FOURIER SERIES FOR GENERALIZED MEIJER FUNCTIONS

BY

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§ 1. Introduction

The Generalized Meijer Function of Two Variables. Recently Dr. Sharma ([6], pp. 26-40) has introduced the generalized Meijer function of two variables by means of a double Mellin-Barnes type contour integral in the form

$$(1.1) \qquad S\left[\begin{matrix} \begin{bmatrix} p & , & 0 \\ A-p, & & B \end{bmatrix} \\ \begin{pmatrix} q & , & r \\ C-q & , & D-r \end{pmatrix} \\ \begin{pmatrix} k & , & l \\ E-k & , & F-l \end{pmatrix} \end{matrix} \middle| \begin{matrix} (a);\ (b) \\ (c);\ (d) \\ (e);\ (f) \end{matrix} \middle| \ x\ ,\ y \ \right] =$$

$$= \frac{q}{(2\pi i)^2} \int_{L_1}\int_{L_2} \Phi(s+t)\,\Psi(s,\ t)\, x^s y^t\, ds\, dt,$$

where

$$\Psi(s,t) = \frac{\prod_{j=1}^{q}\Gamma(1-c_j+s)\prod_{j=1}^{r}\Gamma(d_j-s)\prod_{j=1}^{k}\Gamma(1-e_j+t)\prod_{j=1}^{l}\Gamma(f_j-t)}{\prod_{j=q+1}^{C}\Gamma(c_j-s)\prod_{j=r+1}^{D}(1-d_j+s)\prod_{j=k+1}^{E}\Gamma(e_j-t)\prod_{j=l+1}^{F}\Gamma(1-f_j+t)},$$

$$\Phi(s+t) = \frac{\prod_{j=1}^{p}\Gamma[a_j+s+t]}{\prod_{j=p+1}^{A}\Gamma[1-a_j-s-t]\prod_{j=1}^{B}\Gamma[b_j+s+t]}$$

and L_1 and L_2 are suitable contours.

A, B, C etc. are positive integers and satisfy the following inequalities:

$$D \geqslant 1,\ F \geqslant 1,\ A \geqslant 1,\ B \geqslant 1$$

$$0 \leqslant p \leqslant A,\ 0 \leqslant q \leqslant C,\ 0 \leqslant k \leqslant E,\ 0 \leqslant r \leqslant D,\ 0 \leqslant l \leqslant F,$$

$$A+C \leqslant B+D \text{ and } A+E \leqslant B+F.$$

The above integral converges under the following conditions

(i)
$$2(p+q+r) > A+B+C+D,\ |\arg(x)| < (p+q+r-\frac{1}{2}A-\frac{1}{2}B-\frac{1}{2}C-\frac{1}{2}D)\pi,$$

$$2(p+q+l) > A+B+E+F,\ |\arg(y)| < (p+k+r-\frac{1}{2}A-\frac{1}{2}B-\frac{1}{2}E-\frac{1}{2}F)\pi;$$

(ii) If $A+C=B+D$, $A+E=B+F$, then we must have

$$|(x)| < R_1 \leqslant 1,\quad |(y)| < R_2 \leqslant 1.$$

The values $x=0$ and $y=0$ are excluded.

For sake of brevity and ease in writing, we shall follow the contracted notation. (a) stands for the set of A parameters $a_1, a_2, \ldots, a_A$ with similar notations for (b), (c), (d), (e) and (f). The generalized Meijer function (1.1) is represented symbolically by

$$S(x,y) = \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s+t)\Psi(s,t)\,x^s y^t\,ds\,dt.$$

The symbol $\Delta(m,n)$ represents the set of m-parameters:

$$\frac{n}{m},\ \frac{n+1}{m},\ldots,\frac{n+m-1}{m}.$$

The notation $(\Delta(\delta, a \pm b))$ will stand for $(\Delta(\delta, a+b))$, $(\Delta(\delta, a-b))$ and similarly for $\Gamma(a \pm b)$.

The Double Hypergeometric Function of Higher Order. The double hypergeometric function of higher order in two variables has been studied by Professor J. Kampé de Fériet [3]:

(1.2)
$$F\left[\begin{array}{c|cc|c} m & a_1,\ldots,a_m & & \\ l & b_1,\ldots,b_l\ ; & b'_1,\ldots,b'_l & x \\ n & c_1,\ldots,c_n & & y \\ p & d_1,\ldots,d_p\ ; & d'_1,\ldots,d'_p & \end{array}\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{m}(a_j)_{r+s}\prod_{j=1}^{l}\{(b_j)_r(b'_j)_s\}}{r!\,s!\prod_{j=1}^{n}(c_j)_{r+s}\prod_{j=1}^{p}\{(d_j)_r(d'_j)_s\}}x^r y^s,$$

where for absolute convergence of the series, $|x|<1$, $|y|<1$, $m+l \leqslant n+p+1$, and shall be abbreviated to

$$F\left[\begin{array}{c|cc|c} m & (a_m) & & \\ l & (b_l)\ ; & (b'_l) & x \\ n & (c_n) & & y \\ p & (d_p)\ ; & (d'_p) & \end{array}\right],$$

where thy symbol (a_m) denotes the sequence of elements $a_1, a_2, \ldots, a_m$ and similarly for (b_l), (b'_l), (c_n), (d_p) and (d'_p).

The series for the F-function converges absolutely for all complex values of x and y if $m+l < n+p+1$. In case $m+l=p+n+1$, it converges absolutely for all complex values of x and y such that (Ragab 1963 [4])

$$|x|+|y| < \min(1, 2^{p-m+1}).$$

The parameters m, l, n, p are known as the characteristic indices and the parameters $n+p$ and $[n+p+1-(m+l)]$ indicate respectively the order and class of the function. As in other hypergeometric functions, none of the denominator parameters is allowed to be a negative integer.

The function F in (1. 2) reduces to Appell's [1] four functions F_1, F_2, F_3 and F_4 with proper choice of parameters.

Again, when $m=n$, $l=1$, $p=0$ and $y=x$, the F-function of (1.2) reduces to the generalized hypergeometric function of one variable i. e.

$$(1.3)\qquad F\left(\begin{array}{c|cc|c} p & (a_p) & & \\ l & b\,; & b' & x \\ p & (c_p) & & x \\ 0 & \ldots\ldots & & \end{array}\right) = {}_{p+1}F_p\left[\begin{array}{l} a_1,\ldots,a_p,\quad b+b'\,; \\ \rho_1,\ldots,\rho_p \end{array}\; x\right].$$

The object of this paper is to evaluate some integrals involving generalized Meijer functions of two variables in terms of generalized Meijer functions. These integrals have been employed to establish the Fourier-series for the generalized Meijer functions. Many particular interesting results are also obtained on specializing the parameters. Hence the results established in this paper are of more general character.

