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ECHILIBRUL ELASTO-PLASTIC AL SFEREI PLINE ORTOTROPE

Rezumat

Se studiază echilibrul elasto-plastic al unei sfere pline ortotrope, sub acţiunea presiunii exterioare uniform distribuite pe suprafaţa sa. Tensiunile, deformările şi deplasarea radială în zona elastică $0 \leq r \leq \rho$ sînt date de (2.5), (2.6) şi (2.7). Tensiunile în zona plastică $\rho \leq r \leq a$ sînt date de (2.17)

EXACT SOLUTIONS OF THE ONE-DIMENSIONAL NONSTEADY FLOW OF IDEAL ELASTIC FLUIDS

BY

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1. Introduction. The classical linear theories of mechanics and physics use some general representations for the solutions of the underlying partial differential equations. The fundamental equations become more complicated by generalizing them to include chemical reactions and electromagnetic phenomena, or when one considers substances with unfamiliar equations of state.

In this paper we deal with the nonsteady one-dimensional compressible flow. A survey of this subject has been made by O. Zaldastani [1] and an unified account has been given by R. von Mises [2] ch. III. In the case of the anisentropic flows the general solution has been constructed by M. H. Martin and G. S. S. Ludford [3], for the equations of state that include Karmen-Tsien's approximation to ideal gas. J. A. Giese [4, 5] gives an unified study of the above problem, and finds a general parametric representation of fluids.

Here we intend to find a general solution of the one-dimensional flow for some classes of elastic ideal fluids and to construct a solution for some boundary-value problems using the general representation of the solution. We consider an ideal elastic fluid with the equation of state

$$p = p(\rho), \tag{1.1}$$

where p is the pressure, ρ is the density of the fluid, and $p(\rho)$ is a C^1 function. Newton's equation and the equation of continuity may be written in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{1.2}$$

where $u(x, t)$ is the velocity, x is the abscisse in the direction of motion, and t is a time parameter. Using the one-to-one transformation

$$v = \int_{\rho_1}^{\rho} \frac{a\,d\rho}{\rho}, \tag{1.3}$$

where $a = \sqrt{dp/d\rho}$ is the local sound velocity, we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial v}{\partial x} = 0, \quad a\frac{\partial u}{\partial x} + \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = 0. \tag{1.4}$$

If we interchange the dependent and independent variables, the transformed equations (1.4) are:

$$\frac{\partial}{\partial v}(x - tu) + \frac{a}{\rho}\frac{\partial}{\partial u}(t\rho) = 0, \quad \frac{\partial}{\partial u}(x - tu) + \frac{a}{\rho}\frac{\partial}{\partial v}(t\rho) = 0. \tag{1.5}$$

The first equation implies the existence of a function $z(u, v)$, such that:

$$x - tu = \frac{\partial z}{\partial u}, \quad at = -\frac{\partial z}{\partial v}. \tag{1.6}$$

Now using (1.6), Eq. $(1.5)_2$ becomes

$$\frac{\partial^2 z}{\partial v^2} - \frac{\partial^2 z}{\partial u^2} = -\frac{1}{a}\left(1 - \frac{da}{dv}\right)\frac{\partial z}{\partial v}, \tag{1.7}$$

where $z(u, v)$ is a C^2-function. This equation has been studied by R. von M i s e s ([2], p. 166) in the isentropic case ($v = 5a$, $(-1/a)[1 - da/dv] = -4/5$) and its general solution can be written in the form

$$z = \frac{\varphi(\xi) + \psi(\eta)}{v^3} - \frac{\varphi'(\xi) + \psi'(\eta)}{v^2}. \tag{1.8}$$

Here $\varphi(\xi)$ and $\psi(\eta)$ are arbitrary C^3 functions and the independent variables are given by

$$\xi = v + u, \quad \eta = v - u. \tag{1.9}$$

If we apply the one-to-one transformation (1.9), the equation (1.7) becomes:

$$\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{1}{4a}\left(1 - \frac{da}{dv}\right)\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) = 0, \text{ or } \quad s + (p + q)\cdot A(\xi + \eta) = 0, \tag{1.10}$$

where

$$A(\xi + \eta) = \frac{1}{4a}\left(1 - \frac{da}{dv}\right). \tag{1.11}$$

The differential equations of the characteristics of the system (1.4) have the form

$$\frac{dx}{dt} = u \pm a. \tag{1.12}$$

In the u, v-plane, the characteristics of the system (1.4) are given by

$$v + u = \xi_1 = \text{const.}, \quad v - u = \eta_1 = \text{const.} \tag{1.12'}$$

The study of the nonsteady one-dimensional flow of the elastic ideal fluids is confined to the integration of the equation (1.10). In the following sections we shall determine the class of elastic fluids (1.1) for which the general solution of the equation (1.10) has the form

$$z = f\,[\xi, \eta, \varphi(\xi), \varphi'(\xi), \ldots, \varphi^{(n)}(\xi), \psi(\eta), \psi'(\eta), \ldots, \psi^{(m)}(\eta)], \tag{1.13}$$

where $\varphi(\xi)$ is a C^{n+2}-function and $\psi(\eta)$ is a C^{m+2}-function.

