ON A GENERALIZED FOURIER-TYPE INTEGRAL AND ITS APPLICATION TO THE SOLUTION OF CERTAIN BOUNDARY VALUE PROBLEMS

BY

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§ 1. Introduction

Solutions of linear partial differential equations are frequently expressed in the form

$$(1.1) \qquad Y(X,t) = \int_{-\infty}^{\infty} F(X,\omega) U(\omega) e^{i\omega t} d\omega,$$

in which $X \underset{\text{def}}{=} (x_1, \ldots, x_n)$, $x_1, \ldots, x_n, t$ are the independent variables, F is a kernel dependent upon the differential equation and U is a function which may be chosen to be arbitrary as long as a formal compliance only of (1. 1) with the differential equation is considered. To every choice of U corresponds a particular solution and, conversely, U is fixed, at least formally, by a particular well-posed set of boundary conditions. When, e. g., for a fixed value X_0 of X, Y is to take on the values of a given function y:

$$(1.2) \qquad y(t) = Y(X_0, t) = \int_{-\infty}^{\infty} F(X_0, \omega) U(\omega) e^{i\omega t} d\omega ,$$

then Fourier's Integral Theorem renders

$$(1.3) \qquad U(\omega) = [2\pi F(X_0, \omega)]^{-1} \int_{-\infty}^{\infty} y(\tau) e^{-i\omega\tau} d\tau .$$

Ultimately, therefore, (1. 1) can serve as a solution to the boundary value problem only when the principal values of the integrals in both (1. 3) and (1. 1) exist. However, as shown by the example in §7, the demand already for (1. 3) to converge is not met with in many important applications.

To remove some of the perturbing limitations, examination of a modified form of (1.1) which preserves the linearity properties of the integral representations is near at hand. Let a „truncated function" y_T be defined for every $T > 0$ by

$$y_T(t) = \begin{cases} y(t) & \text{when } |t| \leqslant T, \\ 0 & \text{when } |t| > T. \end{cases} \tag{1.4}$$

Were y, in (1.3), replaced by y_T, a relationship of the form

$$U_{(T)}(\omega) = [2\pi F(X_0,\omega)]^{-1} \int_{-T}^{T} y(\tau)\, e^{-i\omega\tau} d\tau \tag{1.5}$$

would ensue instead, provided y is continuous. (1.1), in turn, would be replaced by

$$Y(X,t,T) = \int_{-\infty}^{\infty} F(X,\omega) U_{(T)}(\omega)\, e^{i\omega t} d\omega\ . \tag{1.6}$$

The set of parameters X for which the integral converges is dependent on F and y. Since $\lim_{T\to\infty} y_T(t) = y(t)$ and since, formally, $Y(X,t,T)$ also satisfies the partial differential equation, three questions are of interest: Does

$$Y_1(X,t) = \lim_{T\to\infty} Y(X,t,T) \tag{1.7}$$

exist? When it exists, do those partial derivatives of Y_1 appearing in the differential equation exist too? Should they exist, can the corresponding differentiations be carried out under the limit sign in (1.7) and the integral sign in (1.6)? For sets $\{F,y,X\}$ for which these questions are answered in the affirmative, a formal solution of the boundary value problem could consequently be converted into a real one.

Unfortunately, however, some applications give rise to functions F for which Y does *not* exist. An example, taken from the theory of viscoelasticity, is cited in §7. To solve the boundary value problem for such cases, a „smoothed truncated function" $y_{T,\varepsilon,j,\eta}$, depending upon additional parameters ε, j and η had to be introduced instead of y_T. $y_{T,\varepsilon,j,\eta}$, too, coincides with y in a certain interval, namely in $-T + \varepsilon \leqslant t \leqslant T - \varepsilon$ and vanishes for $|t| > T$. It *differs*, in general, from y_T, by the possession of j derivatives everywhere. The meaning of η is explained in §2.

When $y_{T,\varepsilon,j,\eta}$ is operated on in a fashion similar to the one described for y_T, a function $Y(X,t,T,\varepsilon,j,\eta)$ — replacing $Y(X,t,T)$ — is produced. $\lim_{T\to\infty} Y(X,t,T,\varepsilon,j,\eta)$ not only exists for a fairly wide class of sets $\{F,y,X\}$, including those which first gave rise to the problem; but, in addition, this limit function possesses the desired differentiability properties mentioned previously. The corresponding results are summarized in Theorems 1 and 2. The (trivial) Lemma 3.1 has only been included to settle the question of the interchageability of the $\int_0^t$ and $\lim_{T\to\infty}$ signs in $\int_0^t \lim_{T\to\infty} Y(X,\tau,T,\varepsilon,j,\eta)\, d\tau$.