§ 2. Integrals

The first integral to be evaluated is

$$(2.1)\qquad \int_0^{\pi/2} \cos 2u\theta\,(\sin\theta)^v\, S\left[\begin{array}{cc|c|c} \begin{bmatrix} p, & 0 \\ A-p, & B \end{bmatrix} & & (a)\,;\ (b) & \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} & & (c)\,;\ (d) & x(\sin\theta)^{2\delta},\ y(\sin\theta)^{2\delta} \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & & (e)\,;\ (f) & \end{array}\right] d\theta =$$

$$= \frac{\Gamma\left(\frac{1}{2}\pm u\right)}{2\sqrt{\pi}\,\delta^{1/2}}\, S\left[\begin{array}{c|l|c} \begin{bmatrix} p+2\delta, & 0 \\ A-p, & B+2\delta \end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right);\ (a)\,; & \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} & \Delta(\delta, 1/2\,v \pm u + 1),\ (b) & x,\ y \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & \qquad (c)\,;\ (d) \quad (e)\,;\ (f) & \end{array}\right]$$

The second integral to be proved is

$$(2.2)\qquad \int_0^{\pi/2} \cos u\theta\,(\cos\theta)^v\, S\left[\begin{array}{c|c|c} \begin{bmatrix} p, & 0 \\ A-p, & B \end{bmatrix} & (a)\,;\ (b) & \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} & (c)\,;\ (d) & x(\cos\theta)^{2\delta};\ y(\cos\theta)^{2\delta} \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & (e)\,;\ (f) & \end{array}\right] d\theta =$$

$$= \frac{\sqrt{\pi}}{2\delta^{1/2}}\, S\left[\begin{array}{c|l|c} \begin{bmatrix} p+2\delta, & 0 \\ A-p, & B+2\delta \end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right),\quad (a)\,; & \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} & \Delta\left(\delta, \frac{1}{2}v \pm \frac{1}{2}u + 1\right),\quad (b)\,, & x,\ y \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & \qquad (c)\,;\ (d) \quad (e)\,;\ (f) & \end{array}\right].$$

where δ is a positive integer, $u = 0, 1, 2,\ldots; 0 < \theta < \pi/2$, and are valid under the following conditions

(i) $$A + B + C + D < 2(p + q + r),\quad |\arg(x)| < (p + q + r - \tfrac{1}{2}A - \tfrac{1}{2}B - \tfrac{1}{2}C - \tfrac{1}{2}D)\pi,$$

$$A + B + E + F < 2(p + k + l),\quad |\arg(y)| < (p + k + l - \tfrac{1}{2}A - \tfrac{1}{2}B - \tfrac{1}{2}E - \tfrac{1}{2}F)\pi\,;$$

$$\mathrm{Re}\,(v + 2\delta d_{h_1} + 2\delta f_{h_2}) > 0,\quad (h_1 = 1, 2,\ldots, r;\ h_2 = 1, 2,\ldots, l).$$

(ii) $A + C < B + D$, $A + E < B + F$, $(A + C = B + D$, $A + E = B + F)$

then $|x|, |y| < 1$, $\mathrm{Re}\,(v + 2\delta d_{h_1} + 2\delta f_{h_2}) > 0$, $(h_1 = 1, 2,\ldots, r;\ h_2 = 1, 2,\ldots, l)$.

Proof :

(A) To prove (2.1), we substitute the value of the generalized Meijer function $S(x, y)$ in the integrand of (2.1) from (1.1), interchange the order of integration; we obtain

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \Phi(s+t)\,\Psi(s, t)\left\{\int_0^{\pi/2} \cos 2u\theta\,(\sin\theta)^{v+2\delta s+2\delta t}\, d\theta\right\} x^s y^t\, ds\, dt.$$

Now evaluating the θ-integral by virtue of the known result ([5], p. 41, ex. (9)):

$$\int_0^{\pi/2} \cos 2u\theta\,(\sin\theta)^v\, d\theta = \frac{\Gamma(v+1)\,\Gamma\left(\frac{1}{2}\pm u\right)}{2^{v+1}\,\Gamma\left(\frac{1}{2}v \pm u + 1\right)},$$

where $u = 0, 1,\ldots;\ \mathrm{Re}\,(v) > 0$.

Regarding the interchange of the order of integrations, it is observed that the θ-integral is absolutely convergent if $\mathrm{Re}\,(v+2\delta s+2\delta t)>0$, $u=0, 1, 2,\ldots$; $0<\theta<\pi/2$, the double contour integral converges absolutely under the conditions referred to earlier, and the convergence of the repeated integral follows from that of the integral in (2.1). Hence the interchange of the order of integrations is justified.

Making use of the following relations

$$(\alpha)_{2n}=2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n,\quad (\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

and Gauss's multiplication formula ([2], p. 4, (11)):

$$\prod_{k=0}^{m-1}\Gamma\left(z+\frac{k}{m}\right)=(2\pi)^{\frac{1}{2}m-\frac{1}{2}}(m)^{\frac{1}{2}-mz}\,\Gamma(mz),\quad m=2, 3, 4,\ldots,\ \text{we have}$$

$$\frac{\Gamma\left(\frac{1}{2}\pm u\right)}{2\sqrt{\pi}\,\delta^{1/2}}\,\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\prod\limits_{j=1}^{p}\Gamma(a_j+s+t)\prod\limits_{j=1}^{q}\Gamma(1-c_j+s)\prod\limits_{j=1}^{r}\Gamma(d_j-s)}{\prod\limits_{j=p+1}^{A}\Gamma(1-a_j-s-t)\prod\limits_{j=1}^{B}\Gamma(b_j+s+t)\prod\limits_{j=q+1}^{C}\Gamma(c_j-s)}$$

(2.3)

$$\frac{\prod\limits_{j=1}^{k}\Gamma(1-e_j+t)\prod\limits_{j=1}^{l}\Gamma(f_j-t)\prod\limits_{i=0}^{\delta-1}\Gamma\left(\frac{v+1+2i}{2\delta}+s+t\right)\prod\limits_{i=0}^{\delta-1}\Gamma\left(\frac{v+2+2i}{2\delta}+s+t\right)}{\prod\limits_{j=r+1}^{D}\Gamma(1-d_j+s)\prod\limits_{j=k+1}^{E}\Gamma(e_j-t)\prod\limits_{j=l+1}^{F}(1-f_j+t)\prod\limits_{i=0}^{\delta-1}\Gamma\left(\frac{v+2u+2+2i}{2\delta}+s+t\right)}\,x^s y^t\,ds\,dt$$

L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops, if necessary to ensure that the poles of $\Gamma(d_j-s)$, $(j=1, 2,\ldots, r)$ lie to the right and the poles of $\Gamma(1-c_j+s)$, $(j=1, 2,\ldots, q)$ and $\Gamma(a_j+s+t)$, $(j=1, 2,\ldots, p)$, $\Gamma\left(\frac{v+1+2i}{2\delta}+s+t\right)$, $\Gamma\left(\frac{v+2+2i}{2\delta}+s+t\right)$, $i=0$, $1,\ldots,(\delta-1)$ to the left of the contour.

