ON THE THEORY OF COUPLED THERMOVISCOELASTICITY

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In this paper we consider the linear theory of coupled termoviscoelasticity for a homogeneous and isotropic solid and establish some general theorems: reciprocity theorems, a variational theorem and an uniqueness theorem.

1. Basic Equations. Throughout this paper we employ a rectangular coordinate system x_k and the usual indicial notation.

Let $\bar{V}$ be a regular region of the space occupied by a thermoviscoelastic material whose boundary is Σ. Let V be the interior of $\bar{V}, n_i$ the components of the unit outward normal to Σ and Σ_α $(\alpha = 1,2,3,4)$ subsets of Σ so that $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \Sigma$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \varnothing$.

The basic equations of the linear theory of coupled thermoviscoelasticity are [3, p. 347]:

equations of motion

$$t_{ji,j} + F_i = \rho \ddot{u}_i, \tag{1.1}$$

equation of entropy

$$T_0 \dot{\eta} = q_{i,i} + r, \tag{1.2}$$

constitutive equations

$$t_{ij} = \int_{-\infty}^{t} G_{ijkl}(t - \tau) \dot{e}_{kl}(x, \tau)\, d\tau - \beta T \delta_{ij}, \tag{1.3}$$

$$q_i = k T_{,i}, \tag{1.4}$$

(1.5) $$\eta = \beta e_{ii} + \frac{c}{T_0} T,$$

geometrical equations

(1.6) $$2e_{ij} = u_{i,j} + u_{j,i} \equiv 2u_{(i,j)}.$$

We have assumed that the medium is isotropic, i. e.

(1.7) $$G_{ijkl} = G_1(\delta_{ik}\delta_{jl} + \delta_{kj}\delta_{il}) + G_2\delta_{kl}\delta_{ij}.$$

In these relations, we have used the following notations: u_i- components of the displacement vector, t_{ij}-components of the stress tensor, F_i-components of the body force vector, η-entropy per unit volume measured from the entropy of the reference state, q_i-components of the heat flux vector, r-heat supplied per unit volume, T-temperature measured from the constant absolute temperature T_0 of the natural state, e_{ij}-components of the strain tensor, G_{ijkl}-components of the relaxation tensor, ρ, β, k, c-characteristic constants of the material, δ_{ij}-Kronecker's delta, the comma denotes partial derivation with respect to the space variables x_k and a dot denotes partial derivation with respect to the time t. It is known that

(1.8) $$k > 0, \quad c > 0.$$

We introduce the notion of an admissible state $S = \{u_i, q_i, e_{ij}, \eta, t_{ij}, T\}$ by which we mean an ordered set of functions $u_i, q_i, e_{ij}, \eta, t_{ij}, T$ defined on $\bar{V}\times(-\infty, \infty)$ with the following properties:

(1.9) $$u_i \in C^{1,0},\ q_i \in C^{1,0},\ e_{ij} \in C^{0,0},\ \eta \in C^{0,1},$$
$$t_{ij} \in C^{1,0},\ T \in C^{1,0},\ e_{ij} = e_{ji},\ t_{ij} = t_{ji},$$

where by $C^{m,n}$ we mean the notation from [6].

If we define addition of states and multiplication of a state by a scalar λ through

$$S + S' = \{u_i + u'_i, q_i + q'_i, ..., T + T'\},\ \lambda S = \{\lambda u_i, \lambda q_i, ..., \lambda T\},$$

then the set of admissible states is a linear space.

Since a viscoelastic material „remembers" its past history, we must prescribe u_i, e_{ij}, t_{ij} in the body up to some initial time t_0. These data consist of the functions $\{\overset{\circ}{u}_i, \overset{\circ}{e}_{ij}, \overset{\circ}{t}_{ij}\} = \overset{\circ}{S}$ defined on $V\times(-\infty, t_0)$; call $\overset{\circ}{S}$ the initial history. If $S_t = \{u_k, e_{kj}, t_{kj}\}$ on $V\times(-\infty, t_0)$, the initial history condition is

(1.10) $$S_t = \overset{\circ}{S},$$

Without loss of generality, we take $t_0 = 0$.