2. The classes of equations with explicit general solution. Let us take

$$f_\xi = \frac{\partial f}{\partial \xi}, \quad f_\varphi = \frac{\partial f}{\partial \varphi}, \ldots, f_{\varphi^{(n)}} = \frac{\partial f}{\partial \varphi^{(n)}}, \quad f_\eta = \frac{\partial f}{\partial \eta},$$

$$f_\psi = \frac{\partial f}{\partial \psi}, \ldots, f_{\psi^{(m)}} = \frac{\partial f}{\partial \psi^{(m)}},$$

where f is a C^3-function. If we differentiate (1.13) and introduce in (1.10) we obtain

$$\begin{aligned} f_{\xi\eta} &+ f_{\xi\psi}\psi' + \cdots + f_{\xi\psi^{(m)}}\psi^{(m+1)} + f_{\varphi\eta}\varphi' + f_{\varphi\psi}\varphi'\psi' + \cdots + f_{\varphi\psi^{(m)}}\varphi'\psi^{(m+1)} + \\ &+ \cdots + f_{\varphi^{(n)}\eta}\varphi^{(n+1)} + f_{\varphi^{(n)}\psi}\varphi^{(n+1)}\psi' + \cdots + f_{\varphi^{(n)}\psi^{(m)}}\varphi^{(n+1)}\psi^{(m+1)} + \\ &+ A\,(f_\xi + f_\varphi\varphi' + \cdots + f_{\varphi^{(n)}}\varphi^{(n+1)} + f_\eta + f_\psi\psi' + \cdots + f_{\psi^{(m)}}\psi^{(m+1)}) = 0. \end{aligned} \tag{2.1}$$

From (2.1) we see that

$$\begin{aligned} &f_{\varphi^{(n)}\psi^{(m)}} = 0, \\ &f_{\xi\psi^{(m)}} + f_{\varphi\psi^{(m)}}\varphi' + \cdots + f_{\varphi^{(n-1)}\psi^{(m)}}\varphi^{(n)} + Af_{\psi^{(m)}} = 0, \\ &f_{\eta\varphi^{(n)}} + f_{\psi\varphi^{(n)}}\psi' + \cdots + f_{\psi^{(m-1)}\varphi^{(n)}}\psi^{(m)} + Af_{\varphi^{(n)}} = 0, \end{aligned} \tag{2.2}$$

$$f_{\xi\eta}+f_{\xi\psi}\psi'+\cdots+f_{\xi\psi^{(m-1)}}\psi^{(m)}+f_{\varphi\eta}\varphi'+f_{\varphi\psi}\varphi'\psi'+\cdots+$$
$$+f_{\varphi\psi^{(m-1)}}\varphi'\psi^{(m)}+\cdots+f_{\varphi^{(n-1)}\eta}\varphi^{(n)}+f_{\varphi^{(n-1)}\psi}\varphi^{(n)}\psi'+\cdots+$$
$$+f_{\varphi^{(n-1)}\psi^{(m-1)}}\varphi^{(n)}\psi^{(m)}+A\,(f_{\xi}+f_{\varphi}\varphi'+\cdots+$$
$$+f_{\varphi^{(n-1)}}\varphi^{(n)}+f_{\eta}+f_{\psi}\psi'+\cdots+f_{\psi^{(m-1)}}\psi^{(m)})=0.$$

Differentiating $(2.2)_2$ with respect to $\varphi^{(n)},\varphi^{(n-1)},\ldots,\varphi'$ and φ, and $(2.2)_3$ with respect to $\psi^{(m)},\psi^{(m-1)},\ldots,\psi'$ and ψ respectively, we have

$$f_{\eta\varphi^{(n)}}+Af_{\varphi^{(n)}}=0,\quad f_{\xi\psi^{(m)}}+Af_{\psi^{(m)}}=0,$$

(2.3)
$$f_{\xi\eta}+f_{\xi\psi}\psi'+\cdots+f_{\xi\psi^{(m-1)}}\psi^{(m)}+f_{\eta\varphi}\varphi'+f_{\eta\varphi'}\varphi''+\cdots+f_{\eta\varphi^{(n-1)}}\varphi^{(n)}+$$
$$+A(f_{\xi}+f_{\varphi}\varphi'+\cdots+f_{\varphi^{(n-1)}}\varphi^{(n)}+f_{\eta}+f_{\psi}\psi'+\cdots+f_{\psi^{(m-1)}}\psi^{(m)})=0$$