Similarly the contour L_2 is in the t-plane and runs from $-i\infty$ to $i\infty$ with loops, if necessary to ensure that the poles of $\Gamma(f_j-t)$, $(j=1, 2,\ldots, l)$ lie to the right and the poles of $\Gamma(1-e_j+t)$, $(j=1, 2,\ldots, k)$ and $\Gamma(a_j+s+t)$, $(j=1, 2,\ldots, p)$, $\Gamma\left(\frac{v+1+2i}{2\delta}+s+t\right)$, $\Gamma\left(\frac{v+2+2i}{2\delta}+s+t\right)$, $i=0, 1,\ldots,(\delta-1)$ to the left of the contour.

Therefore, on interpreting (2.3) in accordance with the definition of the generalized Meijer function $S(x, y)$ (1.1), we obtain the value of the integral (2.1).

(B) The proof of integral (2.2) is similar to (2.1) and using the known integral [5, p. 41, ex. (9)]:

$$\int_0^{\pi/2}\cos m\theta\cos^n\theta\,d\theta=\frac{\pi\,(n+1)}{2^{n+1}\,\Gamma\left(\frac{1}{2}n+\frac{1}{2}m+1\right)},$$

where m is a positive integer.

§ 3. Fourier Series

In this section we have established two Fourier series for generalized Meijer functions $S\left(x\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}^{2\delta}, y\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}^{2\delta}\right)$ of two variables with the use of integrals evaluated in section 2.

First Fourier Series:

$$(\sin\theta)^v\,S\left[\begin{array}{c|c|c}\begin{matrix}p, & 0\\ A-p, & B\end{matrix} & (a);\ \ (b) & \\ \begin{pmatrix}q, & r\\ C-q, & D-r\end{pmatrix} & (c);\ \ (d) & x(\sin\theta)^{2\delta},\ \ y(\sin\theta)^{2\delta}\\ \begin{pmatrix}k, & l\\ E-k, & F-l\end{pmatrix} & (e);\ \ (f) & \end{array}\right]=$$

(3.1)

$$=\frac{2}{\pi^{3/2}\sqrt{\delta}}\sum_{\xi=0}^{\infty}\Gamma\left(\frac{1}{2}\pm\xi\right)S\left[\begin{array}{c|l}\begin{bmatrix}p+2, & 0\\ A-p, & B+2\delta\end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right),\\ \begin{pmatrix}q, & r\\ C-q, & D-r\end{pmatrix} & \Delta\left(\delta, \frac{1}{2}\right)v\pm\xi+1,\\ \begin{pmatrix}k, & l\\ E-k, & F-l\end{pmatrix} & \end{array}\right.$$

$$\left.\begin{array}{ll|c}(a); & & \\ (b); & & \\ (c); & (d) & x,\ \ y\\ (e); & (f) & \end{array}\right]\cos 2\,\xi\theta.$$

Second Fourier Series:

$$
(\cos\theta)^{v} S\left[\begin{array}{c|c|c}
\left|\begin{matrix} p, & 0 \\ A-p, & B \end{matrix}\right| & (a);\ (b) & \\
\left(\begin{matrix} q, & r \\ C-q, & D-r \end{matrix}\right) & (c);\ (d) & x(\cos\theta)^{2\delta},\ y(\cos\theta)^{2\delta} \\
\left(\begin{matrix} k, & l \\ E-k, & F-l \end{matrix}\right) & (e);\ (f) &
\end{array}\right] =
$$

(3.2)

$$
= \frac{2}{\sqrt{\pi}\,\delta^{1/2}} \sum_{\xi=0}^{\infty} S\left[\begin{array}{c|c|c}
\left|\begin{matrix} p+2\delta, & 0 \\ A-p, & B-2\delta \end{matrix}\right| \Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right), & (a); & \\
& (b); & \\
\left(\begin{matrix} q, & r \\ C-q, & D-r \end{matrix}\right) \Delta\left(\delta, \frac{1}{2}v \pm \frac{1}{2}\xi + 1\right), & & x, y \\
& (c);\ (d) & \\
\left(\begin{matrix} k, & l \\ E-k & F-l \end{matrix}\right) & (e);\ (f) &
\end{array}\right] \cos \xi\theta
$$

where δ is a positive integer, $0 < \theta < \pi/2$.

The above expansion formulae are valid under the following set of conditions

(i) $A+B+C+D < 2(p+q+r)$, $|\arg(x)| < (p+k+r - \frac{1}{2}A - \frac{1}{2}B - \frac{1}{2}C - \frac{1}{2}D)\pi$,

$A+B+E+F < 2(p+k+l)$, $|\arg(y)| < (p+k+l - \frac{1}{2}A - \frac{1}{2}B - \frac{1}{2}E - \frac{1}{2}F)\pi$,

$\mathrm{Re}(v + 2\delta d_{h_1} + 2\delta f_{h_2}) > 0$, $h_1 = 1, 2, \ldots, r$; $h_2 = 1, 2, \ldots, l$.

(ii) $A+C < B+D$, $A+E < B+F$ $(A+C = B+D$, $A+E = B+F)$, then $|x|, |y| < 1$, $\mathrm{Re}(v + 2\delta d_{h_1} + 2\delta f_{h_2}) > 0$, $\{h_1 = 1, 2, \ldots, r;\ h_2 = 1, 2, \ldots, l\}$.

Proofs:

(A) For $0 < \theta < 2 < \pi/2$, let

(3.3)

$$
f(\theta) \equiv (\sin\theta)^{v} S\left[\begin{array}{c|c|c}
\left|\begin{matrix} p, & 0 \\ A-p, & B \end{matrix}\right| & (a);\ (b) & \\
\left(\begin{matrix} q, & r \\ C-q, & D-r \end{matrix}\right) & (c);\ (d) & x(\sin\theta)^{2\delta},\ y(\sin\theta)^{2\delta} \\
\left(\begin{matrix} k & l \\ E-k & D-l \end{matrix}\right) & (e);\ (f) &
\end{array}\right] =
$$

$$
= \sum_{\xi=0}^{\infty} P_{\xi} \cos 2\xi\theta,
$$

where $0 < \theta < \pi/2$ and δ is a positive integer.