The definitions of some of the concepts employed are formulated in §2 while the proofs of the propositions of §3 are carried through in §6, followed by remarks concerning the theorems. The proofs are preceded by further definitions in §4 and two lemmas in §5, introduced to facilitate the proofs of §6. In §7, as mentioned already, an application of the theorems to a particular boundary value problem arising in the theory of viscoelasticity is delineated in some detail.

§ 2. Definitions of terms employed in the statement of the theorems

Definition 2.1 : *Definition of the N-fold smoothed truncated function of a given function $f \in C^N$.*

Let N be a non-negative integer, b a real number or $+\infty$, I the infinite interval $]-\infty, b[$ and $f \in C^N(I)$. For $T > 0, \varepsilon > 0, \eta > 0$ satisfying, in addition $T > \varepsilon$, an N-fold smoothed truncated function $f_{T,\varepsilon,N,\eta}$ of f is to stand for any function from I into $\mathfrak{C}$ satisfying the following conditions:

a) $f_{T,\varepsilon,N,\eta} \in C^N(I)$.

b) *For* $|t| < T - \varepsilon$, $t \in I$ $f_{T,\varepsilon,N,\eta}(t) = f(t)$,

c) *For* $|t| > T, t \in I$ $f_{T,\varepsilon,N,\eta}(t) = 0$,

d) *For* $-T \leqslant t \leqslant -T + \varepsilon$, $t \in I$ $|f_{T,\varepsilon,N,\eta}(t)| < |f(-T+\varepsilon)+\eta$.

Functions $f_{T,\varepsilon,N,\eta}$ obviously exist.

Definition 2.2 : *Suppose $\Phi \in C(R)$.*

$$U(\Phi,z) \underset{\text{def}}{=} \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} \Phi(\tau) e^{-iz\tau} d\tau\,, \tag{2.1}$$

provided the expression on the right-hand side of (2.1) exists.

§ 3. Theorems 1 and 2. Lemma 3.1

Theorem 1. 1. *Let $v \in C^{(1)}(R)$ take values in $\mathbb{C}$ and satisfy*

$$|v(x)| < \alpha\, |x|^{c-\lambda}$$

for $x < -x_0$, with suitable positive α, x_0, c and λ for which $c > \lambda$.

2. *Let f be defined in the complement D of the set*

$$\{z \in \mathbb{C} \mid |z - qi| \leqslant b,\ \mathrm{Im}(z) \geqslant q\} \cup [0, qi] -$$

where b, q are positive — by

$$f(z) = f_1(z) + e^{-i\gamma z} f_2(z).$$

γ is supposed to be positive, f_1 and f_2 to be holomorphic in D. Let, in addition

a) $\lim\limits_{r\to\infty} \max\limits_{|z|=r} |f_k(z)| = 0$ *for* $k = 1,2,$

b) the one-sided limit-functions

$$f_{(1)}(it) = \lim_{\substack{z\to ti\\ \mathrm{Re}(z)>0}} f(z), \qquad f_{(2)}(it) = \lim_{\substack{z\to ti\\ \mathrm{Re}(z)<0}} f(z)$$

exist in $[0,p]$ for a suitable positive p smaller than q and be expressible as sums

$$f_{(k)}(it) = g(t) + h_k(t)$$

of a function $g \in C([0,p])$ and functions h_1, h_2, respectively, satisfying, for a suitable s,

$$|h_k(t)| < st^c.$$

c) Let f be bounded in some disc centered at the origin.

Then the functions

$$u_1(t) \underset{\mathrm{def}}{=} \lim_{T\to\infty} \lim_{r\to\infty} \int_{-r}^{r} U(v_{T,\varepsilon,1,\eta}, z) f(z) e^{izt}\, dz,$$

$$u_2(t) \underset{\mathrm{def}}{=} \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} v_{T,\varepsilon,1,\eta}(\tau) \lim_{r\to\infty} \int_{-r}^{r} f(z) e^{iz(t-\tau)}\, dz\, d\tau,$$

$$u_3(t) \underset{\mathrm{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} v(\tau) \lim_{r\to\infty} \int_{-r}^{r} f(z) e^{iz(t-\tau)}\, dz\, d\tau$$

exist in R; they are equal and represent a continuous function.

