THE WEAK FIELD OF FRACTIONS OF A WEAK INTEGRAL DOMAIN

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Communicated at the Scientific Session of the „Al. I. Cuza" University, 26–28 october 1970

§ 1. A weak ring $R^{\cdot}$, [1], is an additive abelian group R_+ in which a second composition law is defined — let us call it multiplication „." — related to the addition by the weak distributive laws

$$a.(b_1+b_2)+a.0=a.b_1+a.b_2, \tag{1}$$

$$(b_1+b_2).a+0.a=b_1.a+b_2.a, \tag{2}$$

which imply

(3) $\quad a.0=a\varphi+c,\qquad$ (4) $\quad 0.a=a\psi+c,\ \forall a\in R_+,$

where φ, ψ are group endomorphisms of R_+ and c a fixed element of R_+.

From an ordinary ring $R^\times$ (not necessarily associative) — let us note its multiplication by „$\times$" — one can construct a weak ring $R^{\cdot}$ [1], by choosing two endomorphisms φ, ψ and an element c of its additive group R_+ and defining a new multiplication law „." by

$$a.b=a\times b+a\varphi+b\psi+c,\ \forall a,b\in R_+. \tag{5}$$

Conversely, to any weak ring $R^{\cdot}$ we can associate an ordinary ring $R^\times$ with the same additive group and the multiplication „$\times$" defined by

$$a\times b=a.b-a.0-0.b+0.0. \tag{6}$$

The weak ring $R^\cdot$ is a *weak field of II type (weak field II)* if $R^\cdot$ contains an element d with the property

(7) $$a.d = d.a = d, \ \forall a \in R^\cdot,$$

and $R^\cdot - \{d\}$ is a multiplicative group.

The weak ring $R^\cdot$ is a weak integral domain [4], if it possesses the element d, from (7), („$R^\cdot$ is a ring with d"), and the property

$$a.b = d \Rightarrow a = d \quad \text{or} \quad b = d.$$

A weak ring $R^\cdot$, with d, is a weak integral domain if and only if its associated ordinary ring $R^\times$ is an integral domain (in the sense: ring without zero divisors).

For a weak ring with d, (5) and (6) become respectively:

(5_1) $$a.b = (a - d) \times (b - d) + d,$$

(6_1) $$a \times b = (a + d).(b + d) - d$$

and the mapping

(8) $$\eta : R^\cdot \to R^\times_\times, \quad a\eta = a - d$$

is an isomorphism of the multiplicative groupoids of the rings $R^\cdot$ and $R^\times$.

From this isomorphism it follows, in particular, that the weak ring $R^\cdot$ is a weak field of II type, if and only if its associated ordinary ring is an ordinary field (or skew field).

§ **2.** We shall now define the concept of *weak quotient field of a commutative weak integral domain.* Let $R^\cdot$ be a weak integral domain, included in the commutative weak field II, $L^\cdot \supset R^\cdot$. The ordinary weak rings $L^\times$, $R^\times$ — respectively associated to $L^\cdot$, $R^\cdot$ by (6_1) — $L^\times \supset R^\times$, [3], are respectively an ordinary field and an ordinary integral domain, [4]; the unity elements $u \in L^\cdot$, $e \in L^\times$ are related by

$$e = u - d.$$

Each $r \in R^\cdot$, $r \neq d$, has a weak inverse $r^{-i} \in L^\cdot$ and, for every $a, b \in R^\cdot$, $b \neq d$, there exists a weak quotient $a.b^{-i} \in L^\cdot$. From the isomorphism η, (8), r^{-i} and $a.b^{-i}$ are related with the ordinary inverse and quotient of the elements of $R^\times$ in $L^\times$ by the formulas

(9) $$r^{-i} = (r - d)^{-1} + d, \quad r \neq d, \quad \text{or}$$

(9′) $$r_1^{-1} = (r_1 + d)^{-i} - d, \quad r_1 \neq 0.$$

(10) $$a.b^{-i} = \frac{a - d}{b - d} + d, \quad b \neq d, \quad \text{or}$$

(10′) $$\frac{a_1}{b_1} = (a_1 + d).(b_1 + d)^{-i} - d, \quad b_1 \neq 0.$$

Let $\mathcal{C}_0$ be the set of the weak quotients $a.b^{-i} \in L^\cdot$; we have the set inclusion

$$\mathcal{C}_0 \supset R^\cdot,$$

because, for every $r \in R^\cdot$, we have

$$r = (r.z).z^{-i}, \ \forall z \neq d,$$

in particular, $d = (d.z).z^{-i}$.