(2.4)
$$f_{\varphi^{(n)}\psi^{(m)}}=f_{\varphi^{(n-1)}\psi^{(m)}}=\cdots=f_{\varphi\psi^{(m)}}=0,$$
$$f_{\varphi^{(n)}\psi^{(m-1)}}=f_{\varphi^{(n)}\psi^{(m-2)}}=\cdots=f_{\varphi^{(n)}\psi}=0.$$

From (2.4), it results that f must have the form

(2.5)
$$f=g(\xi,\eta,\varphi,\varphi',\ldots,\varphi^{(n)})+h(\xi,\eta,\psi,\psi',\ldots,\psi^{(m)}),$$

where g and h are C^3-functions. Using (2.5), the relations (2.3) take the from

$$g_{\eta\varphi^{(n)}}+Ag_{\varphi^{(n)}}=0,\quad h_{\xi\psi^{(m)}}+Ah_{\psi^{(m)}}=0,$$

(2.6)
$$g_{\xi\eta}+h_{\xi\eta}+g_{\eta\varphi}\varphi'+\cdots+g_{\eta\varphi^{(n-1)}}\varphi^{(n)}+h_{\xi\psi}\psi'+\cdots+h_{\xi\psi^{(m-1)}}\psi^{(m)}+$$
$$+A(g_{\xi}+h_{\xi}+g_{\varphi}\varphi'+\cdots+g_{\varphi^{(n-1)}}\varphi^{(n)}+g_{\eta}+h_{\eta}+h_{\psi}\psi'+$$
$$+\cdots+h_{\psi^{(m-1)}}\psi^{(m)})=0.$$

If we integrate $(2.6)_1$ and $(2.6)_2$, we obtain

(2.7)
$$g_{\eta}+Ag=C_1(\xi,\eta,\varphi,\varphi',\ldots,\varphi^{(n-1)}),$$
$$h_{\xi}+Ah=C_2(\xi,\eta,\psi,\psi',\ldots,\psi^{(m-1)}),$$

where C_1 and C_2 are arbitrary C^3-functions of their arguments.

Now, we can write $(2.6)_3$, in the form

(2.8)
$$C_{1\xi}+C_{1\varphi}\varphi'+\cdots+C_{1\varphi^{(n-1)}}\varphi^{(n)}+C_{2\eta}+C_{2\psi}\psi'+\cdots+$$
$$+C_{2\psi^{(m-1)}}\psi^{(m)}+A\,(C_1+C_2)-(A'+A^2)\,(g+h)=0.$$

If we differentiate (2.8) with respect to $\varphi^{(n)}$ and $\psi^{(m)}$, we have the following relations:

(2.9)
$$C_{1\varphi^{(n-1)}}=(A'+A^2)g_{\varphi^{(n)}},\quad C_{2\psi^{(m-1)}}=(A'+A^2)h_{\psi^{(m)}}.$$

a) *Suppose that* $A'+A^2\neq 0$. Integrating (2.9), we obtain

(2.10)
$$g=\frac{1}{A'+A^2}C_{1\varphi^{(n-1)}}\varphi^{(n)}+C_3(\xi,\eta,\varphi,\varphi',\ldots,\varphi^{(n-1)}),$$
$$h=\frac{1}{A'+A^2}C_{2\psi^{(m-1)}}\psi^{(m)}+C_4(\xi,\eta,\psi,\psi',\ldots,\psi^{(m-1)}).$$

Here, C_3 and C_4 are arbitrary C^3 functions. By replacing (2.10) into (2.8), we can write

(2.11)
$$C_{1\xi}+C_{1\varphi}\varphi'+\cdots+C_{1\varphi^{(n-2)}}\varphi^{(n-1)}+C_{2\eta}+C_{2\psi}\psi'+\cdots+$$
$$+C_{2\psi^{(m-2)}}\psi^{(m-1)}+A(C_1+C_2)-(A'+A^2)\,(C_3+C_4)=0.$$

Substituting (2.10) into (2.7) and differentiating the obtained relations with respect to $\varphi^{(n)}$ and $\psi^{(m)}$, we have

(2.12)
$$C_{1\eta\varphi^{(n-1)}}+\frac{A^3-AA'-A''}{A'+A^2}C_{1\varphi^{(n-1)}}=0,$$
$$C_{2\xi\psi^{(m-1)}}+\frac{A^3-AA'-A''}{A'+A^2}C_{2\psi^{(m-1)}}=0,$$
$$C_{3\eta}+AC_3=0,\quad C_{4\xi}+AC_4=0.$$