Theorem 2. *Let x_1 be a real number and let $f_1(\cdot, x)$, $f_2(\cdot, x)$, $f(\cdot, x)$ be complex functions defined for each fixed $x \geqslant x_1$ in the set D introduced in* **Theorem 1** *so as to satisfy the relevant conditions of that theorem. γ is supposed to be independent of x.*

In addition,

1. *Let N be non-negative, let $v \in C^{(N+2)}(R)$ take values in $\mathbb{C}$ and satisfy*

$$|v(x)| < \alpha\, |x|^{c-\lambda}$$

for $x < -x_0$, where α, x_0, c and λ are positive and $c > \lambda$.
For $k = 0, \ldots, N$, $x \geqslant x_1$ let 2. — 7. be satisfied.

2. $f^{(k)} \underset{\mathrm{def}}{=} \dfrac{\partial^k f}{\partial x^k}$ *exists and is continuous in $D \times [x_1, \infty[$,*

3. *$f^{(k)}$ is holomorphic in D.*

4. *$f^{(k)}$ is bounded in some disc centered at the origin.*

5. *Let $X \underset{\mathrm{def}}{=} \{(x,y) \mid x \geqslant x_1,\ y \geqslant x_1,\ x < y\}$. For some suitable positive function b_k defined in X, $f^{(k)}$ is bounded in $\{]pi, pi + b_k(\xi_1, \xi_2)] \cup [pi - b_k(\xi_1, \xi_2), pi[\} \times [\xi_1, \xi_2]$ for each $(\xi_1, \xi_2) \in X$.*

6. $f_{k,1}(it, x) \underset{\mathrm{def}}{=} \lim\limits_{\substack{z\to ti\\ Re(z)>0}} f^{(k)}(z, x)$, $f_{k,2}(it, x) \underset{\mathrm{def}}{=} \lim\limits_{\substack{z\to ti\\ Re(z)<0}} f^{(k)}(z, x)$

exist in $[0, p]$ and are expressible as sums, respectively,

$$f_{k,l}(it, x) = g_k(t, x) + h_{k,l}(t, x)$$

of functions $g_k(\cdot, x) \in C([0,p])$ and functions $h_{k,l}$ which, in $[0,p] \times [\xi_1, \xi_2]$ satisfy

$$|h_{k,l}(t, x)| < s_k^*(\xi_1, \xi_2)\, t^c$$

for each $(\xi_1, \xi_2) \in X$ and suitable positive functions s_k^* defined in X.

7. *Positive functions c_k and Π_k, defined in X, exist so that*

$$|f^{(k)}(z, x)| < c_k(\xi_1, \xi_2)\, |z|^k$$

whenever $\xi_1 \leqslant x \leqslant \xi_2$, $z \in R$, $|z| > \Pi_k(\xi_1, \xi_2)$.

Define

$$u(x,t) \underset{\mathrm{def.}}{=} \lim_{T\to\infty} \lim_{r\to\infty} \int_{-r}^{r} f(z, x)\, U(v_{T,\varepsilon,N+2,\eta}, z)\, e^{izt} dz, \quad U_{m,n} \underset{\mathrm{def}}{=} \frac{\partial^{m+n} u}{\partial t^m\, \partial x^n},$$

$$u^*_{m,n} = \lim_{T\to\infty} \lim_{r\to\infty} \int_{-r}^{r} \frac{\partial^{m+n}}{\partial t^m\, \partial x^n} [f(z, x) U(v_{T,\varepsilon,N+2,\eta}, z)\, e^{itz}]\, dz .$$

Then $u_{m,n}$, $u^*_{m,n}$ *exist, are equal and represent a continuous function in* $[x_1,\infty[\times R$, *provided* $m+n\leqslant N$.

Lemma 3.1. *Let t be a real number and T_0 a positive number. Let* $\Phi\in C(R\times[0,t]\times]T_0,\infty[)$ *take values in* $\mathfrak{C}$ *and satisfy :*

1. *For every fixed T greater than T_0,* $\lim\limits_{s\to\infty}\int_{-s}^{s}\Phi(x,\tau,T)\,dx$ *converges uniformly in* $[0,t]$.

2. *For* $(\tau,T)\in[0,t]\times]T_0,\infty[$, $\lim\limits_{s\to\infty}\int_{-s}^{s}\Phi(x,\tau,T)\,dx$ *can be expressed in the form* $f(\tau)-g(\tau,T)$, *with suitable functions* $f\in C([0,t])$, $g(.,T)\in C(R)$ *and* $\lim\limits_{T\to\infty}g(\tau,T)=0$, *uniformly in* $[0,t]$.