It is easy verified that $\mathcal{C}_0 - \{d\}$ is a subgroup of the multiplicative group $L^\cdot - \{d\}$.

Let $\{R^\cdot\}$, $\{\mathcal{C}_0\}$ be the weak subfields generated, respectively, by $R^\cdot$ in $L^\cdot$ and $\mathcal{C}_0$ in $L^\cdot$; obviously

$$\mathcal{C}_0 \subseteq \{R^\cdot\} \Rightarrow \{\mathcal{C}_0\} \subseteq \{R^\cdot\}.$$

As the weak subfield $\{\mathcal{C}_0\}$ contains the weak ring $R^\cdot$ and $\{R^\cdot\}$ is the minimal weak field containing $R^\cdot$, it follows

$$\{R^\cdot\} = \{\mathcal{C}_0\}.$$

On the other hand, let $\mathcal{C}^\times$ be the quotient field of the ordinary ring $R^\times$ in the field $L^\times$

(11) $$R^\times \subset \mathcal{C}^\times \subset L^\times$$

and let $\mathcal{C}^\cdot$ be the corresponding weak field II of $\mathcal{C}^\times$ (constructed by the method indicated in § 1, that is using (5_1) for the ordinary quotients of $\mathcal{C}^\times$, with $d = \frac{dy}{y} \in \mathcal{C}^\times$, $d \in R^\cdot$).

Then, from (11) it follows ([3], Lemma 1).

$R^\cdot$ weak subring $\subset$ *$\mathcal{C}^\cdot$ weak subfield* $\subset$ *$L^\cdot$ weak field.* Moreover, from (10)

$$\{\mathcal{C}_0\} \subseteq \mathcal{C}^\cdot.$$

As $\mathcal{C}^\times$ and $\mathcal{C}^\cdot$ have the same additive group, each element of $\mathcal{C}^\cdot$ can be written as an ordinary quotient $\frac{a_1}{b_1}$ in $\mathcal{C}^\times$ and, by (10′), we hawe

$$\mathcal{C}^\cdot \subseteq \{\mathcal{C}_0\}, \quad \text{hence} \quad \mathcal{C}^\cdot = \{\mathcal{C}_0\}.$$

We hawe thus proved:

Theorem 1. *If $R^\cdot$ is a weak integral domain contained in the weak field II $L^\cdot$, then:*

1° *The set of the weak quotients* $a \cdot b^{-1}$ *of the elements of* $R^\cdot$, *from which we exclude* $d = (d \cdot z) \cdot z^{-1}$, *is an abelian multiplicative group* $\mathcal{Q}_0 - \{d\}$, *which contains the subsemigroup* $R^\cdot - \{d\}$, *and is a subgroup of the multiplicative group* $L^\cdot - \{d\}$.

2° *The weak subfield generated by* $R^\cdot$ *in* $L^\cdot$ *coincides with the one generated by* $\mathcal{Q}_0$

$$\{R^\cdot\} = \{\mathcal{Q}_0\};$$

we call it the weak quotient field of $R^\cdot$ *in* $L^\cdot$.

3° *This weak quotient field coincides with the weak field associated to the ordinary quotient field* $\mathcal{Q}^\times$ *of* $R^\times$, *the ordinary ring associated to* $R^\cdot$.

§ 3. We shall expose now the embedding problem for a given commutative and associative weak integral domain $R^\cdot$ in a weak field II. The ordinary ring $R^\times$, associated to $R^\cdot$ by (6_1), being an ordinary integral domain (associative and commutative), one can construct its field of fractions $\mathcal{Q}^\times_{R^\times}$, whose fractions are equivalence classes

$$\frac{r_1}{r_2} = \overline{(r_1, r_2)}, \ \forall r_1, r_2 \in R, \quad r_2 \neq 0$$

and the embedding monomorphism ζ

$$\zeta : R^\times \to \mathcal{Q}^\times_{R^\times}$$

is defined by

$$a\zeta = \frac{ay}{y} \in \mathcal{Q}^\times_{R^\times}, \quad a, y \in R, \quad y \neq 0, \tag{12}$$