If $S = \{u_i, q_i, e_{ij}, \eta, t_{ij}, T\}$ is an admissible state which satisfies the initial history condition (1.10) then u_i and $\dot{u}_i$ automatically satisfies the initial conditions

(1.11) $$u_i(x,0) = \lim_{t\to 0-} \overset{\circ}{u}_i(x,t) \equiv a_i(x),\ \dot{u}_i(x,0) = \lim_{t\to 0-} \frac{d\overset{\circ}{u}(x,t)}{dt} \equiv b_i.$$

We adjoin the initial condition

$$\eta(x,0) = \eta_0(x),$$

and the boundary conditions

(1.13) $$\begin{aligned} u_i &= \tilde{u}_i, \quad \text{on} \quad \Sigma_1\times[0,\infty),\\ t_i &\equiv t_{ji}n_j = \tilde{t}_i, \quad \text{on} \quad \Sigma_2\times[0,\infty),\\ q &\equiv q_i n_i = \tilde{q}, \quad \text{on} \quad \Sigma_3\times[0,\infty),\\ T &= \tilde{T}, \quad \text{on} \quad \Sigma_4\times[0,\infty), \end{aligned}$$

where $\eta_0, \tilde{u}_i, \tilde{t}_i, \tilde{q}, \tilde{T}$ are prescribed functions and we have used the abbreviation $x = (x_1, x_2, x_3)$.

We introduce the notion of a thermoviscoelastic state $\mathcal{T} = \{u_i, q_i, e_{ij}, \eta, t_{ij}, T\}$ by which we mean an admissible state for which $u_i, q_i, e_{ij}, \eta, t_{ij}, T$ satisfy the field equations (1.1)—(1.6).

By a solution of the mixed problem we mean a thermoviscoelastic state which satisfies the initial history condition (1.10), the initial condition (1.12) and the boundary conditions (1.13).

2. Alternative Formulation of the Problem. Let v and ω be functions defined on $\bar{V}\times[0,\infty)$, continuous on $[0,\infty)$ with respect to the time t for each $x \in \bar{V}$. We denote by $v * \omega$ the convolution of v and ω

$$v * \omega = \int_0^t v(x, t-\tau)\omega(x,\tau)d\tau.$$

We will have to use the following well-known properties of the convolution

(2.1) $$v*\omega = \omega*v,\ v*\omega = 0 \text{ implies } v = 0 \text{ or } \omega = 0,$$
$$u*(v*\omega) = (u*v)*\omega = u*v*\omega,\ u*(v+\omega) = u*v + u*\omega.$$

Proceeding as in [12] we obtain

Lema 2.1. *$\mathcal{T}$ is a thermoviscoelastic state with the initial history $\overset{\circ}{S}$ if and only if $\mathcal{T}$ is admissible and satisfies the initial history condition (1.10), the equation (1.1), (1.2), (1.4)—(1.6) and*

(2.2)
$$t_{ij} = \bar{t}_{ij} + \frac{d}{dt}[G_{ijkl} * e_{kl}] - \beta T \delta_{ij},$$

where

(2.3)
$$\bar{t}_{ij} = \int_0^\infty \dot{G}_{ijkl}(t+\tau)\overset{\circ}{e}_{kl}(x_s, -\tau)d\tau .$$

Henceforth we will denote by l, g the functions defined on $[0,\infty)$ by

(2.4)
$$l(t) = 1, \; g(t) = t.$$

Let f_i, W be functions defined on $\bar{V}\times[0,\infty)$ by

(2.5)
$$f_i = g * F_i + \rho(tb_i + a_i), \quad W = l * r + T_0\eta_0 .$$

From [10], [6], [7], we have

Theorem 2.1. *Let* $u_i \in C^{0,2}$, $q_i \in C^{1,0}$, $\eta \in C^{0,1}$, $t_{ij} \in C^{1,0}$, $t_{ij} = t_{ji}$. *Then* u_i, q_i, η, t_{ij} *satisfy the equations* (1.1), (1.2) *and the initial conditions* (1.11), (1.12) *if and only if*

(2.6)
$$g * t_{ij,j} + f_i = \rho u_i ,$$

(2.7)
$$T_0 \eta = l * q_{i,i} + W.$$

Lemma 2.1 and Theorem 2.1 imply

Theorem 2.2. *Let S be an admissible state. Then S is a solution of the mixed problem if and only if S satisfies the equations* (2.6), (2.7), (2.2), (1.4)–(1.6), *the initial history condition* (1.10) *and the boundary conditions* (1.13).