Then $\int_0^t\lim\limits_{T\to\infty}\lim\limits_{s\to\infty}\int_{-s}^{s}\Phi(x,\tau,T)\,dx d\tau$ *and* $\lim\limits_{T\to\infty}\lim\limits_{s\to\infty}\int_{-s}^{s}\int_0^t\Phi(x,\tau,T)d\tau dx$ *both exist and are equal.*

Proof : $\lim\limits_{T\to\infty}\lim\limits_{s\to\infty}\int_{-s}^{s}\Phi(x,.,T)dx$ *exists and equals f, a continuous function.*

Hence $\lim\limits_{T\to\infty}\lim\limits_{s\to\infty}\int_{-s}^{s}\Phi(x,\tau,T)dxd\tau$ *exists and equals* $\int_0^t f(\tau)d\tau=\lim\limits_{T\to\infty}\int_0^t f(\tau)\,d\tau=$

$$=\lim_{T\to\infty}\int_0^t\Big[\lim_{s\to\infty}\int_{-s}^{s}\Phi(x,\tau,T)dx+g(\tau,T)\Big]d\tau=\lim_{T\to\infty}\int_0^t\lim_{s\to\infty}\int_{-s}^{s}\Phi(x,\tau,T)dxd\tau=$$

$$=\lim_{T\to\infty}\lim_{s\to\infty}\int_{-s}^{s}\int_0^t\Phi(x,\tau,T)d\tau dx\,.$$

§ 4. Definitions (conclusion)

Definition 4.1 : *For complex* α,β *let* (α,β) *denote the path* : $z=\alpha+t(\beta-\alpha)$, *with* $0\leqslant t\leqslant 1$.

Definition 4.2 : *For* $a\in R$, $r>0$, $j=1,2$, *let* $\Gamma^j_{a,r}$ *stand for the path* : $z=ai+re^{i(-1)^j t+t}$, *with* $0\leqslant t\leqslant\pi$. *Let* $\Gamma^j_r \underset{\text{def}}{=} \Gamma^j_{0\,r}$.

Definition 4.3 : *For* $r>0$, $j=1,2$ $L^j_r\underset{\text{def}}{=}\Gamma^j_r\cup(-r,r)$.

Definition 4.4 : *For* $0<s<r$, $j=1,2$,

$$\Delta_{r,s}\underset{\text{def}}{=}(-r,-s)\cup(-\Gamma^2_s)\cup(s,r),\quad L^j_{r,s}\underset{\text{def}}{=}\Delta_{r,s}\cup\Gamma^j_r.$$

Definition 4.5 : *Let* $s>0$, $z\in\mathfrak{C}$ *and let f be a* $C(]-\infty,-s]\cup(-\Gamma^2_s)\cup[s,\infty[)$ *mapping into* $\mathfrak{C}$. *For* $z\in\mathfrak{C}$ *define*

$$J(f,z)=\lim_{r\to\infty}\int_{\Delta_{r,s}}f(\omega)e^{iz\omega}\,d\omega \tag{4.1}$$

whenever the last expression has a meaning.

§ 5. Two lemmas employed in the proofs of theorems 1 and 2

The assertions made in Theorems 1 and 2 are identical, essentially, with enunciations concerning the interchageability of certain integrations, differentiations and other limit operations. It will therefore be natural that proofs lean, inter alia, on a lemma, 5. 1., which provides an estimate for certain integrals and another lemma, 5. 2., which warrants the interchangeability of certain integrations.

Lemma 5.1 1. *Let f, g, h_1 and h_2 be functions having the same domains and ranges, respectively, as in the statement of Theorem 1, conditions 2b) and 2c) of that theorem being satisfied.*

2. *Let f be holomorphic in D and let $J(f,x)$ exist and be equal to* $\int_{L^1_{\rho,\sigma}}f(z)e^{ixz}dz$ *for x larger than some positive x_0 and ρ which satisfies* $L^1_{\rho,\sigma}\subset D$.