in particular

$$d\zeta = \frac{dy}{y} \in \mathcal{Q}^\times_{R^\times}.$$

By keeping the additive group of $\mathcal{Q}^\times_{R^\times}$ and defining for the fractions r_1/r_2 a new multiplication „." by (5_1), where d is $d\zeta$, one obtains a weak field II, $\mathcal{Q}^\cdot_{R^\times}$. The weak ring $R^\cdot$ can be embedded in $\mathcal{Q}^\cdot_{R^\times}$, since the ring monomorphism ζ, (12), induces a weak ring monomorphism

$$\zeta^\cdot : R^\cdot \to \mathcal{Q}^\cdot_{R^\times} \text{ with } a\zeta^\cdot = a\zeta = \frac{a \times y}{y} \in \mathcal{Q}^\cdot.$$

In fact, $\zeta^\cdot_+$ is group monomorphism

$$\zeta^\cdot_+ : R^\cdot_+ = R^\times_+ \to \mathcal{Q}^\cdot_+ = \mathcal{Q}^\times_+$$

and

$$(a \cdot b)\,\zeta = \frac{(a \cdot b) \times v}{v} + \frac{[ab - (a+b)\,d + c]\,v}{v}$$

whereas

$$\frac{ay}{y} \cdot \frac{bz}{z} = \frac{[ab - (a+b)\,d + c]\,yz}{yz};$$

hence $(a \cdot b)\,\zeta^\cdot = a\zeta^\cdot \cdot b\zeta^\cdot$.

Hence, every $r \neq d$, $r \in R^\cdot$ has a weak inverse $r^{-1} \in \mathcal{Q}^\cdot_{R^\times}$ given by (9). Using the expression (12) for the elements of $R^\cdot$ and applying to the fractions the weak multiplication law given by (5_1), we find again (10) for the weak quotient $a \cdot b^{-1}$. In the same manner as in the proof of Th. 1.3°, we find that $\mathcal{Q}^\cdot_{R^\times}$ is the weak quotient field of the weak integral domain $R^\cdot\zeta^\cdot$; we shall denote it by $\mathcal{Q}^\cdot_{R^\cdot}$ and call it the *weak field of fractions of the commutative weak integral domain* $R^\cdot$.

§ 4. Let us study now the uniqueness problem for the weak field of fractions. Previously, let us observe that, by using the concepts given in [3], one can easily verify

Lemma 1. *The associated ordinary rings of the homomorphic (isomorphic) weak rings are homomorphic (isomorphic); or, more explicitely, a weak ring homomorphism (isomorphism)* $\xi^\cdot : R^\cdot_1 \to R^\cdot_2$ *induces a unique homomorphism (isomorphism)*

$$\xi^\times : R^\times_1 \to R^\times_2$$

of their associated ordinary rings, so that $\xi^\cdot$ *and* $\xi^\times$ *have coincident restrictions to the additive group* R_{1+}.

Obviously, the converse does not hold, in general, but one can easily prove:

Lemma 2. *If* $\xi : R_1 \to R_2$ *is a ring isomorphism (epimorphism) and we construct the weak rings* $R^\cdot_i$ *by* (5_1), *with* d_i *for* d, *where* $d_2 = d_1\xi$, $d_i \in R_i$ *then* ξ *induces a unique weak ring isomorphism (epimorphism)* $\xi^\cdot : R^\cdot_1 \to R^\cdot_2$, *so that* ξ, $\xi^\cdot$ *have coincident restrictions to the additive group* R_{1+}.

We can now prove

Theorem 2. *If the commutative weak integral domains* $R^\cdot_i$ *are respectively included in the weak fields II* $K^\cdot_i$, *which are their quotient fields,*

$$R^\cdot_i \subset K^\cdot_i = \{R^\cdot_i\}, \quad i = 1,2,$$

and we have the weak ring isomorphism

$$\xi : R^\cdot_1 \to R^\cdot_2 \ (\Rightarrow d_2 = d_1\xi),$$

then $\xi^\cdot$ *can be uniquely extended to the weak field isomorphism*

$$\bar{\xi}^{\cdot}: K_1^{\cdot} \to K_2^{\cdot}.$$

Proof. Considering the ordinary associated fields $K_i^{\times}$, $i=1,2$ (by (6) or (6_1)), the weak subrings $R_i^{\cdot}$ shall be transformed in the ordinary rings $R_i^{\times}$, which are ordinary commutative integral domains