The equations $(2.12)_1$ and $(2.12)_2$ become, by integration

(2.13)
$$C_{1\eta}+\frac{A^3-AA'-A''}{A'+A^2}C_1=a_1(\xi,\eta,\varphi,\varphi',\ldots,\varphi^{(n-2)}),$$
$$C_{2\xi}+\frac{A^3-AA'-A''}{A'+A^2}C_2=a_2(\xi,\eta,\psi,\psi',\ldots,\psi^{(m-2)}),$$

where a_1 and a_2 are arbitrary functions of their arguments. The equations (2.13) have the solutions

(2.14)
$$C_1=b_1(\xi,\varphi,\ldots,\varphi^{(n-1)})\exp\int_{\tau_1}^{\tau}\frac{A''+AA'-A^3}{A'+A^2}d\tau+H_1(\xi,\eta,\varphi,\ldots,\varphi^{(n-2)}),$$
$$C_2=b_2(\eta,\psi,\ldots,\psi^{(m-1)})\exp\int_{\tau_1}^{\tau}\frac{A''+AA'-A^3}{A'+A^2}d\tau+H_2(\xi,\eta,\psi,\ldots,\psi^{(m-2)}).$$

Here b_1 and b_2 are arbitrary functions, $\tau=2v$ and H_1 and H_2 are given by

$$H_1 = \left[\int_{\eta_1}^{\eta} a_1 \exp\left\{\int_{\eta_1}^{\eta} \frac{A^3 - AA' - A''}{A' + A^2}\, d\eta\right\} d\eta\right] \exp\int_{\eta_1}^{\eta} \frac{A'' + AA' - A^3}{A' + A^2}\, d\eta,$$

$$H_2 = \left[\int_{\xi_1}^{\xi} a_2 \exp\left\{\int_{\xi_1}^{\xi} \frac{A^3 - AA' - A''}{A' + A^2}\, d\xi\right\} d\xi\right] \exp\int_{\xi_1}^{\xi} \frac{A'' + AA' - A^3}{A' + A^2}\, d\xi.$$

If we take into consideration the arguments of the functions C_1, C_2, C_3, and C_4 we see that the relation (2.11) becomes

$$AC_1 + C_{1\xi} + C_{1\varphi}\varphi' + \cdots + C_{1\varphi^{(n-2)}}\varphi^{(n-1)} - (A' + A^2)\, C_3 = -AC_2 -$$
$$- C_{2\eta} - C_{2\psi}\psi' - \cdots - C_{2\psi^{(m-2)}}\psi^{(m-1)} + (A' + A^2)\, C_4 = \alpha(\xi, \eta),$$

where $\alpha(\xi, \eta)$ is an arbitrary function of its arguments. Hence, we find

$$\text{(2.15)}\quad \begin{aligned} C_3 &= \frac{1}{A' + A^2}\left[AC_1 + C_{1\xi} + C_{1\varphi}\varphi' + \cdots + C_{1\varphi^{(n-2)}}\varphi^{(n-1)} - \alpha(\xi, \eta)\right], \\ C_4 &= \frac{1}{A' + A^2}\left[AC_2 + C_{2\eta} + C_{2\psi}\psi' + \cdots + C_{2\psi^{(m-2)}}\psi^{(m-1)} + \alpha(\xi, \eta)\right]. \end{aligned}$$

Using (2.14) and (2.15), the equations $(2.12)_3$ and $(2.12)_4$ become

$$\text{(2.16)}\quad \begin{aligned} &\left\{\left[\frac{(A' + A^2)'}{A' + A^2}\right]' - (A' + A^2)\right\} b_1 \exp\int_{\tau_1}^{\tau} \frac{A'' + AA' - A^3}{A' + A^2}\, d\tau + \\ &+ \frac{A^3 - AA' - A''}{A' + A^2}\left[H_{1\xi} + H_{1\varphi}\varphi' + \cdots + H_{1\varphi^{(n-2)}}\varphi^{(n-1)} - \alpha\right] + AH_{1\eta} - \\ &- \frac{A(A' + A^2)'}{A' + A^2} H_1 + H_{1\xi\eta} + H_{1\eta\varphi}\varphi' + \cdots + H_{1\eta\varphi^{(n-2)}}\varphi^{(n-1)} - \alpha_\eta = 0, \\ &\left\{\left[\frac{(A' + A^2)'}{A' + A^2}\right]' - (A' + A^2)\right\} b_2 \exp\int_{\tau_1}^{\tau} \frac{A'' + AA' - A^3}{A' + A^2}\, d\tau + \\ &+ \frac{A^3 - AA' - A''}{A' + A^2}\left[H_{2\eta} + H_{2\psi}\psi' + \cdots + H_{2\psi^{(m-2)}}\psi^{(m-1)} + \alpha\right] + AH_{2\xi} - \\ &- \frac{A(A' + A^2)'}{A' + A^2} H_2 + H_{2\xi\eta} + H_{2\xi\psi}\psi' + \cdots + H_{2\xi\psi^{(m-2)}}\psi^{(m-1)} + \alpha_\xi = 0. \end{aligned}$$