3. Reciprocity Theorems. In this section we use the method given in [8], [9] to obtain the reciprocity theorems without using the Laplace transform.

We consider the body subjected to two different systems of thermoviscoelastic loadings

(3.1)
$$L^{(\alpha)} = \{F_i^{(\alpha)}, r^{(\alpha)}, \hat{u}_i^{(\alpha)}, \tilde{q}^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \tilde{T}^{(\alpha)}, \overset{\circ}{S}^{(\alpha)}, \eta_0^{(\alpha)}\}, \quad (\alpha = 1, 2),$$

and the two corresponding thermoviscoelastic configurations $C^{(\alpha)} = \{u_i^{(\alpha)}, T^{(\alpha)}\}$.

Theorem 3.1. *If a thermoviscoelastic solid is subjected to two different systems of thermoviscoelastic loadings $L^{(\alpha)}(\alpha = 1,2)$, then, between the corresponding configurations $C^{(\alpha)}(\alpha = 1,2)$ there is the following reciprocity relation*

$$\int_V \left[(f_i^{(1)} - \bar{f}_i^{(1)}) * u_i^{(2)} - \frac{1}{T_0} g * W^{(1)} * T^{(2)}\right] dx + \int_\Sigma g * \left[(t_i^{(1)} - \bar{t}_i^{(1)}) * u_i^{(2)} - \right.$$

(3.2)
$$\left. - \frac{1}{T_0} l * q^{(1)} * T^{(2)}\right] dx = \int_V \left[(f_i^{(2)} - \bar{f}_i^{(2)}) * u_i^{(1)} - \frac{1}{T_0} g * W^{(2)} * T^{(1)}\right] dx +$$
$$+ \int_\Sigma g * \left[(t_i^{(2)} - \bar{t}_i^{(2)}) * u_i^{(1)} - \frac{1}{T_0} l * q^{(2)} * T^{(1)}\right] dx ,$$

where

(3.3)
$$\bar{f}_i^{(\alpha)} = - g * \bar{t}_{ij,j}^{(\alpha)}, \quad \bar{t}_i^{(\alpha)} = \bar{t}_{ij}^{(\alpha)} n_j .$$

Proof. On the basis of relations (2.1), (1.7) we obtain

(3.4)
$$e_{ij}^{(1)} * \frac{d}{dt}[G_{ijkl} * e_{kl}^{(2)}] = e_{ij}^{(2)} * \frac{d}{dt}[G_{ijkl} * e_{kl}^{(1)}].$$

From (1.5), (2.2), (3.4) we get

(3.5)
$$[t_{ij}^{(1)} - \bar{t}_{ij}^{(1)} + \beta T^{(1)}\delta_{ij}] * e_{ij}^{(2)} = [t_{ij}^{(2)} - \bar{t}_{ij}^{(2)} + \beta T^{(2)}\delta_{ij}] * e_{ij}^{(1)},$$
$$[\beta e_{rr}^{(1)} - \eta^{(1)}] * T^{(2)} = [\beta e_{rr}^{(2)} - \eta^{(2)}] * T^{(1)}.$$

Adding these relations we obtain

(3.6)
$$(t_{ij}^{(1)} - \bar{t}_{ij}^{(1)}) * e_{ij}^{(2)} - \eta^{(1)} * T^{(2)} = (t_{ij}^{(2)} - \bar{t}_{ij}^{(2)}) * e_{ij}^{(1)} - \eta^{(2)} * T^{(1)}.$$

If we introduce the notation

(3.7)
$$L_{\alpha\beta} = \int_V g * [(t_{ij}^{(\alpha)} - \bar{t}_{ij}^{(\alpha)}) * e_{ij}^{(\beta)} - \eta^{(\alpha)} * T^{(\beta)}]dx, \quad (\alpha, \beta = 1, 2),$$

from (3.6) we have

(3.8)
$$L_{12} = L_{21} .$$

Using the relations (1.4), (1.6), (2.6), (2.7) we obtain

$$g * [(t_{ij}^{(\alpha)} - \bar{t}_{ij}^{(\beta)}) * e_{ij}^{(\alpha)} - \eta^{(\alpha)} * T^{(\beta)}] = g * (t_{ij}^{(\alpha)} - \bar{t}_{ij}^{(\alpha)}) * u_{i,j}^{(\beta)} +$$
$$- \frac{1}{T_0} g * l * q_{i,i}^{(\alpha)} * T^{(\beta)} - \frac{1}{T_0} g * W^{(\alpha)} * T^{(\beta)} = g * [(t_{ij}^{(\alpha)} - \bar{t}_{ij}^{(\alpha)}) * u_i^{(\beta)}]_{,j} -$$