$$R_i^{\cdot} \subset K_i^{\cdot} \Rightarrow R_i^{\times} \subset K_i^{\times}, \quad i=1,2$$

and, by Lemma 1, $\xi^{\cdot}$ induces the ordinary ring isomorphism

$$\xi^{\times}: R_1^{\times} \to R_2^{\times}, \tag{13}$$

which coincides with $\xi^{\cdot}$ in the additive group

$$r\xi^{\times} = r\xi^{\cdot} \in R_{2+}^{\cdot} = R_{2+}^{\times}, \forall r \in R_{1+}^{\times} = R_{1+}^{\cdot}. \tag{13_1}$$

By theorem 1.3°, $K_i^{\times}$ is the quotient field of the commutative ntegral domain $R_i^{\times}$, $i=1,2$; then we know that the isomorphism $\xi^{\times}$, 13), (13_1) can be uniquely extended to the quotient field isomorphism $\bar{\xi}^{\times}$

$$\bar{\xi}^{\times}: K_1^{\times} - K_2^{\times}.$$

Now, applying Lemma 2 (where we put $K_i^{\times}$ for $R_i^{\times}$), $\bar{\xi}^{\times}$ induces a unique weak field isomorphism $\bar{\xi}^{\cdot}$

$$\bar{\xi}^{\cdot}: K_1^{\cdot} \to K_2^{\cdot},$$

$\bar{\xi}^{\times}$ and $\bar{\xi}^{\cdot}$ having coincident restrictions to the additive group

$$K_{1+}^{\times} = K_{1+}^{\cdot} \to K_{2+}^{\times} = K_{2+}^{\cdot}.$$

Moreover $\bar{\xi}^{\cdot}$ is the extension of the given weak ring isomorphism $\xi: R_1^{\cdot} \to R_2^{\cdot}$, because, by the definition of $\bar{\xi}^{\cdot}$, we have

$$k\bar{\xi}^{\cdot} = k\bar{\xi}^{\times} \in K_{2+}^{\times} = K_{2+}^{\cdot}, \forall k \in K_{1+}^{\cdot} = K_{1+}^{\times},$$

in particular

$$r\bar{\xi}^{\cdot} = r\bar{\xi}^{\times} = r\xi^{\times} = r\xi^{\cdot} \in R_{2+}^{\cdot} = R_{2+}, \forall r \in R_{1+}^{\cdot} = R_{1+}^{\times}.$$

The uniqueness of $\bar{\xi}^{\cdot}$ is proved by *reductio ad absurdum*, using Lemma 1 and the uniqueness of $\bar{\xi}^{\cdot}$

In particular, if $R_1^{\cdot} = R_2^{\cdot} = R^{\cdot}$, $\xi = I$ the identity, $K_1^{\cdot} = \mathcal{O}_{R^{\cdot}}^{\cdot}$ the weak field of fractions for $R^{\cdot}$ and $K_2^{\cdot} = \{R^{\cdot}\}$, then we obtain:

Theorem 3. *If $R^{\cdot}$ is a commutative weak integral domain $\mathcal{O}_{R^{\cdot}}^{\cdot}$ its weak field of fractions and $K^{\cdot}$ another weak field II so that $K^{\cdot} = \{R^{\cdot}\}$, then there exists a unique $R^{\cdot}$-isomorphism*

$$\bar{\xi}^{\cdot}: \mathcal{O}_{R^{\cdot}}^{\cdot} \to K^{\cdot}.$$

The same theorem can be stated also:

The uniqueness theorem for the weak field of fractions. *The weak field of fractions, $\mathcal{O}_{R^{\cdot}}^{\cdot}$, of a commutative weak integral domain $R^{\cdot}$ is defined up to $R^{\cdot}$-isomorphisms.*

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CORPUL SLAB AL FRACŢIILOR UNUI DOMENIU DE INTEGRITATE SLAB

Rezumat

Se rezolvă problema scufundării unui domeniu de integritate slab asociativ şi comutativ, $R^{\cdot}$, într-un corp slab de specia II; se demonstrează unicitatea acestuia, abstracţie făcînd de $R^{\cdot}$-izomorfisme.