Equation (3.3) is valid since $f(\theta)$ is continuous and of bounded variation in the open interval $(0, \pi/2)$. Now multiply both sides of (3.3) by $\cos 2u\theta$ and integrate with respect to θ from 0 to $\pi/2$ and change the order of integration and summation (which we suppose to be permissible) on the right, we obtain

$$
\int_0^{\pi/2} \cos 2u\theta\, (\sin\theta)^{v} S\left[\begin{array}{c|c|c}
\left|\begin{matrix} p & 0 \\ A-p, & B \end{matrix}\right| & (a);\ (b) & \\
\left(\begin{matrix} q & r \\ C-q, & D-r \end{matrix}\right) & (c);\ (d) & (\sin\theta)^{2\delta},\ y(\sin\theta)^{2\delta} \\
\left(\begin{matrix} k, & l \\ E-k, & F-l \end{matrix}\right) & (e);\ (f) &
\end{array}\right] d\theta =
$$

(3.4)

$$
= \sum_{\xi=0}^{\infty} P_{\xi} \int_0^{\pi/2} \cos 2u\theta \cos 2\xi\theta\, d\theta.
$$

Using the orthogonality-property for the cosine-functions on the right and the result (2.1) on the left of (3.4), we get

$$
P_u = \frac{2\Gamma\left(\frac{1}{2} \pm u\right)}{\pi^{3/2}\,\delta^{1/2}} S\left[\begin{array}{c}
\left[\begin{matrix} p+2\delta, & 0 \\ A-p, & B+2 \end{matrix}\right] \Delta\left(\delta, \frac{v+1}{2}\right), \quad \Delta\left(\delta, \frac{v+2}{2}\right), \\
\left(\begin{matrix} q, & r \\ C-q, & D-r \end{matrix}\right) \Delta\left(\delta, \frac{1}{2}v \pm u + 1\right), \\
\left(\begin{matrix} k, & l \\ E-k, & F-l \end{matrix}\right)
\end{array}\right.
$$

(3.5)

$$\left.\begin{matrix}(a)\ ; & \\ (b)\ ; & \\ (c)\ ; & (d) \\ (e)\ ; & (f)\end{matrix}\ \right|\ x,\ y$$

The Fourier-series (3.1) is obtained with the help of (3.3) and (3.5).

(B) To establish (3.2), let

$$(3.6)\qquad f(\theta) \equiv (\cos\theta)^{v} S \left|\begin{matrix} \left|\begin{matrix} p, & 0 \\ A-p, & B \end{matrix}\right| & (a)\ ; & (b) & \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} & (c)\ ; & (d) & x(\cos\theta)^{2\delta},\ y(\cos\theta)^{2\delta} \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & (e)\ ; & (f) & \end{matrix}\right| = \sum_{\xi=0}^{\infty} P_{\xi} \cos\xi\theta,$$

where $0 < \theta < \pi/2$ and δ is a positive integer.

Proceeding on the same lines as above, multiply both sides of (3.6) by $\cos u\theta$ and integrate with respect to θ over $(0, \pi/2)$ and change the order of integration and summation which is easily seen to be justified and using the orthogonal-relationship for cosine functions on the right and result (2.2) on the left of (3.6), we have

$$(3.7)\qquad P_u = \frac{2}{\sqrt{\pi}\,\delta^{1/2}} S \left|\begin{matrix} \left|\begin{matrix} p+2 & 0 \\ A-p, & B+2\delta \end{matrix}\right| & \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right), & \\ \begin{pmatrix} q & r \\ C-q, & D-r \end{pmatrix} & \Delta\left(\delta, \frac{1}{2}v \pm \frac{1}{2}u+1\right), & \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & & \\ & (a)\ ; & \\ & (b)\ ; & x,\ y \\ & (c)\ ;\quad (d) & \\ & (e)\ ;\quad (f) & \end{matrix}\right|.$$

With the help of (3.6) and (3.7), we obtain the Fourier series (3.2).

§ 4. Applications

Particular cases of (2.1), (2.2), (3.1) and (3.2):

a) Substituting $A = B = 0$, we obtain

$$(4.1)\qquad \int_0^{\pi/2} \cos 2u\theta\,(\sin\theta)^{v}\, G^{r,q}_{C,D}\left[x\{\sin\theta\}^{2\delta}\ \middle|\ \begin{matrix}(c)\\(d)\end{matrix}\right] G^{l,k}_{E,F}\left[y\{\sin\theta\}^{2\delta}\ \middle|\ \begin{matrix}(e)\\(f)\end{matrix}\right] d\theta =$$

$$= \frac{\Gamma\left(\frac{1}{2} \pm u\right)}{2\sqrt{\pi}\,\delta^{1/2}} S \left|\begin{matrix} \left|\begin{matrix} 2\delta, & 0 \\ 0, & 2\delta \end{matrix}\right| & \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right)\ ; & \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} & \Delta\left(\delta, \frac{1}{2}v \pm u+1\right) & \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & \begin{matrix}(c)\ ; & (d) \\ (e)\ ; & (f)\end{matrix} & x,\ y \end{matrix}\right|,$$

$$(4.2)\qquad \int_0^{\pi/2} \cos u\theta\,(\cos\theta)^{v}\, G^{r,q}_{C,D}\left[x\{\cos\theta\}^{2\delta}\ \middle|\ \begin{matrix}(c)\\(d)\end{matrix}\right] G^{l,k}_{E,F}\left[y\{\cos\theta\}^{2\delta}\ \middle|\ \begin{matrix}(e)\\(f)\end{matrix}\right] d\theta =$$

$$= \frac{\sqrt{\pi}}{2\delta^{1/2}} S \left|\begin{matrix} \begin{bmatrix} 2\delta, & 0 \\ 0, & 2\delta \end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right)\ ; & \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} & \Delta\left(\delta, \frac{1}{2}v \pm u+1\right) & x,\ y \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & \begin{matrix}(c)\ ; & (d) \\ (e)\ ; & (f)\end{matrix} & \end{matrix}\right|.$$

$$(4.3)\qquad (\sin\theta)^{v}\, G^{r,q}_{C,D}\left(x\{\sin\theta\}^{2\delta}\ \middle|\ \begin{matrix}(c)\\(d)\end{matrix}\right) G^{l,k}_{E,F}\left(y\{\sin\theta\}^{2\delta}\ \middle|\ \begin{matrix}(e)\\(f)\end{matrix}\right) =$$

$$\frac{2}{\pi^{3/2}\,\delta^{1/2}} \sum_{\xi=0}^{\infty} \Gamma\left(\frac{1}{2} \pm \xi\right) S \left|\begin{matrix} \begin{bmatrix} 2\delta, & 0 \\ 0, & 2\delta \end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right)\ ; & \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} & \Delta\left(\delta, \frac{1}{2}v \pm \xi+1\right) & x,\ y \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} & \begin{matrix}(c)\ ; & (d) \\ (e)\ ; & (f)\end{matrix} & \end{matrix}\right| \cos 2\xi\theta$$

$$(\cos\theta)^{v}\, G^{r,q}_{C,D}\left(x\,(\cos\theta)^{2\delta}\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right) G^{l,k}_{E,F}\left(y\,(\cos\theta)^{2}\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right)=$$