(3.9)
$$- \frac{1}{T_0} g * l * (q_i^{(\alpha)} * T^{(\beta)})_{,i} + (f_i^{(\alpha)} - \bar{f}_i^{(\alpha)}) * u_i^{(\beta)} - \frac{1}{T_0} g * W^{(\alpha)} * T^{(\beta)} -$$
$$- \rho u_i^{(\alpha)} * u_i^{(\beta)} + \frac{k}{T_0} g * l * T_{,i}^{(\alpha)} * T_{,i}^{(\beta)},$$

so that

$$L_{\alpha\beta} = \int_V [(f_i^{(\alpha)} - \bar{f}_i^{(\alpha)}) * u_i^{(\beta)} - \frac{1}{T_0} g * W^{(\alpha)} * T^{(\beta)}] \, dx +$$

$$(3.10) \qquad + \int_\Sigma g * \left[(t_i^{(\alpha)} - \bar{t}_i^{(\alpha)}) * u_i^{(\beta)} - \frac{1}{T_0} g * l * q^{(\alpha)} * T^{(\beta)} \right] dx -$$

$$- \int_V \left[\rho u_i^{(\alpha)} * u_i^{(\beta)} - \frac{k}{T_0} g * l * T_{,i}^{(\alpha)} * T_{,i}^{(\beta)} \right] dx.$$

From (3.8), (3.10) we obtain (3.2).

If instead of the relation (1.5) we assume that $\eta = (c/T_0)T$, then we obtain the uncoupled problem of the linear thermoviscoelasticity.

We seek to obtain the relation between the thermoviscoelastic configurations $C^{(1)}$ and $C^{(2)}$ corresponding to the coupled problem $P^{(1)}$ and to the uncoupled problem $P^{(2)}$, respectively. We assume that $L^{(\alpha)}$ are the system of thermoviscoelastic loadings corresponding to the problems $P^{(\alpha)} (\alpha = 1, 2)$. By a proof analogous to that of Theorem 3.1 we obtain

Theorem 3.2. *If a thermoviscoelastic solid is subject to two different systems of loadings $L^{(\alpha)} (\alpha = 1, 2)$, then between the thermoviscoelastic configurations $C^{(\alpha)} (\alpha = 1, 2)$ corresponding to the problem $P^{(\alpha)} (\alpha = 1, 2)$, there exists the following reciprocity relation*

$$\int_V \left[(f_i^{(1)} - \bar{f}_i^{(1)}) * u_i^{(2)} - \frac{1}{T_0} g * W^{(1)} * T^{(2)} \right] dx + \int_\Sigma g * \left[(t_i^{(1)} - \bar{t}_i^{(1)}) * u_i^{(2)} - \right.$$

$$(3.11) \qquad \left. - \frac{1}{T_0} l * q^{(1)} * T^{(2)} \right] dx = \int_V \left[(f_i^{(2)} - \bar{f}_i^{(2)}) * u_i^{(1)} - \frac{1}{T_0} g * W^{(2)} * T^{(1)} \right] dx +$$

$$+ \int_\Sigma g * \left[(t_i^{(2)} - \bar{t}_i^{(2)}) * u_i^{(1)} - \frac{1}{T_0} l * q^{(2)} * T^{(1)} \right] dx - \beta \int_V g * e_{rr}^{(2)} * T^{(1)} dx.$$

In the following, we consider some applications of the preceding theorems, assuming $\Sigma_1 = \Sigma_3 = 0$.

