EXPANSION FORMULA FOR THE G-FUNCTION OF TWO VARIABLES

BY

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§ 1. **Introduction**. In this paper we have established an expansion formula for the G-function of two variables [1] involving Legendre functions. Later some interesting particular cases are obtained which seem to be new.

§ 2. **Expansion Formula**

$$(1)\quad (1-x^2)^{\lambda-1}(1-y^2)^{\lambda_1-1}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix} z(1-x^2)^{\delta} \\ z_1(1-y^2)^{\delta_1} \end{matrix}\middle|\begin{matrix} (\varepsilon_p) \\ (\nu_t);(\nu'_{t'}) \\ (\delta_s) \\ (\beta_q);(\beta'_{q'}) \end{matrix}\right] =$$

$$= 2^{\mu+\mu_1-2}\pi^2\delta^{-1}\delta_1^{-1}\sum_{r=0}^{\infty}\sum_{r_1=0}^{\infty}\frac{(r-\mu)!\,(r_1-\mu_1)!\,(2r+1)(2r_1+1)}{(r+\mu)!\,(r_1+\mu_1)!\,\Gamma\left(\frac{2-\mu+r}{2}\right)\Gamma\left(\frac{2-\mu_1+r_1}{2}\right)}\times$$

$$\times\left[\Gamma\left(\frac{1-\mu-r}{2}\right)\left(\frac{1-\mu_1-r_1}{2}\right)\right]^{-1}P_r^{\mu}(x)\,P_{r_1}^{\mu_1}(y)\times$$

$$G^{n,\nu_1+2\delta,\nu_2+2\delta_1,m_1,m_2}_{p,[t+2\delta:t'+2\delta_1],s,[q+2\delta:q'+2\delta_1]}\left|\begin{matrix} z \\ z_1 \end{matrix}\right.$$

$$\left.\begin{matrix} (\varepsilon_p) \\ \Delta[\delta,\lambda+\mu/2],\Delta[\delta,\lambda-\mu/2],(q_t);\Delta[\delta_1,\lambda_1+\mu_1/2],\Delta[\delta_1,\lambda_1-\mu_1/2],(q'_{t'}) \\ (\delta_s) \\ (p_q),\Delta[\delta,-\lambda-r/2],\Delta[\delta,1-\lambda+r/2];(\beta'_{q'}),\Delta[\delta,-\lambda_1-r_1/2],\Delta[\delta,1-\lambda_1+r_1/2] \end{matrix}\right|$$

where
$$p+q+s+t<2(m_1+\nu_1+n),$$
$$p+q'+s+t'<2(m_2+\nu_2+n)$$
and
$$|\arg z|<\pi\left[m_1+\nu_1+n-\frac{1}{2}(p+q+s+t)\right],$$
$$|\arg z_1|<\pi\left[m_2+\nu_2+n-\frac{1}{2}(p+q'+s+t')\right],$$
$$2\,\mathrm{Re}\,(\lambda+\beta_j)>|\mathrm{Re}\,\mu|,\quad j=1,2,\ldots,m_1,\ \lambda\geqslant 1 \text{ and}$$
$$2\,\mathrm{Re}\,(\lambda_1+\beta_j')>|\mathrm{Re}\,\mu_1|,\quad j=1,2,\ldots,m_2,\ \lambda_1\geqslant 1.$$

Proof. Let

$$f(x,y)=(1-x^2)^{\lambda-1}(1-y^2)^{\lambda_1-1}\,G^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix} z(1-x^2)^{\delta}\\ z_1(1-y^2)^{\delta_1}\end{matrix}\,\middle|\,\begin{matrix}(\varepsilon_p)\\ (q_t);(q'_{t'})\\ (\delta_s)\\ (\beta_q);(\beta'_{q'})\end{matrix}\right]=$$

(2)
$$=\sum_{r=0}^{\infty}\sum_{r_1=0}^{\infty}C_r\,C'_{r_1}\,P_r^{\mu}(x)\,P_{r_1}^{\mu_1}(y),\ldots\ -1<x,y<1.$$

Equation (2) is valid since $f(x,y)$ is continuous and of bounded variation in the open interval. Proceeding on the similar lines of S. D. Bajpai[4] and after little simplification (using the orthogonality property of Legendre function), (1) can be established.

Particular cases. Many more results can be derived from (1) by specializing the parameters as it involves the G-function of two variables introduced by Agarwal [1].

By putting $n=0$, $p=0$, $s=0$ and using [1, (3.1)], we get

$$(1-x^2)^{\lambda-1}(1-y^2)^{\lambda_1-1}\,G^{m_1,\nu_1}_{t,q}\left(z(1-x^2)^{\delta}\,\middle|\,\begin{matrix}(1-\nu_t)\\(\beta_q)\end{matrix}\right)G^{m_2,\nu_2}_{t',q'}\left(z_1(1-y^2)^{\delta_1}\,\middle|\,\begin{matrix}(1-q'_{t'})\\(\beta'_{q'})\end{matrix}\right)=$$
$$=2^{\mu+\mu_1-2}\pi^2\,\delta^{-1}\delta_1^{-1}\sum_{r=0}^{\infty}\sum_{r_1=0}^{\infty}\frac{(r-\mu)!\,(r_1-\mu_1)!\,(2r+1)(2r_1+1)}{(r+\mu)!\,(r_1+\mu_1)!\,\Gamma\left(\frac{2-\mu+r}{2}\right)\Gamma\left(\frac{2-\mu_1+r_1}{2}\right)}\times$$
$$\times\left[\Gamma\left(\frac{1-\mu-r}{2}\right)\Gamma\left(\frac{1-\mu_1-r_1}{2}\right)\right]^{-1}P_r^{\mu}(x)\,P_{r_1}^{\mu_1}(y)\times$$
$$G^{m_1,\nu_1+2\delta}_{t+2\delta,\,q+2\delta}\left(z\,\middle|\,\begin{matrix}\Delta[\delta,1-\lambda-\mu/2],\Delta[\delta,1-\lambda+\mu/2],(1-q_t)\\(\beta_q),\Delta[\delta,-\lambda-r/2],\Delta[\delta,1-\lambda+r/2]\end{matrix}\right)\times$$
$$\times G^{m_2,\nu_2+2\delta_1}_{t'+2\delta_1,\,q'+2\delta_1}\left(z_1\,\middle|\,\begin{matrix}\Delta[\delta_1,1-\lambda_1-\mu_1/2],\Delta[\delta_1,1-\lambda_1+\mu_1/2],(1-q'_{t'})\\(\beta'_{q'}),\Delta[\delta,-\lambda_1-r_1/2],\Delta[\delta_1,-1-\lambda_1-r_1/2]\end{matrix}\right),$$

where
$$t+q<2(m_1+\nu_1),$$
$$t'+q'<2(m_2+\nu_2)$$
and
$$|\arg z|<\pi\left[m_1+\nu_1-\frac{1}{2}(t+q)\right],\quad |\arg z_1|<\pi\left[m_2+\nu_2-\frac{1}{2}(t'+q')\right].$$

It includes the expansion obtained by Bajpai [4] as particular case and thus, in turn, it includes all the expansions which are special cases of [4].

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O FORMULĂ DE DEZVOLTARE PENTRU O FUNCȚIE G DE DOUĂ VARIABILE

Rezumat

Se demonstrează formula de dezvoltare din § 2.