Differentiating these relations with respect to $\varphi^{(n-1)}$ and $\psi^{(m-1)}$, and remarking that the functions b_1 and b_2 depend on $\varphi^{(n-1)}$ and $\psi^{(m-1)}$ respectively, we obtain

$$\text{(2.17)}\quad \left[\frac{(A' + A^2)'}{A' + A^2}\right]' - (A' + A^2) = 0.$$

Hence, if we integrate (2.17), we find the various functions $A(2v)$, and from (1.11), we obtain the differential equations for $a(v)$.

b) Take the second case, when $A' + A^2 = 0$.

If we integrate this condition, we find the other differential equations for $a(v)$. Using the relation (1.3) and the definition of a ($a = \sqrt{dp/d\rho}$) we obtain the equations of state of the fluid.

3. The form of the general solution. a) We now consider the first case: $A' + A^2 \neq 0$. Then from (2.16) and (2.17) we obtain:

$$\text{(3.1)}\quad \begin{aligned} &\left[A - \frac{(A' + A^2)'}{A' + A^2}\right]\left[H_{1\xi} + H_{1\varphi}\varphi' + \cdots + H_{1\varphi^{(n-2)}}\varphi^{(n-1)} - \alpha\right] + AH_{1\eta} - \\ &- \frac{A(A' + A^2)'}{A' + A^2} H_1 + H_{1\xi\eta} + H_{1\eta\varphi}\varphi' + \cdots + H_{1\eta\varphi^{(n-2)}}\varphi^{(n-1)} - \alpha_\eta = 0, \\ &\left[A - \frac{(A' + A^2)'}{A' + A^2}\right]\left[H_{2\eta} + H_{2\psi}\psi' + \cdots + H_{2\psi^{(m-2)}}\psi^{(m-1)} + \alpha\right] + AH_{2\xi} - \\ &- \frac{A(A' + A^2)'}{A' + A^2} H_2 + H_{2\xi\eta} + H_{2\xi\psi}\psi' + \cdots + H_{2\xi\psi^{(m-2)}}\psi^{(m-1)} + \alpha_\xi = 0. \end{aligned}$$

By differentiating $(3.1)_1$ with respect to $\varphi^{(n-1)}$, we obtain

$$\text{(3.2)}\quad \left[A - \frac{(A' + A^2)'}{A' + A^2}\right] H_{1\varphi^{(n-2)}} + H_{1\eta\varphi^{(n-2)}} = 0.$$

The solution of this equation is

$$\text{(3.3)}\quad H_1 = (A' + A^2)\, K_1(\xi, \varphi, \ldots, \varphi^{(n-2)}) \exp\left(-\int_{\tau_1}^{\tau} A\, d\tau\right) + L_1(\xi, \eta, \varphi, \ldots, \varphi^{(n-3)}),$$

where K_1 and L_1 are arbitrary functions of their arguments.

From Eqs. $(3.1)_1$, (3.3), and (2.17) we find

$$\left[A - \frac{(A' + A^2)'}{A' + A^2}\right]\left[L_{1\xi} + L_{1\varphi}\varphi' + \cdots + L_{1\varphi^{(n-3)}}\varphi^{(n-2)} - \alpha\right] + AL_{1\eta} -$$
$$- \frac{A(A' + A^2)'}{A' + A^2} L_1 + L_{1\xi\eta} + L_{1\eta\varphi}\varphi' + \cdots + L_{1\eta\varphi^{(n-3)}}\varphi^{(n-2)} - \alpha_\eta = 0.$$

This last equation is the same as Eq. $(3.1)_1$ and we easily see that its solution is given by

$$L_1 = (A' + A^2)\exp\left(-\int_{\tau_1}^{\tau} A d\tau\right) M_1(\xi, \varphi, \ldots, \varphi^{(n-3)}) + N_1(\xi, \eta, \varphi, \ldots, \varphi^{(n-4)}).$$

Therefore, (3.3) is equivalent to

$$(3.3') \qquad H_1 = (A' + A^2)\exp\left(-\int_{\tau_1}^{\tau} A d\tau\right) X_1(\xi, \varphi, \ldots, \varphi^{(n-2)}) + Y_1(\xi, \eta),$$

where

$$X_1 = K_1(\xi, \varphi, \ldots, \varphi^{(n-2)}) + M_1(\xi, \varphi, \ldots, \varphi^{(n-3)}) + \cdots + U_1(\xi, \varphi).$$

Returning to Eq. $(3.1)_2$, and using a similar way, we obtain

$$(3.4) \qquad H_2 = (A' + A^2)\exp\left(-\int_{\tau_1}^{\tau} A d\tau\right) X_2(\eta, \psi, \ldots, \psi^{(m-2)}) + Y_2(\xi, \eta).$$