Let us assume that the following loading system

$$(3.12) \qquad F_j^{(1)} = \delta(x - \xi)\delta(t)\delta_{ij}, \; r^{(1)} = \overset{\circ}{S}{}^{(1)} = \eta_0^{(1)} = 0,$$

where δ is Dirac function, will generate the displacement and thermal fields $U_j^{(i)}, T^{(i)} (i = 1, 2, 3)$. We also assume that the loading system

$$(3.13) \qquad F_j^{(1)} = 0, \; r^{(1)} = \delta(x - \xi)\delta(t), \; \overset{\circ}{S}{}^{(1)} = \eta_0^{(1)} = 0,$$

will generate the displacement and thermal fields $U^{(4)}, T^{(4)}$. Let $P_j^{(s)}, Q^{(s)}$ be the components of tractions and heat flux generated by $U^{(s)}, T^{(s)}$ on Σ, $(s = 1, 2, 3, 4)$.

If we use the following notations

$$(3.14) \qquad u_i = \tau^{(i)}, \quad l * T = - T_0 \tau^{(4)},$$

from (3.2) we obtain

Theorem 3.3. *The components of displacements vector and the temperature, u_i, T, corresponding to the system $L = \{F_i, r, \tilde{t}_i, \tilde{T}, \overset{\circ}{S}, \eta_0\}$, are given by the following relations*

$$[g * \tau^{(s)}](\xi, t) = \int_V \left[(f_k - \bar{f}_k) * U_k^{(s)} - \frac{1}{T_0} g * W * T^{(s)} \right] dx +$$

$$(3.15) \qquad + \int_\Sigma \left[g * (t_k - \bar{t}_k) * U_k^{(s)} - \frac{1}{T_0} l * q * T^{(s)} - g * P_k^{(s)} * u_k + \right.$$

$$\left. + \frac{1}{T_0} l * Q^{(s)} * T \right] dx.$$

If to the loadings systems (3.12), (3.13) we adjoin the boundary conditions

$$\tilde{t}_i^{(1)} = \tilde{T}^{(1)} = 0, \text{ on } \Sigma,$$

then the functions $U_j^{(s)}, T^{(s)} (s = 1, 2, 3, 4)$ are „Green functions" corresponding to the domain considered.

Let us consider a consequence of the Theorem 3.2. We assume that the uncoupled problem $P^{(2)}$ corresponds to the following thermoviscoelastic loading system

$$(3.16) \qquad F_i^{(2)} = 0, \quad r^{(2)} = \delta(x - \xi)\delta(t), \quad \tilde{t}_i^{(2)} = \tilde{T}^{(2)} = \overset{\circ}{S}{}^{(2)} = \eta_0^{(2)} = 0.$$

Let U_j, Θ, Q be the components of displacement vector, temperature and heat flux on Σ generated by the loading system (3.16). Using the relation (3.11), we obtain

Theorem 3.4. *The problem of coupled thermoviscoelasticity corresponding to the loading system $L = \{F_i, r, \tilde{t}_i, \tilde{T}, \overset{\circ}{S} = \eta_0 = 0\}$ is reduced to solving an associated problem of uncoupled thermoviscoelasticity corresponding to the loading (3.16) and to solving the following integral equation*

$$T(\xi,t) = -T_0\int_V\left(F_i * \dot{U}_i - \frac{1}{T_0} r * \Theta\right)dx - T_0\int_{\dot{\Sigma}}\left(\widetilde{t}_i * \dot{U}_i + \right.$$

(3.17)

$$\left. + \frac{1}{T_0} l * \widetilde{T} * Q\right)dx - \beta\int_{\dot{V}} g * \dot{U}_{r,r} * T dx.$$

These theorems generalize the results from [11].

4. Variational Theorem. In [6], Gurtin deduces variational principles which fully characterize the solution of the mixed problem of linear elastodynamics. We have extended [7] these principles to the dynamic theory of coupled thermoelasticity. In [12], the variational principles established in [5], [6] were extended to dynamic viscoelasticity theory.

In what follows we give a variational theorem in the linear theory of thermoviscoelasticity.

Let $\mathfrak{L}$ be a linear space, $\mathfrak{K}$ a subset of $\mathfrak{L}$ and Ω a functional defined on $\mathfrak{K}$. Further, let

(4.1) $$y, y' \in \mathfrak{L},\ y + \lambda y' \in \mathfrak{K},\ \text{for every real } \lambda,$$

and formally define $\delta_{y'}\Omega\{y\}$ on $\mathfrak{K}$ by

(4.2) $$\delta_{y'}\Omega\{y\} = \frac{d}{d\lambda}\Omega\{y + \lambda y'\}\big|_{\lambda=0}.$$

We say that the variation of Ω is zero at y over $\mathfrak{K}$, and write

(4.3) $$\delta\Omega\{y\} = 0,\ \text{over } \mathfrak{K},$$

if and only if, $\delta_{y'}\Omega\{y\}$ exists and equals zero for all y' consistent with (4.1).