$$(4.4)\qquad =\frac{2}{\sqrt{\pi}\,\delta^{1/2}}\sum_{\xi=0}^{\infty} S\left[\begin{matrix}\begin{bmatrix}2\delta, & 0\\ 0 & 2\delta\end{bmatrix}\\ \begin{pmatrix}q, & r\\ C-q, & D-r\end{pmatrix}\\ \begin{pmatrix}k, & l\\ E-k, & F-l\end{pmatrix}\end{matrix}\middle|\begin{matrix}\Delta\left(\delta,\frac{v+1}{2}\right),\ \Delta\left(\delta,\frac{v+2}{2}\right);\\ \Delta\left(\delta,\frac{1}{2}v\pm\frac{1}{2}\xi+1\right)\\ (c);\quad (d)\\ (e);\quad (f)\end{matrix}\middle|\, x, y\right]\cos\xi\theta,$$

where $G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)$ is the Meijer G-function ([2], p. 207, [1]) and δ is a positive integer, $u=0, 1, 2,\ldots;\ 0<\theta<\pi/2$, and valid under the following conditions

(i) $2(q+r)>C+D,\quad |\arg(x)|<\left(q+r-\frac{1}{2}C-\frac{1}{2}D\right)\pi,$

$2(l+k)>E+F,\quad |\arg(y)|<\left(l+k-\frac{1}{2}E-\frac{1}{2}F\right)\pi,$

$\mathrm{Re}\,(v+2\delta d_{h_1}+2\delta f_{h_2})>0,\ (h_1=1, 2,\ldots, r;\ h_2=1, 2,\ldots, l).$

(ii) $C<D,\ E<F$ (If $C=D,\ E=F$, we must have $|x|<1,\ |y|<1$),

$\mathrm{Re}\,(v+2\delta d_{h_1}+2\delta f_{h_2})>0,\ (h_1=1, 2,\ldots, r;\ h_2=1, 2,\ldots, l).$

b) Setting $A=p,\ E=k,\ l=1,\ f_1=0$ and replace $A+C$ by A, $B+D$ by B, $A+q$ by s together with the appropriate changes in the parameters and there let $y\to 0$ etc., we have

$$(4.5)\qquad \int_0^{\pi/2}\cos 2u\theta\,(\sin\theta)^{v}\, G^{r,s}_{A,B}\left(x\,(\sin\theta)^{2\delta}\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right)d\theta=$$

$$=\frac{\Gamma\left(\frac{1}{2}\pm u\right)}{2\sqrt{\pi}\,\delta^{1/2}}\, G^{r,s+2\delta}_{A+2\delta,B+2\delta}\left(x\,\middle|\,\begin{matrix}\nabla\left(\delta,\frac{v+1}{2}\right),\ \nabla\left(\delta,\frac{v+2}{2}\right),\ (a)\\ (b),\ \nabla\left(\delta,\frac{1}{2}v\pm u+1\right)\end{matrix}\right).$$

$$(4.6)\qquad \int_0^{\pi/2}\cos u\theta\,(\cos\theta)^{v}\, G^{r,s}_{A,B}\left(x\,(\cos\theta)^{2\delta}\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right)d\theta=$$

$$=\frac{\sqrt{\pi}}{2\delta^{1/2}}\, G^{r,s+2\delta}_{A+2\delta,B+2\delta}\left(x\,\middle|\,\begin{matrix}\nabla\left(\delta,\frac{v+1}{2}\right),\ \nabla\left(\delta,\frac{v+2}{2}\right),\ (a)\\ (b),\ \nabla\left(\delta,\frac{1}{2}v\pm\frac{1}{2}u+1\right)\end{matrix}\right).$$

$$(4.7)\qquad (\sin\theta)^{v}\, G^{r,s}_{A,B}\left(x\,(\sin\theta)^{2\delta}\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right)=$$

$$=\frac{2}{\delta^{1/2}\pi^{3/2}}\sum_{\xi=0}^{\infty}\Gamma\left(\frac{1}{2}\pm\xi\right) G^{r,s+2\delta}_{A+2\delta,B+2\delta}\left(x\,\middle|\,\begin{matrix}\nabla\left(\delta,\frac{v+1}{2}\right),\ \nabla\left(\delta,\frac{v+2}{2}\right),\ (a)\\ (b),\ \nabla\left(\delta,\frac{1}{2}v\pm\xi+1\right)\end{matrix}\right)\cos 2\xi\theta.$$

$$(4.8)\qquad (\cos\theta)^{v}\, G^{r,s}_{A,B}\left(x\,(\cos\theta)^{2\delta}\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right)=$$

$$=\frac{2}{\sqrt{\pi}\,\delta^{1/2}}\sum_{\xi=0}^{\infty} G^{r,s+2\delta}_{A+2\delta,B+2\delta}\left(x\,\middle|\,\begin{matrix}\nabla\left(\delta,\frac{v+1}{2}\right),\ \nabla\left(\delta,\frac{v+2}{2}\right),\ (a)\\ (b),\ \nabla\left(\delta,\frac{1}{2}v\pm\xi+1\right)\end{matrix}\right)\cos 2\xi\theta$$

valid for δ positive and $u=0, 1, 2,\ldots;\ 0<\theta<\pi/2$.

(i) $2(r+s)>A+B,\quad |\arg(x)|<(r+s-(1/2)A-(1/2)B)\pi,$

$\mathrm{Re}\,(v+2\delta b_h)>0,\ h=1, 2,\ldots, r.$

(ii) $A<B$ (or $A=B,\ |x|<1$).

$\mathrm{Re}\,(v+2\delta b_h)>0,\ h=1, 2,\ldots, r.$

The symbol $\nabla(m, n)$ stands for a set of m-parameters:

$$1-\frac{n}{m},\quad 1-\frac{n+1}{m},\ldots, 1-\frac{n+m-1}{m}.$$

c) Taking $p=A=m,\ B=n,\ q=k=C=E=l,\ r=l=1,\ D=F=p+1,\ d_1=f_1=0$ and replace $b_j,\ 1-c_j,\ 1-d_j,\ 1-e_j$ and $1-f_j$ by $c_j,\ b_j,\ d_j,\ b'_j$ and d'_j respectively, the generalized Meijer function $S(x, y)$ reduces to Kampé de Fériet's double hypergeometric function (1.2). Thus we have

$$\int_0^{\pi/2} \cos 2u\theta\, (\sin\theta)^v F\left\{\begin{array}{ll|l} m & (a_m) & \\ l & (b_l)\,;\ (b'_l) & x\,(\sin\theta)^{2\delta},\ y\,(\sin\theta)^{2\delta} \\ n & (c_n) & \\ p & (d_p)\,;\ (d'_p) & \end{array}\right\} d\theta =$$

