Die Lösungsmethode der Grundprobleme einmal festgesetzt, bietet die Lösung des Planungsproblems $\mathfrak{P}$ mit Hilfe der Branch aud Bound-Prozedur (bzw. S.E.P. bzw. Radós Algorithmus)keine prinzipiellen Schwierigkeiten mehr.

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PLANIFICAREA LUCRULUI ÎN MAI MULTE ECHIPE

Rezumat

Se studiază problema programării în timp a fabricației într-un atelier care funcționează în mai multe echipe. Se propune un model matematic al problemei și se indică metoda de rezolvare.

ON THE REDUCED FORM OF THE LINEAR BOOLEAN AUTOMATA

BY

DAN A. SIMOVICI and CORINA REISCHER

1. Introduction. In our paper [4] we studied a new class of finite automata: the linear Boolean automata (L.B.A.). This sort of automata can be regarded as useful models for the theory of the communication networks for the switching theory and, also for the study of programming systems.

We shall denote below, following the usual notations [3], [4] by B_2 the two-element Boolean algebra, $\{0, 1, \cup, \cap, -\}$ and with $B_2^{(p,q)}$ the matrix Boolean algebra, whose elements are matrices with p rows and q columns with entries belonging to B_2.

A linear Boolean automaton is a 5-uple: $< B_2^{(l,1)}, B_2^{(n,1)}, B_2^{(m,1)}, f, g >$ where:

$$f : B_2^{(n,1)} \times B_2^{(l,1)} \to B_2^{(n,1)}, \tag{1}$$

$$g : B_2^{(n,1)} \times B_2^{(l,1)} \to B_2^{(m,1)}. \tag{2}$$

The finite sets $B_2^{(l,1)}$, $B_2^{(n,1)}$, $B_2^{(m,1)}$, are respectively the set of inputs the set of states and the set of outputs. The mapping f is the next state function; g is the output function. These functions are given by:

$$f(s(t), u(t)) = s(t+1) = As(t) \cup Bu(t), \tag{1'}$$

$$g(s(t), u(t)) = y(t) = Cs(t) \cup Du(t), \tag{2'}$$

where $s(t), s(t+1) \in B_2^{(n,1)}$, $u(t) \in B_2^{(l,1)}$ $y(t) \in B_2^{(m,1)}$, A, B, C, D the characteristic matrices of the circuit belonging respectively to $B_2^{(n,n)}$, $B_2^{(n,l)}$, $B_2^{(m,n)}$, $B_2^{(m,l)}$.

We shall denote the L.B.A. automata with characteristic matrices A, B, C, D by $< A, B, C, D >$.

In the sequel E_n will be the symbol of the unit matrix from $B_2^{n,n}$.

We shall give an algorithm which is similar to Gill's algorithm for the minimization of linear sequential circuits [1], pp. 164–175.

II. The physical structure of the L.B.A. and their exteriorized form. The equations (1) and (2) indicate the physical structure of an L.B.A.

We notice that the implementation of these equations needs disjunctive and conjunctive gates and also, delayers.

The way of acting of the first two elements is well known. Regarding the delayers, if their input at time t is $\sigma'(t)$, the output in the next moment is $\sigma(t+1) = \sigma'(t)$, where $\sigma, \sigma' \in B_2$.

The sequential mode of acting is realised by using the delayers. If we consider that the state of the automaton is given by the output of the n delayers:

(3)
$$s(t) = \begin{pmatrix}\sigma_1(t)\\ \sigma_2(t)\\ \vdots\\ \sigma_n(t)\end{pmatrix},$$

we find the next-state with (1). Therefore we can consider the „exteriorized" form of the automaton (see fig. 2) in wich the circuits which implement the functions f and g are presented as a „blackbox" and the delayers are drown outwardly.

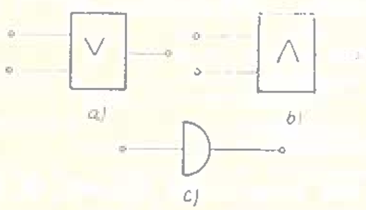

Fig. 1 — Constitutive elements a, b) „or" and „and" elements c) delayers.