Theorem 4.1. *Let $\mathfrak{M}$ be the set of all admissible states that satisfy the initial history condition (1.10). Let $S = \{u_i, q_i, e_{ij}, \eta, t_{ij}, T\} \in \mathfrak{M}$ and for each $t \in [0, \infty]$ define the functional $\Omega_t\{S\}$ on $\mathfrak{M}$ through*

$$\Omega_t\{S\} = \frac{1}{2}\int_V\left[l * G_{ijkl}\, e_{ij} * e_{kl} + g * \frac{T_0}{c}(\eta - \beta e_{rr}) * (\eta - \beta e_{rr})\right]dx -$$

(4.4) $$-\int_V g * [(t_{ij} - \bar{t}_{ij}) * e_{ij} + \eta * T]dx + \frac{1}{2}\int_{\dot{V}} \rho u_i * u_i dx -$$

$$-\int_{\dot{V}} (g * t_{ij,j} + f_i) * u_i dx + \frac{1}{T_0}\int_{\dot{V}} g * W * T dx +$$

$$+ \frac{1}{kT_0}\int_{\ddot{A}} g * l * q_i * \left(\frac{1}{2} q_i - kT_{,i}\right)dx + \int_{\Sigma_1} g * t_i * \widetilde{u}_i dx +$$

$$+ \int_{\dot{\Sigma}_2} g * (t_i - \widetilde{t}_i) * u_i dx + \frac{1}{T_0}\int_{\dot{\Sigma}_3} g * l * T * \widetilde{q} dx + \frac{1}{T_0}\int_{\dot{\Sigma}_4} g * l * (T - \widetilde{T}) * q dx.$$

Then

(4.5) $$\delta\Omega_t\{S\} = 0,\ \textit{over } \mathfrak{M},\ t \in [0, \infty),$$

if, and only if, S is a solution of the mixed problem.

Proof. Let $S' = \{u_i', q_i', e_{ij}', \eta', t_{ij}', T'\}$ be chosen so that $S + \lambda S' \in \mathfrak{M}$, for every real λ. *Then* (4.4), (4.2) and the divergence theorem imply

$$\delta_{S'}\Omega_t\{S\} = \int_V g * \left[\frac{d}{dt}(G_{ijkl} * e_{kl}) - (t_{ij} - \bar{t}_{ij}) - \beta\frac{T_0\delta_{ij}}{c}(\eta - \beta e_{rr})\right] * e_{ij}' dx$$

$$+ \int_V g * \left[\frac{T_0}{c}(\eta - \beta e_{rr}) - T\right] * \eta' dx - \int_{\dot{V}} (g * t_{ij,j} + f_i - \rho u_i) * u_i' dx +$$

(4.6) $$+ \frac{1}{T_0}\int_V g * (W - T_0\eta + l * q_{i,i})dx + \frac{1}{kT_0}\int_{\dot{V}} g * l * (q_i - kT_{,i}) * q_i' dx +$$

$$+ \int_V g * (u_{(i,j)} - e_{ij}) * t_{ij}' dx - \int_{\Sigma_1} g * (u_i - \widetilde{u}_i) * t_i' dx +$$

$$+ \int_{\dot{\Sigma}_2} g * (t_i - \widetilde{t}_i) * u_i' dx - \frac{1}{T_0}\int_{\dot{\Sigma}_3} g * l * (q - \widetilde{q}) * T' dx + \frac{1}{T_0}\int_{\dot{\Sigma}_4} g * l * (T - \widetilde{T}) * q' dx.$$

First suppose that S is a solution of the mixed problem. Then by Theorem 2.2, (4.6) and the choice of S', (4.5) holds and the „if" part of the theorem is proved. Conversely, to show that the „only if" part is true, assume that (4.5) holds. Then

(4.7) $$\delta_{S'}\Omega_t\{S\} = 0,\ t \in [0, \infty),$$

for every S' such that $S + \lambda S' \in \mathfrak{M}$ for all real λ.