From Eqs. (2.5), (2.10), (2.14), (2.15), (3.3), and (3.4) we see that the function f must have the form

$$f = \exp\left(-\int_{\tau_1}^{\tau} A d\tau\right)\left[\varphi'^*(\xi) + \psi'^*(\eta) + \frac{(A'+A^2)'}{A'+A^2}(\varphi^*(\xi) + \psi^*(\eta))\right] + P(\xi, \eta),$$

where

$$\varphi^*(\xi) = \left(\frac{1}{A' + B^2}\right)_{\tau=\tau_1} b_1 + X_1, \quad \psi^*(\eta) = \left(\frac{1}{A' + A^2}\right)_{\tau=\tau_1} b_2 + X_2,$$

$$P(\xi, \eta) = \frac{1}{A' + A^2}[Y_{1\xi} + Y_{2\eta} + A(Y_1 + Y_2)].$$

Here $P(\xi, \eta)$ is a particular solution of Eq. (1.10) and we can take $P = 0$ Hence when $A' + A^2 \neq 0$, the general solution of the equation (1.10), in a simplified notation, has the form

$$(3.5) \qquad f = \left[\varphi'(\xi) + \psi'(\eta) + \frac{(A'+A^2)'}{A'+A^2}(\varphi(\xi) + \psi(\eta))\right]\exp\left(-\int_{\tau_1}^{\tau} A d\tau\right).$$

b) Let us now examine the second case, when $A' + A^2 = 0$. By differentiating (2.8) with respect to $\varphi^{(n)}, \ldots, \varphi, \psi^{(m)}, \ldots, \psi'$ and ψ we see that C_1 and C_2 are functions of ξ and η. From $(2.7)_1$ and $(2.7)_2$, it follows

$$(3.6) \qquad \begin{aligned} g &= \varphi^*(\xi)\exp\left(-\int_{\tau_1}^{\tau} A d\tau\right) + a_1(\xi, \eta), \\ h &= \psi^*(\eta)\exp\left(-\int_{\tau_1}^{\tau} A d\tau\right) + a_2(\xi, \eta), \end{aligned}$$

where $\varphi^*(\xi)$, $\psi^*(\eta)$, $a_1(\xi, \eta)$ and $a_2(\xi, \eta)$ are arbitrary functions.

If we introduce (3.6) into (2.5) the solution may be written as follows:

$$z = f = [\varphi^*(\xi) + \psi^*(\eta)]\exp\left(-\int_{\tau_1}^{\tau} A d\tau\right) + a(\xi, \eta),$$

where $a(\xi, \eta) = a_1(\xi, \eta) + a_2(\xi, \eta)$ is a particular solution of the Eq. (1.10). Therefore, in the case $A' + A^2 = 0$, the general solution of the Eq. (1.10) is

$$(3.7) \qquad z = f = [\varphi^*(\xi) + \psi^*(\eta)]\exp\left(-\int_{\tau_1}^{\tau} A d\tau\right).$$

4. Determination of the function $A(\xi + \eta)$. If we take $A' + A^2 = B$ the Eq. (2.17) can be written $[B''/B']' = B'$.

Then, by integration, we obtain the equation,

$$(4.1) \qquad \frac{2B'}{B^2 + 2C_1 B + 2C_2} = 1, \qquad (\tau = 2v),$$

where C_1 and C_2 are constants. From (4.1) we have

$$(4.2) \qquad \begin{aligned} &A' + A^2 = \frac{2k_1^2}{\cos^2(k_1\tau + k_2)}, \\ &A' + A^2 = \frac{2k_3^2 e^{k_3\tau + k_4}}{(1 - e^{k_3\tau + k_4})^2}, \qquad A' + A^2 = \frac{2}{(\tau + k_5)^2}, \end{aligned}$$

where k_1, k_2, k_3, k_4 and k_5 are arbitrary constants.

By integrating the above Eqs., we obtain, by (1.3), (1.11) and $a = \sqrt{dp/d\rho}$ three classes of ideal elastic fluids. If $k_5 = 0$, $(4.2)_3$ gives:

$$(4.3) \qquad A' + A^2 = \frac{2}{\tau^2}.$$

This differential equation has the particular solution: $A = 1/v$. If, in (3.5), we put $A = 1/v$, then we find the solution given by R. von Mises for isentropic fluid flow (1.8).