(4.9)

$$= \frac{\Gamma\left(\frac{1}{2} \pm u\right) \prod_{i=0}^{\delta-1} \Gamma\left(\frac{v+1+2i}{2\delta}\right) \prod_{i=0}^{\delta-1} \left(\frac{v+2+2i}{2\delta}\right)}{2\,\delta^{1/2} \sqrt{\pi} \prod_{i=0}^{\delta-1} \left(\frac{v \pm 2u + 2 + 2i}{2\delta}\right)}\,.$$

$$\cdot F \left|\begin{array}{l|c|l} m+2\delta & (a_m),\ \Delta\left(\delta,\ \frac{v+1}{2}\right),\ \Delta\left(\delta,\ \frac{v+2}{2}\right) & \\ l & (b_l)\,;\ \ (b'_l) & x,\ y \\ n+2\delta & (c_n),\ \Delta\,(\delta,\ 1/2v \pm u + 1) & \\ p & (d_p)\,;\ \ (d'_p) & \end{array}\right| .$$

$$\int_0^{\pi/2} \cos u\theta\ (\cos\theta)^v F \left|\begin{array}{l|l|l} m & (a_m) & \\ l & (b_l)\,;\ \ (b'_l) & x\,(\cos\theta)^{2\delta},\ y\,(\cos\theta)^{2\delta} \\ n & (c_n) & \\ p & (d_p)\,;\ \ (d'_p) & \end{array}\right| d\theta =$$

(4.10)

$$= \frac{\sqrt{\pi} \sum_{i=0}^{\delta-1} \Gamma\left(\frac{v+1+i}{2\delta}\right) \prod_{i=0}^{\delta-1} \Gamma\left(\frac{v+2+i}{2\delta}\right)}{2\,\delta^{1/2} \sum_{i=0}^{\delta-1} \Gamma\left(\frac{v \pm u + 2 + 2i}{2}\right)}\,.$$

$$\cdot F \left|\begin{array}{l|c|l} m+2 & (a_m),\ \Delta\left(\delta,\ \frac{v+1}{2}\right),\ \Delta\left(\delta,\ \frac{v+2}{2}\right) & \\ l & (b_l)\,;\ \ (b'_l) & x,\ y \\ n+2 & (c_n),\ \Delta\,(\delta,\ 1/2\,v \pm u + 1) & \\ p & (d_p)\,;\ \ (d'_p) & \end{array}\right| .$$

$$(\sin\theta)^v F \left|\begin{array}{l|l|l} m & (a_m) & \\ l & (b_l)\,;\ \ (b'_l) & x\,(\sin\theta)^{2\delta},\ y\,(\sin\theta)^{2\delta} \\ n & (c_n) & \\ p & (d_p)\,;\ \ (d'_p) & \end{array}\right| =$$

(4.11)

$$= \frac{2 \sum_{i=0}^{\delta-1} \Gamma\left(\frac{v+1+2i}{2\delta}\right) \prod_{i=0}^{\delta-1} \Gamma\left(\frac{v+2+2i}{2\delta}\right)}{\pi^{3/2}\,\delta^{1/2}} \sum_{\xi=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} \pm \xi\right)}{\sum_{i=0}^{\delta-1} \Gamma\left(\frac{v + 2\xi + 2 + 2i}{2\delta}\right)}\,.$$

$$\cdot F \left|\begin{array}{l|c|l} m+2 & (a_m),\ \Delta\left(\delta,\ \frac{v+1}{2}\right),\ \Delta\left(\delta,\ \frac{v+2}{2}\right) & \\ l & (b_l)\,;\ \ (b'_l) & x,\ y \\ n+2 & (c_n),\ \Delta\,(\delta,\ 1/2\,v \pm \xi + 1) & \\ p & (d_p)\,;\ \ (d'_p) & \end{array}\right| \cos 2\xi\theta.$$

(4.12)

$$(\cos\theta)^v F \left|\begin{array}{l|l|l} m & (a_m) & \\ l & (b_l)\,;\ \ (b'_l) & x\,(\cos\theta)^{2\delta},\ y\,(\cos\theta)^{2\delta} \\ n & (c_n) & \\ p & (d_p)\,;\ \ (d'_p) & \end{array}\right| =$$

$$= \frac{2 \sum_{i=0}^{\delta-1} \Gamma\left(\frac{v+1+2i}{2\delta}\right) \prod_{i=0}^{\delta-1} \Gamma\left(\frac{v+2+2i}{2\delta}\right)}{\pi^{1/2}} \sum_{\xi=0}^{\infty} \frac{1}{\sum_{i=0}^{\delta-1} \Gamma\left(\frac{v + \xi + 2 + 2i}{2\delta}\right)}$$

$$\cdot F \left|\begin{array}{l|c|l} m+2 & (a_m),\ \Delta\left(\delta,\ \frac{v+1}{2}\right),\ \Delta\left(\delta,\ \frac{v+2}{2}\right) & \\ l & (b_l)\,;\ \ (b'_l) & x,\ y \\ n+2 & (c_n),\ \Delta\left(\delta,\ \frac{1}{2}\,v \pm \frac{1}{2}\,\xi + 1\right) & \\ p & (d_p)\,;\ \ (d'_p) & \end{array}\right| \cos \xi\theta\,.$$

where δ is a positive integer, $u = 0, 1, 2, \dots$; $0 < \theta < \pi/2$ and valid if $(m + l) < (n + p + 1)$, $\{m + l = n + p + 1$; then $|x|, |y| < 1\}$, Re $(v) > 0$.
Or if $m + l + 1 > n + p$, then

$$|\arg(y)|,\ |\arg(x)| < (m + l - n - p)\pi/2.$$

Special cases of (4.9), (4.10), (4.11) and (4.12):

(i) By setting the parameters suitably, the double hypergeometric functions F can be reduced to Appell's [1] four functions F_1, F_2, F_3 and F_4 and subsequently we obtain the results involving Appell's functions.