Using the Lemmas established by Gurtin in [6], as in [6], [12], [7], we can show that S satisfies the equations (2.6), (2.7), (2.2), (1.4)—(1.6), the initial history condition (1.10) and the boundary conditions (1.13). From Theorem 2.2 we conclude that S is a solution of the mixed problem.

5. Uniqueness Theorem. Theorem 5.1. *Suppose that $G_{ijkl}(t)$ is initially positive definite and possesses a Laplace transform.*
Then there is at most one solution of the mixed problem which has a Laplace transform with respect to t.

Proof. Let us consider two solutions of the stated problem $u_i^{(\alpha)}, q_i^{(\alpha)}, e_{ij}^{(\alpha)}, \eta^{(\alpha)}, t_{ij}^{(\alpha)}, T^{(\alpha)}$ $(\alpha = 1, 2)$ and let

$$u_i = u_i^{(1)} - u_i^{(2)},\ q_i = q_i^{(1)} - q_i^{(2)}, \dots, T = T^{(1)} - T^{(2)}. \tag{5.1}$$

According to the linearity of the problem, the differences (5.1) satisfy the mixed problem corresponding to null data.

Following [1], [4] we take the Laplace transform of the field equations and boundary conditions satisfied by the state $\{u_i, q_i, e_{ij}, \eta, t_{ij}, T\}$. We denote by $f^*(x, s)$ the Laplace transform with respect to t of the function $f(x, t)$.

From (1.1), (1.2), (2.2), (1.4)—(1.6) we obtain

$$t^*_{ji,j} = \rho s^2 u^*_i, \tag{5.2}$$

$$T_0 s \eta^* = q^*_{i,i}, \tag{5.3}$$

$$t^*_{ij} = s G^*_{ijkl} e^*_{kl} - \beta T^* \delta_{ij}, \tag{5.4}$$

$$q^*_i = k T^*_{,i}, \tag{5.5}$$

$$\eta^* = \beta e^*_{rr} + \frac{c}{T_0} T^*, \tag{5.6}$$

$$2e^*_{ij} = u^*_{i,j} + u^*_{j,i}. \tag{5.7}$$

Using (5.2)—(5.7) we obtain

$$\begin{aligned} t^*_{ij} e^*_{ij} + T^* \eta^* &= t^*_{ij} u^*_{i,j} + \frac{1}{T_0 s} T^* q^*_{i,i} = \\ &= (t^*_{ij} u^*_i)_{,j} + \frac{1}{T_0 s} (T^* q^*_i)_{,i} - \rho s^2 u^*_i u^*_i - \frac{1}{T_0 s} q^*_i T^*_{,i}, \end{aligned} \tag{5.8}$$

$$t^*_{ij} e^*_{ij} + T^* \eta^* = s G^*_{ijkl} e^*_{ij} e^*_{kl} + \frac{c}{T_0} T^{*2}. \tag{5.9}$$

From (5.8), (5.9) we get

$$\int_V s^2 G^*_{ijkl} e^*_{ij} e^*_{kl}\, dx + \frac{c}{T_0} \int_V s T^{*2} dx + \rho \int_V s^3 u^*_i u^*_i\, dx = -\frac{k}{T_0} \int_V T^*_{,i} T^*_{,i}\, dx \leq 0. \tag{5.10}$$

As in [4], we can prove that there exists a number $s_1 > 0$ such that if $s \geq s_1$, the integrals on the left-hand side of (5.10) are non-negative. Hence

$$u^*_i(x, s) = 0,\quad T^*(x, s) = 0, \text{ on } V \times [s_1, \infty).$$

This implies [2, p. 74] that

$$u_i(x, t) = 0,\quad T(x, t) = 0, \text{ on } \bar{V} \times [0, \infty).$$

The proof is complete.

This theorem generalises Weiner's uniqueness theorem from the classical thermoelasticity [13].

The analogous theorems are derived in [9] where we have assumed another form of the constitutive equations.

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ASUPRA TEORIEI TERMOVISCOELASTICITĂŢII CUPLATE

Rezumat

Se consideră ecuaţiile teoriei liniare a termoviscoelasticităţii cuplate în cazul mediilor omogene şi izotrope şi se stabilesc unele teoreme generale: teoreme de reciprocitate, o teoremă variaţională şi o teoremă de unicitate a soluţiei.