5. The solution of a Riemann problem. The general solution of Eq. (4.3) is

$$A = \frac{2}{\tau} + \frac{3}{3C\tau^4 - \tau},$$

where $\tau = \xi + \eta = 2v$, and C is a constant. When $C = 1/24$, we find $A = \frac{1}{v} + \frac{3}{2(v^4 - v)}$ and the general solution (3.5) can be written

(5.2) $$z = \frac{v}{v^3 - 1}\left[\varphi'(\xi) + \psi'(\eta) - \frac{1}{v}(\varphi(\xi) + \psi(\eta))\right].$$

From Eqs. (5.2), (1.6), and (1.4), we obtain

(5.3)
$$x - tu = \frac{v}{v^3 - 1}\left(\varphi'' - \psi'' - \frac{1}{v}(\varphi' - \psi')\right),$$
$$at = \frac{2v^3 + 1}{v^3 - 1}\left[\varphi' + \psi' - \frac{1}{v}(\varphi + \psi)\right] - \frac{v}{v^3 - 1}\left[\varphi'' + \psi'' + \frac{1}{v^2}(\varphi + \psi) - \frac{1}{v}(\varphi' + \psi')\right].$$

In the last part of this article, we shall deal with the determination of the arbitrary functions $\varphi(\xi) = \varphi(v + u)$ and $\psi(\eta) = \psi(v - u)$ when.

(5.4)
$$x - tu = \alpha(v), \quad \text{along} \quad v = \frac{1}{2}(\xi + \eta_1), \quad \eta_1 = \text{const.}$$
$$x - tu = \beta(v), \quad \text{along} \quad v = \frac{1}{2}(\xi_1 + \eta), \quad \xi_1 = \text{const.}$$

where $\alpha(v)$ and $\beta(v)$ are given functions. We remark that $\alpha(v_1) = \beta(v_1)$, and this value may be taken to be zero by locating the origin suitably on the x-axis.

Substituting (5.4) into (5.3), we obtain

(5.5)
$$\varphi'' - \frac{1}{v}\varphi' = \frac{v^3 - 1}{v}\alpha(v) + \psi_1'' - \frac{\psi_1'}{v}, \quad v = \frac{1}{2}(\xi + \eta_1),$$
$$\psi'' - \frac{1}{v}\psi' = -\frac{v^3 - 1}{v}\beta(v) + \varphi_1'' - \frac{\varphi_1'}{v}, \quad v = \frac{1}{2}(\xi_1 + \eta),$$

where $\varphi_1 = \varphi(\xi_1)$ and $\psi_1 = \psi(\eta_1)$. Integrating (5.5) we have

$$\varphi'(\xi) = C_1 v^2 + 2v^2\int_{v_1}^{v}\frac{v^3 - 1}{v^3}\alpha(v)dv - 2\psi_1'' v + \psi_1', \quad v = \frac{1}{2}(\xi + \eta_1),$$

$$\psi'(\eta) = C_2 v^2 - 2v^2\int_{v_1}^{v}\frac{v^3 - 1}{v^3}\beta(v)dv - 2\varphi_1'' v + \varphi_1', \quad v = \frac{1}{2}(\xi_1 + \eta),$$

where C_1 and C_2 are the constants of the integration. If we introduce the functions

(5.6)
$$\mathcal{A}(v) = \int_{v_1}^{v}\frac{v^3 - 1}{v^3}\alpha(v)dv, \quad v = \frac{1}{2}(\xi + \eta_1),$$
$$\mathcal{B}(v) = \int_{v_1}^{v}\frac{v^3 - 1}{v^3}\beta(v)dv, \quad v = \frac{1}{2}(\xi_1 + \eta),$$

the functions $\varphi'(\xi)$ and $\psi'(\eta)$ may be written

(5.7)
$$\varphi'(\xi) = C_1 v^2 - 2\psi_1'' v + \psi_1' + 2v^2\mathcal{A}(v), \quad v = \frac{1}{2}(\xi + \eta_1),$$
$$\psi'(\eta) = C_2 v^2 - 2\varphi_1'' v + \varphi_1' - 2v^2\mathcal{B}(v), \quad v = \frac{1}{2}(\xi_1 + \eta).$$

We find now

(5.8)
$$\varphi(\xi) = 4\int_{v_1}^{v} v^2\mathcal{A}(v)dv + \frac{2}{3}C_1 v^3 - 2\psi_1'' v^2 + 2\psi_1' v + K_1, \quad v = \frac{1}{2}(\xi + \eta_1),$$
$$\psi(\eta) = -4\int_{v_1}^{v} v^2\mathcal{B}(v)dv + \frac{2}{3}C_2 v^3 - 2\varphi_1'' v^2 + 2\varphi_1' v + K_2, \quad v = \frac{1}{2}(\xi_1 + \eta).$$

We get seven restricting conditions for the constants if we evaluate (5.5) (5.7), (5.8), and $(5.3)_2$ for $\xi = \xi_1$ and $\eta = \eta_1$,

$$\varphi_1''' + \psi_1''' = C_1 v_1, \quad \varphi_1'' + \psi_1'' = C_2 v_1, \quad \varphi_1' = C_1 v_1^2 - 2\psi_1'' v_1 + \psi_1',$$