(ii) Taking $m = n$, $l = 1$, $p = 0$ and $y = x$, the F-functions are reduced to the generalized hypergeometric functions of one variable, we get

$$\int_0^{\pi/2} \cos 2u\theta\,(\sin\theta)^v\,{}_{p+1}F_p\left[\begin{matrix} a_1,\dots,a_p,\ b+b' \\ c_1,\dots,c_p \end{matrix};\ x(\sin\theta)^{2\delta}\right] d\theta =$$

(4.13)

$$\frac{\Gamma\left(\frac{1}{2} \pm u\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+1+2i}{2\delta}\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+2+2i}{2\delta}\right)}{2\sqrt{\pi}\,\delta^{1/2}\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v \pm 2u + 2 + 2i}{2\delta}\right)}\cdot$$

$${}_{p+2\delta+1}F_{p+2\delta}\left[\begin{matrix} a_1,\dots,a_p,\ \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right),\ b+b' \\ c_1,\dots,c_p,\ \Delta(\delta,\ 1/2\,v \pm u + 1) \end{matrix};\ x\right].$$

$$\int_0^{\pi/2} \cos u\theta\,(\cos\theta)^v\,{}_{p+1}F_p\left[\begin{matrix} a_1,\dots,a_p,\ b+b' \\ c_1,\dots,c_p \end{matrix};\ x(\cos\theta)^{2\delta}\right] d\theta =$$

(4.14)

$$= \frac{\sqrt{\pi}\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+1+2i}{2\delta}\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+2+2i}{2\delta}\right)}{2\delta^{1/2}\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v \pm u + 2i}{2\delta}\right)}$$

$${}_{p+2\delta+1}F_{p+2\delta}\left[\begin{matrix} a_1,\dots,a_p,\ \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right),\ b+b' \\ c_1,\dots,c_p,\ \Delta(\delta,\ 1/2\,v + 1/2\,u + 1) \end{matrix};\ x\right].$$

(4.15)
$$(\sin\theta)^v\,{}_{p+1}F_p\left[\begin{matrix} a_1,\dots,a_p,\ b+b' \\ c_1,\dots,c_p \end{matrix};\ x(\sin\theta)^{2\delta}\right] =$$

$$= \frac{2\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+1+2i}{2\delta}\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+2+2i}{2\delta}\right)}{\delta^{1/2}\,\pi^{1/2}}\sum_{\xi=0}^{\infty}\frac{\Gamma\left(\frac{1}{2} \pm \xi\right)}{\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v \pm 2\xi + 2 + 2i}{2\delta}\right)}$$

$${}_{p+2\delta+1}F_{p+2\delta}\left[\begin{matrix} a_1,\dots,a_p,\ \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right),\ b+b' \\ c_1,\dots,c_p,\ \Delta(\delta,\ 1/2\,v \pm \xi + 1) \end{matrix};\ x\right]\cos 2\xi\theta.$$

(4.16)
$$(\cos\theta)^v\,{}_{p+1}F_p\left[\begin{matrix} a_1,\dots,a_p,\ b+b' \\ c_1,\dots,c_p \end{matrix};\ x(\cos\theta)^{2\delta}\right] =$$

$$\frac{2\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+1+2i}{2\delta}\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+2+2i}{2\delta}\right)}{\sqrt{\pi}\,\delta^{1/2}}\sum_{\xi=0}^{\infty}\frac{1}{\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v \pm \xi + 2 + 2i}{2\delta}\right)}$$

$${}_{p+2\delta+1}F_{p+2\delta}\left[\begin{matrix} a_1,\dots,a_p,\ \Delta\left(\delta, \frac{v+1}{2}\right),\ \Delta\left(\delta, \frac{v+2}{2}\right),\ b+b'; \\ c_1,\dots,c_p,\ \Delta(\delta,\ 1/2\,v + 1/2\,\xi + 1) \end{matrix}\ x\right]\cos\xi\theta$$

valid for δ a positive integer, $u = 0, 1, 2, \dots$; $0 < \theta < \pi/2$ and Re $(v) > 0$.

§ 5. The double-integral and expansion analogues

In this section we have established some double-integral and expansion-analogues involving generalized Meijer functions with the help of the preceeding sections

$$(5.1)\qquad \int_0^{\pi/2}\int_0^{\pi/2} \cos 2u\theta \cos 2m\varphi\, (\sin\theta)^r (\sin\varphi)^n$$

$$S\left[\begin{array}{c|c} \begin{bmatrix} p\ , & 0 \\ A-p\ , & B \end{bmatrix} & (a)\ ;\ (b) \\ \begin{pmatrix} q\ , & r \\ C-q\ , & D-r \end{pmatrix} & (c)\ ;\ (d) \\ \begin{bmatrix} k\ , & l \\ E-k\ , & F-l \end{bmatrix} & (e)\ ;\ (f) \end{array}\ \middle|\ x(\sin\theta)^{2\rho},\ y(\sin\varphi)^{2\sigma}\right] d\theta\, d\varphi =$$

$$= \frac{(-1)^{u+m}\pi}{4\sqrt{(\rho\sigma)}} S\left[\begin{array}{c|c} & (a)\ ;\ (b) \\ \begin{bmatrix} p\ , & 0 \\ A-p\ , & B \end{bmatrix} & \Delta\left(\rho, -\frac{1}{2}v+\frac{1}{2}\right), \Delta\left(\rho, -\frac{1}{2}v\right), (c) \\ \begin{pmatrix} q+2\rho, & r \\ C-q, & D-r+2\rho \end{pmatrix} & (d), \Delta\left[\rho, -\frac{1}{2}v \pm u\right] \\ \begin{pmatrix} k+2\sigma, & l \\ E-k, & F-l+2\sigma \end{pmatrix} & \Delta\left(\sigma, -\frac{1}{2}n+\frac{1}{2}\right), \Delta\left(\sigma, -\frac{1}{2}n\right), (e)\ ; \\ & (f), \Delta\left[\sigma, -\frac{1}{2}n \pm m\right] \end{array}\ \middle|\ x, y\right]$$

$$(5.2)\qquad \int_0^{\pi/2}\int_0^{\pi/2} \cos u\theta \cos m\varphi\, (\cos\theta)^r (\cos\varphi)^n\, d\theta d\varphi$$

$$S\left[\begin{array}{c|c} \begin{bmatrix} p\ , & 0 \\ A-p\ , & B \end{bmatrix} & (a)\ ;\ (b) \\ \begin{bmatrix} q\ , & r \\ C-q\ , & D-r \end{bmatrix} & (c)\ ;\ (d) \\ \begin{bmatrix} k\ , & l \\ E-k\ , & F-l \end{bmatrix} & (e)\ ;\ (f) \end{array}\ \middle|\ x(\cos\theta)^{2\rho},\ y(\cos\varphi)^{2\sigma}\right] d\theta\, d\varphi =$$

$$= \frac{\pi}{4\sqrt{(\rho\sigma)}} S\left[\begin{array}{c|c} & (a)\ ;\ (b) \\ \begin{bmatrix} p\ , & 0 \\ A-p\ , & B \end{bmatrix} & \Delta\left(\rho, -\frac{1}{2}v+\frac{1}{2}\right), \Delta\left(\rho, -\frac{1}{2}v\right), (c)\ ; \\ \begin{bmatrix} q+2, & r \\ C-q, & D-r+2\rho \end{bmatrix} & (d), \Delta\left(\rho, -\frac{1}{2}v \pm \frac{1}{2}u\right) \\ \begin{pmatrix} k+2, & l \\ E-k, & F-l+2\sigma \end{pmatrix} & \Delta\left(\sigma, -\frac{1}{2}n+\frac{1}{2}\right), \Delta\left(\sigma, -\frac{1}{2}n\right), (e)\ ; \\ & (f), \Delta\left(\sigma, -\frac{1}{2}n \pm \frac{1}{2}m\right) \end{array}\ \middle|\ x, y\right]$$