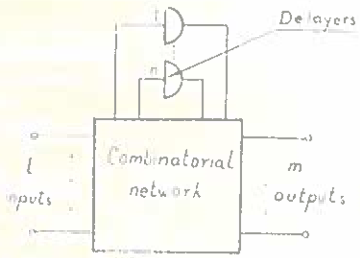

Fig. 2 — The exteriorized form of a L.B.A.

III. The ideal of the states which are equivalent with the 0-state. We shall give firstly a lemma which is proved in [2], p. 44.

Lemma 1. *Let $A \in B_2^{n,n}$. There exists the matrix A^{-1} iff the matrix A is obtained from the unit matrix through a permutation of the rows or of the columns.*

Let A_i be the i-column of the matrix A. Then A will be an invertible matrix iff we can write:

$$A_1 = E_{\xi_1}, \ldots, A_n = E_{\xi_n},$$

where:

$$P = \begin{pmatrix}1 & 2 & \ldots & n\\ \xi_1 & \xi_2 & \ldots & \xi_n\end{pmatrix}$$

is a permutation on the set $\{1, 2, \ldots, n\}$.

Then the matrix A^{-1} will be obtained from A by transposing it: $A^{-1} = A^T$.

From (3) it is clear that the state-vector depends on the order of the delayers. Further, we consider the effect of the renumbering of the delayers on the state-vector.

Lemma 2. *The renumbering of the delayers with the permutation*

$$\begin{pmatrix}1 & 2 & \ldots & n\\ \xi_1 & \xi_2 & \ldots & \xi_n\end{pmatrix}$$

acts on the state-vector as follows:

$$\tilde{s} = (E_{\xi_1}, \ldots, E_{\xi_n})^T s = R^T s,$$

where R is a permutation matrix.

The proof is immediate.

By using Lemma 2, it is clear that:

(4)
$$s = (E_{\xi_1}, \ldots, E_{\xi_n})\tilde{s} = R\tilde{s}.$$

Now, let us infer the form of the characteristic matrices of the circuit when a remembering of delayers is occuring.

From the equations (1′) and (2′), by using (4), we find:

(1″)
$$R\tilde{s}(t+1) = AR\tilde{s}(t) \cup Bu(t),$$

(2″)
$$y(t) = CR\tilde{s}(t) \cup Du(t).$$

By using Lemma 1 these equations could be written:

(1‴)
$$\tilde{s}(t+1) = R^T AR\tilde{s}(t) \cup R^T Bu(t),$$

(2‴)
$$y(t) = CR\tilde{s}(t) \cup Du(t).$$

We conclude that the new characteristic matrices become:

(5)
$$\tilde{A} = R^T AR, \quad \tilde{B} = R^T B, \quad \tilde{C} = CR, \quad \tilde{D} = D.$$

We proved in [4] that the set Z of the states which are equivalent with the 0-state is an ideal in the $B_2^{(n,1)}$ algebra. The equivalence of the states is regarded there from the point of view of the external behaviour of the automaton and it was proved that this equivalence is the same with the ideal-equivalence ([5], p. 77).

Two states will be equivalent ($\omega \varepsilon_Z \tilde{\omega}$) iff $\omega \oplus \tilde{\omega} \in Z$, where „$\oplus$" denotes the exclusive disjunction: $\omega \oplus \tilde{\omega} = (\omega \cap \bar{\tilde{\omega}}) \cup (\bar{\omega} \cap \tilde{\omega})$. Let $\zeta = \bigcup_{\omega \in Z} \omega$. Suppose that ζ has k nonzero components. We shall say that the automaton has the canonical form iff the units of ζ are in its first k entries.

If it happens that these k units are not in the first k entries let us consider a permutation matrix R which allows to arrange the units in these positions. Then we shall renumber the delayers by the method which was indicated earlier (in the Lemma 2). Now it is self evident that every L.B.A. is equivalent with a canonical L.B.A. because the renumbering of the delayers do not affect the external behavior.