(5.9)
$$\psi_1' = C_2 v_1^2 - 2\varphi_1'' v_1 + \varphi_1', \quad \varphi_1 = \frac{2}{3} C_1 v_1^3 - 2\psi_1'' v_1^2 + 2\psi_1' v_1 + K_1,$$
$$\psi_1 = \frac{2}{3} C_2 v_1^3 - 2\varphi_1'' v_1^2 + 2\varphi_1' v_1 + K_2,$$

$$a_1 t_0 = -\frac{v_1}{v_1^3 - 1}(\varphi_1'' + \psi_1'') + \frac{3v_1^3}{(v_1^3-1)^2}(\varphi_1' + \psi_1') - \frac{3v_1^2}{(v_1^3-1)^2}(\varphi_1 + \psi_1),$$

where a_1 is the local sound velocity, and t_0 is the initial moment. Of the ten constants φ_1, ψ_1, φ_1', ψ_1', φ_1'', ψ_1'', C_1, C_2, K_1 and K_2, which appear in (5.9), only four can be chosen arbitrarily.

If we choose

(5.10)
$$\varphi_1 = \alpha, \ \varphi_1' = \beta, \ \psi_1' = \gamma, \ C_1 = \delta,$$

where α, β, γ and δ are constants, from (5.9) we obtain

$$\varphi_1'' = \frac{1}{2}\left(\delta v_1 + \frac{\beta - \gamma}{v_1}\right), \quad \psi_1'' = \frac{1}{2}\left(\delta v_1 - \frac{\beta - \gamma}{v_1}\right),$$

(5.11)
$$\psi_1 = -\alpha - \frac{v_1^3 - 1}{3}\delta + (\beta + \gamma)v_1 - \frac{(v_1^3-1)^2}{3v_1^2} a_1 t_0,$$

$$K_1 = \alpha + v_1^3\frac{\delta}{3} - (\beta + \gamma)v_1, \quad K_2 = -\alpha + \frac{\delta}{3} - \frac{(v_1^3-1)^2}{3v_1^2} a_1 t_0,$$

$$C_1 = C_2 = \delta.$$

From (5.10) and (5.11) we find

(5.12)
$$\varphi(\xi) = 4\int_{v_1}^{v} v^2 \Phi(v)dv + \frac{K}{4}\xi^3 + M\xi^2 + L\xi + N, \quad v = \frac{1}{2}(\xi + \eta_1),$$
$$\psi(\eta) = -4\int_{v_1}^{v} v^2 \Psi(v)dv + \frac{K}{4}\eta^3 - M\eta^2 + L\eta - N + K - \frac{(v_1^3-1)^2}{3v_1^2} a_1 t_0,$$
$$v = \frac{1}{2}(\xi_1 + \eta),$$

where K, L, M, and N are arbitrary constants. If we introduce φ, ψ, φ' and ψ' in (5.2), using (5.12), we obtain

(5.13)
$$x = \frac{v}{v^3-1}\left[\frac{1}{2}(\xi+\eta_1)^2 \Phi\left(\frac{\xi+\eta_1}{2}\right) - \frac{1}{2}(\xi_1+\eta)^2 \Psi\left(\frac{\xi_1+\eta}{2}\right)\right] - \frac{1}{v^3-1}\left[\int_{v_1}^{\frac{1}{2}(\xi+\eta_1)} (\xi+\eta_1)^2 \Phi\left(\frac{\xi+\eta_1}{2}\right) d\left(\frac{\xi+\eta_1}{2}\right) - \int_{v_1}^{\frac{1}{2}(\xi_1+\eta)} (\xi_1+\eta)^2 \Psi\left(\frac{\xi_1+\eta}{2}\right) d\left(\frac{\xi_1+\eta}{2}\right) - \frac{(v_1^3-1)^2}{3v_1^2} a_1 t_0\right] + K.$$

The constant K has no effect on x and t as functions of u and v.

By using (1.6) and (1.9) we obtain the parametric equations of the characteristics.

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SOLUŢII EXACTE PENTRU MIŞCAREA UNIDIMENSIONALĂ NESTAŢIONARĂ A FLUIDELOR IDEALE ELASTICE

Rezumat

În această Notă se caută soluţii exacte, de forma (1.13), pentru ecuaţiile (1.10) care descriu mişcarea unidimensională nestaţionară a fluidelor ideale elastice. Forma soluţiei este dată de (3.5). Ecuaţiile (4.2) determină clasele de fluide elastice a căror mişcare unidimensională nestaţionară este descrisă de soluţia (3.5). Într-un caz particular, se rezolvă o problemă la limită.