dfrac{\partial \psi}{\partial \xi} + M(\sigma_1 + \sigma_2)\xi\eta, \\
\tau_{12} = D(\sigma_1 - \sigma_2)\xi\zeta,
\end{cases}
$$

the first and the second equations (22) will be satisfied.

The third equation will be satisfied if the function ψ is such that

$$
(24)\quad M\frac{\partial^2 \psi}{\partial \xi^2} + L\frac{\partial^2 \psi}{\partial \eta^2} + E\eta = 0 \quad \text{in } S.
$$

Also, the conditions (21) will be satisfied if the function satisfies the condition

$$
(25)\quad M\frac{\partial \psi}{\partial \xi}\cos\alpha + L\frac{\partial \psi}{\partial \eta}\cos\beta = 2M\sigma_1\xi\eta\cos\alpha - E(\xi\eta\cos\alpha - \xi^2\cos\beta) \quad \text{on } \Gamma.
$$

It is easy to prove that the condition of existence for the function ψ is satisfied.

From (20) and (23) we have

$$
(26)\quad \begin{cases}
\sigma_{11} = 0, \quad \sigma_{22} = 0, \quad \sigma_{33} = -aE\xi, \quad \sigma_{12} = 0, \\[2mm]
\sigma_{23} = kaL\dfrac{\partial \psi}{\partial \eta} + kL\sigma_1\xi^2, \quad \sigma_{31} = kaM\dfrac{\partial \psi}{\partial \xi}.
\end{cases}
$$

The strain components corresponding to the stresses (26) verify the conditions of compatibility (12).

At the end $x_3 = h$ we must have one bending couple $\mathfrak{M}_2$. In this purpose first we must take

$$
a = \frac{\mathfrak{M}_2}{IE},
$$

where

$$
I = \iint_S \xi^2\, d\xi\, d\eta
$$

and after this, it remains to correct the boundary conditions at the end $x_3 = h$ by superposition of solutions of some adequate problems for the cylindrical bar (3) (see e.g. [2] or [4]).

REFERENCES

1. Borș C. I. — *Torsion of orthotropic naturally twisted bars*. Bull. Inst. Polit. Iași, t. 16 (20), fasc. 1—2, ser. 1, 1970.
2. Borș C. I. — *Teoria elasticității corpurilor anizotrope*. Edit. Acad. R. S. România, București 1970.
3. Khatiashvili G. M. — *Saint-Venant's problems for homogeneous anisotropic bodies approximately similar to prismatic*. Theory of Plates and Shells, Bratislava. 1966.
4. Lechnitzky S. G. — *Theory of elasticity of anisotropic body* (in Russian), Moskva-Leningrad, 1950.
5. Riz P. M. — *Strains and stresses of naturally twisted bars* (in Russian). Izv. Akad. Nauk SSSR, ser. math., 4, 1939.

ÎNTINDEREA ȘI ÎNCOVOIEREA BARELOR ORTOTROPE UȘOR RĂSUCITE

Rezumat

Se consideră o bară mărginită de două plane $x_3 = 0$, $x_3 = h$ și de o suprafață dată de (1). Pentru astfel de bare se studiază întinderea de către o forță axială și încovoierea de către cupluri.

În lucrare se arată cum se poate găsi o soluție care să verifice ecuațiile de echilibru și să satisfacă condițiile la limită (9). În cazul întinderii, soluția este elementară, iar în cazul încovoierii trebuie găsită o funcție ψ ce satisface ecuația (24) cu condițiile la limită (25).

Pentru satisfacerea condițiilor de la capătul $x_3 = h$, peste soluția găsită, trebuie suprapuse soluțiile unor probleme adecvate referitoare la bara cilindrică (3), probleme pe care știm să le rezolvăm.