Henceforth, in this paragraph the L.B.A. *will be considered as being in the canonical form.*

It is clear that $\omega \in Z$ iff it has the form $\omega = \begin{pmatrix} \mu \\ 0 \end{pmatrix}$, where $\mu \in B_2^{(k,1)}$, $0 \in B_2^{(n-k,1)}$.

Lemma 3. *The states* $\omega = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ *and* $\omega' = \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix}$ *where* $\sigma_1 \in B_2^{(k,1)}$, $\sigma_2 \in B_2^{(n-k,1)}$ $0 \in B_2^{(k,1)}$ *are* ε_Z*-equivalent.*

Indeed:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ 0 \end{pmatrix} \in Z.$$

Lemma 4. *If* $\omega \in Z$, *then* $A\omega \in Z$.

In [4] we pointed out that a necessary and sufficient condition in order that $\omega \in Z$ is:

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \omega = 0. \tag{6}$$

This is equivalent with the equalities:

$$C\omega = CA\omega = \ldots = CA^{n-1}\omega = 0. \tag{6'}$$

By using the associativity of the Boolean matrix product, we obtain

$$C(A\omega) = \ldots = CA^{n-2}(A\omega) = 0.$$

But, we proved in [4] that:

$$C(A^n\omega) \leqslant C(E \cup A \cup \ldots \cup A^e)\,\omega = 0,$$

where $1 \leqslant e \leqslant n-1$. Hence $CA^{n-1}(A\omega) = C(A^n\omega) = 0$, which proves the lemma.

IV. The reduced form of a L.B.A. Theorem 1. *Let R the invertible matrix which carries the k units of $\zeta = \bigcup_{\omega \in Z} \omega$ into the first k positions and $\tilde{A} = R^T AR$ the matrix of the canonical form of the L.B.A. If $\tilde{A} = \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{pmatrix}$, where $\tilde{A}_1 \in B_2^{(k,k)}$, then: $\tilde{A}_3 = (0)_{n-k \times k}$.*

Proof. Let $\omega = \begin{pmatrix} \mu \\ 0 \end{pmatrix} \in \tilde{Z}$, $\mu \in B_2^{(k,1)}$. Using lemma 4, $\tilde{A}\omega \in \tilde{Z}$ and $\tilde{A}\omega = \begin{pmatrix} \tilde{A}_1 \mu \\ \tilde{A}_3 \mu \end{pmatrix}$. Hence $\tilde{A}_3\mu = 0$ for every $\mu \in B_2^{(k,1)}$. Therefore $\tilde{A}_3 = (0)_{n-k \times k}$, q. e. d.

Theorem 2. *Let $\tilde{C} = CR$ the corresponding matrix of the canonical orm of the L.B.A. and let $\tilde{C} = (\tilde{C}_1, \tilde{C}_2)$,* where $\tilde{C}_1 \in B_2^{(m,k)}$. *Then* $\tilde{C}_1 = (0)_{m \times k}$.

Proof. Let $\omega \in \tilde{Z}$. Then $\omega = \begin{pmatrix} \mu \\ 0 \end{pmatrix}$ with $\mu \in B_2^{(k,1)}$.

From $\tilde{C}\omega = 0$ it follows $\tilde{C}_1\mu = 0$ for every $\mu \in B_2^{(k,1)}$. We conclude then $\tilde{C}_1 = (0)_{m \times k}$ q. e. d.

Theorem 3. *Let $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $C = (0, C_2)$, D the characteristic matrices of a L.B.A. in the canonical form, where $A_1 \in B_2^{(k,k)}$, $B_1 \in B_2^{(k,l)}$, $C_2 \in B_2^{(m,n-k)}$ $D \in B_2^{(m,l)}$.*

Then, the L.B.A. with the characteristic matrices A_4, B_2, C_2, D is equivalent with the first automaton from the point of view of the external behavior.