$$(5.3)\quad (\sin\theta)^r (\sin\varphi)^n\, S\left[\begin{array}{c|c} \begin{bmatrix} p\ , & 0 \\ A-p\ , & B \end{bmatrix} & (a)\ ;\ (b) \\ \begin{pmatrix} q\ , & r \\ C-q\ , & D-r \end{pmatrix} & (c)\ ;\ (d) \\ \begin{pmatrix} k\ , & l \\ E-k\ , & F-l \end{pmatrix} & (e)\ ;\ (f) \end{array}\ \middle|\ x(\sin\theta)^{2\rho}, y(\sin\varphi)^{2\sigma}\right] =$$

$$= \frac{4}{\pi\sqrt{(\sigma\rho)}} \sum_{\delta=0}^{\infty}\sum_{\xi=0}^{\infty} (-1)^{\delta+\xi}\, .$$

$$S\left[\begin{array}{c|c} & (a)\ ;\ (b) \\ \begin{pmatrix} p\ , & 0 \\ A-p, & B \end{pmatrix} & \Delta\left(\rho, -\frac{1}{2}v+\frac{1}{2}\right), \Delta\left(\rho, -\frac{1}{2}v\right), (c)\ ; \\ \begin{pmatrix} q+2\rho & r \\ C-q, & D-r+2\rho \end{pmatrix} & (d), \Delta\left(\rho, -\frac{1}{2}v \pm \delta\right) \\ \begin{pmatrix} k+2\sigma, & l \\ E-k, & F-l+2\sigma \end{pmatrix} & \Delta\left(\sigma, -\frac{1}{2}n+\frac{1}{2}\right), \Delta\left(\sigma, -\frac{1}{2}n\right), (e)\ ; \\ & (f), \Delta\left(\sigma, -\frac{1}{2}n \pm \xi\right) \end{array}\ \middle|\ x, y\right] \cos 2\theta \cos 2\xi\varphi.$$

$$(5.4)\qquad (\cos\theta)^{m}(\cos\varphi)^{n}\, S\left[\begin{array}{c|c|c} \begin{bmatrix} p, & 0\\ A-p, & B\end{bmatrix} & (a) \; ; \; (b) & \\ \begin{pmatrix} q, & r\\ C-q, & D-r\end{pmatrix} & (c) \; ; \; (d) & x(\cos\theta)^{2\rho},\ y(\cos\varphi)^{2\sigma}\\ \begin{pmatrix} k, & l\\ E-k, & F-l\end{pmatrix} & (e) \; ; \; (f) & \end{array}\right] =$$

$$= \frac{4}{\pi\Gamma(\sigma\rho)}\sum_{\delta=0}^{\infty}\sum_{\xi=0}^{\infty} S\left[\begin{array}{c|c} \begin{pmatrix} p, & 0\\ A-p, & B\end{pmatrix} & \\ \begin{pmatrix} q+2\rho, & r\\ C-q, & D-r+2\rho\end{pmatrix} & \\ \begin{pmatrix} k+2\sigma, & l\\ E-k, & F-l+2\sigma\end{pmatrix} & \\ \Delta\left(\rho, -\frac{1}{2}v+\frac{1}{2}\right), \Delta\left(\rho, -\frac{1}{2}v\right), (c) ; & \\ (d), \Delta\left(\rho, -\frac{1}{2}v+\frac{1}{2}\delta\right) & x, y \\ \Delta\left(\sigma, -\frac{1}{2}n+\frac{1}{2}\right), \Delta\left(\sigma, -\frac{1}{2}n\right), (e) ; & \\ (f), \Delta\left(\sigma, -\frac{1}{2}n+\frac{1}{2}\xi\right) & \end{array}\right] \cos\delta\theta\cos\xi\varphi .$$

These double-integral and expansion-analogues are similar to our previous results (2.1), (2.2), (3.1) and (3.2) respectively.

The aforesaid results are valid under the following sets of conditions

$$2(p+q+r) > [A+B+C+D],$$
$$2(p+k+l) > [A+B+E+F],$$

$$\text{(i)}\qquad |\arg(x)| < \pi\left[p+q+r-\frac{1}{2}(A+B+C+D)\right],$$
$$|\arg(y)| < \pi\left[p+k+l-\frac{1}{2}(A+B+E+F)\right],$$
$$\operatorname{Re}[v+2\rho d_{h_1}] > 0,\ h_1 = 1,2,\ldots,r,$$
$$\operatorname{Re}[n+2\sigma f_{h_2}] > 0,\ h_2 = 1,2,\ldots,l.$$

$$A+C<B+D,\ A+E<B+F,$$

or else $A+C=B+D$, $A+E=B+F$ with $|x|<1$, $|y|<1$,

$$\text{(ii)}\qquad \operatorname{Re}[v+2\rho d_{h_1}] > 0,\ h_1 = 1,2,\ldots,r,$$
$$\operatorname{Re}[n+2\sigma f_{h_2}] > 0,\ h_2 = 1,2,\ldots,l,$$

where ρ and σ are positive integers, $n=0,1,2,\ldots$; $m=0,1,2,\ldots$; $0<\theta<\pi/2$ and $0<\varphi<\pi/2$.

I wish to express my sincere thanks to Dr. V. M. Bhise for his kind supervision during the preparation of this paper.

REFERENCES

1. Appel Paul and Kampé de Fériet J. — *Fonctions hypergéométriques et hypersphériques. Polynomes d'Hermite.* Paris, Gauthier-Villars, 1926.
2. Erdélyi A. — *Higher transcendental functions.* Vol. I. McGraw-Hill, NewYork.
3. Kampé de Fériet J. — *La fonction hypergéométrique.* Páris, Gauthier-Villars, 1937.
4. Sneddon Ian. I., Ragab F. M. — *Expansion of Kampé de Fériet's double hypergeometric functions of higher order.* J. f. reine u. angew. Math., 212 (1963), pp. 113—119.
5. Sneddon Ian. I. — *Special functions of Mathematical Physics and Chemistry.* Oliver and Boyd, Edinburgh and London, New York, Interscience Publishers, Inc., 1956.
6. Sharma B. L. — *On the generalized function of two variables.* Annales de la Soc. Sci. de Bruxelles, 79 (1965), pp. 10—26.

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SERII FOURIER PENTRU FUNCŢII MEIJER GENERALIZATE

Rezumat

Se evaluează unele integrale care cuprind funcţii Meijer generalizate; acestea se utilizează pentru a stabili unele dezvoltări în serii Fourier pentru funcţii Meijer generalizate. Se stabilesc unele cazuri particulare interesante.