Proof. We shall prove that the state $\omega = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ of the automaton $\langle A, B, C, D \rangle$ is equivalent to the state $\tilde{s} = \sigma_2$ of the automaton $\langle A_4, B_2, C_2, D \rangle$.

The output equation of the first automaton is:

$$y(t) = C\omega(t) \cup Du(t) = (0 \;\; C_2) \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \cup Du = C_2\sigma_2 \cup Du.$$

For the second automaton we have:

$$\tilde{y}(t) = C_2\sigma_2 \cup Du = y(t).$$

A time after t the state of the automaton will be:

$$\omega' = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \cup \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u = \begin{pmatrix} A_1 \sigma_1 \cup A_2 \sigma_2 \cup B_1 u \\ A_4 \sigma_2 \cup B_2 u \end{pmatrix} = \begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix}.$$

The next state in the automaton $< A_4, B_2, C_2, D >$ will be:

$$s' = A_4 \sigma_2 \cup B_2 u.$$

We notice that the states ω' and $\widetilde{s'}$ are also equivalent because the „second part" of the state ω' is precisely $\widetilde{s'}$.

Theorem 4. *The automaton* $< A_4, B_2, C_2, D >$ *is a minimal one.*

Proof. Let us suppose that σ_1 and σ_2 are equivalent states. Then, in the automaton $< A, B, C, D >$ the states $\begin{pmatrix} \sigma_1' \\ \sigma_1 \end{pmatrix}$ and $\begin{pmatrix} \sigma_2' \\ \sigma_2 \end{pmatrix}$ are ε_Z-equivalent.

Hence, by using the condition of state equivalence, we obtain

$$\begin{pmatrix} \sigma_1' \\ \sigma_1 \end{pmatrix} \oplus \begin{pmatrix} \sigma_2' \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_1' \oplus \sigma_2' \\ \sigma_1 \oplus \sigma_2 \end{pmatrix} \in Z.$$

This is possible only if $\sigma_1 \oplus \sigma_2 = 0$ which means $\sigma_1 = \sigma_2$.

Corollary. *Every L.B.A. is equivalent with a minimal L.B.A.*

V. Consequences from the Theorems 3-4.

1°. If $A_4 > E_{n-k}$ then, after a sufficient large time the state of the L.B.A. becomes a constant σ (in a free evolution of the reduced automaton) [4].

We can infer now, that the state of the automaton $< A, B, C, D >$ will be further:

$$s(t) = \begin{pmatrix} \mu(t) \\ \sigma \end{pmatrix},$$

where $\sigma =$ const. and $\mu(t) \in B_2^{k,1}$.

This fact indicates how to determine through an experience with the L.B.A. the index $n - k$ of its reduced automaton.

2° In order to find k we shall use the properties of the ideal Z. We shall prove if any column of the unit matrix E_n is equivalent with the state $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

If there exists k columns of the unit matrix E_n in the ideal Z, for instance $E_{\xi_1}, \dots, E_{\xi_k}$, then $\zeta = \bigcup_{i=1}^{k} E_{\xi_i} \zeta_i$ will contain k units and $n - k$ zeros because every unit in ζ corresponds to a column $E_{\xi_i} \in Z$.

The column E_i belongs to Z iff the i-th column of the matrices $C, CA, \dots, CA^e$ are null vectors. This happens iff the i-column from the matrix $C \cup CA \cup \dots \cup CA^e = C(E \cup A \cup \dots \cup A^e)$ is the null vector. We remember that e was defined in [4].

We obtain:

Theorem 5. *The index of the minimal automaton is equal to the number of the non-zero columns of the matrix* $C(E \cup A \cup \dots \cup A^e)$.

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ASUPRA FORMEI REDUSE A AUTOMATELOR LINIARE BOOLEENE

Rezumat

Principalul rezultat conţinut în lucrare este stabilirea unui algoritm de minimizare pentru automatele liniare booleene. În legătură cu acest algoritm se găsesc o serie de proprietăţi ale idealului stărilor echivalente cu starea nulă precum şi unele rezultate referitoare la diagnosticarea acestui tip de automate.