Then, for $j=0,1,2,\ldots$, $x>x_0$, *the estimate*

$$\Big|\frac{d^j}{dx^j}J(f,x)\Big|<\mu_j x^{-j-c-1} \tag{5.1}$$

holds, where μ_j *is introduced by* :

$$\mu_j\underset{\text{def}}{=}2s\Gamma(j+c+1)+\rho(\pi+2)M_j\left(\frac{j+c+1}{p}\right)^{j+c+1}e^{-j-c-1}, \tag{5.2}$$

$$M_j\underset{\text{def.}}{=}\sup_{z\in L_2}|f(z)z^j|\,,$$

$$L_2\underset{\text{def}}{=}(pi,\rho+pi)\cup\Gamma^1_{\rho,\sigma}\cup(-\rho+pi,pi).$$

Proof: It is trivial to prove that $\dfrac{d^j J(f,x)}{dx^j}$ exists and is equal to $\int\limits_{L^1_{\rho,\sigma}} (iz)^j f(z) e^{ixz}\, dz$, *for* $j = 0, 1, 2, \ldots, x > x_0$.

Let $L_1 \underset{\text{def.}}{=} (0, pi) \cup (pi, 0)$, $L \underset{\text{def.}}{=} L_1 \cup L_2$, $f(z) = f_1(z)$ on $(0, pi)$, $f(z) = f_2(z)$ on $(pi, 0)$. Then application of a generalized version of Cauchy's Theorem — valid because of conditions 2b), 2c) of Theorem 1 — renders

$$\int\limits_{L^1_{\rho,\sigma}} (iz)^j f(z) e^{ixz}\, dz = \int\limits_{L} (iz)^j f(z) e^{ixz}\, dz. \tag{5.3}$$

The following two estimates imply (5.1):

1. For $x > 0$, by 2b) of Theorem 1

$$\left| \int\limits_{L_1} f(z)(iz)^j e^{ixz}\, dz \right| = \left| \int\limits_0^p f_1(it)(-t)^j e^{-xt}\, dt + \int\limits_p^0 f_2(it)(-t)^j e^{-xt}\, dt \right| =$$

$$= \left| \int\limits_0^p h_1(t)(-t)^j e^{-xt}\, dt + \int\limits_p^0 h_2(t)(-t)^j e^{-xt}\, dt \right| \leqslant 2s \int\limits_0^p t^{j+c} e^{-xt}\, dt = \tag{5.4}$$

$$= 2s x^{-j-c-1} \int\limits_0^{px} u^{j+c} e^{-u}\, du < 2s\, \Gamma(j + c + 1) x^{-j-c-1}.$$

2. For $x > 0$

$$\left| \int\limits_{L_2} f(z)(iz)^j e^{ixz}\, dz \right| \leqslant (\pi + 2)\, \rho M_j\, e^{-\rho x}. \tag{5.5}$$

3. By an elementary lemma the sum of the right-hand members of (5. 4) and (5. 5) is not greater, in absolute value, than the right-hand member of (5.1).

Lemma 5.2 1. *Let* f *be holomorphic in the domain* D *defined in Theorem* 1 *and satisfy* $\lim\limits_{r\to\infty} \max\limits_{|z|=r} |f(z)| = 0$.

2. *For some real interval* $[a, d]$ *let* $v \in C^{(1)}([a, d])$ *take values in* $\mathbb{C}$. *Then, for* t *real, the expressions*

$$\lim_{r\to\infty} \int\limits_{\Delta_{r,\sigma}} f(z) \int\limits_a^d v(x) e^{iz(t-x)}\, dx\, dz, \quad \int\limits_a^d v(x) \lim_{r\to\infty} \int\limits_{\Delta_{r,\sigma}} f(z) e^{iz(t-x)}\, dz\, dx$$

exist and are equal.

Proof: It is sufficient to assume $a < d$, $t \notin]a, d[$. For $t \in]a, d[$ the demonstration reduces to the proofs of two cases in each of which $t \notin]a, d]$ does apply.

By applying contour integration and 1. the existence of $\int\limits_a^d v(x) J(f, t - x)\, dx$ and the equality of this expression to $\int\limits_a^d v(x) \int\limits_{L^j_{\rho,\sigma}} f(z) e^{iz(t-x)}\, dz\, dx$ are obtained, with $j = 1$ for $t \geqslant d$, $j = 2$ for $t \leqslant a$, and ρ which satisfies $L^j_{\rho,\sigma} \subset D$.

Because of the continuity of f and v the order of integration in the last-mentioned integral is immaterial and this integral therefore equals

$$\lim_{\rho\to\infty} \left[\int\limits_{\Delta_{\rho,\sigma}} f(z) \int\limits_a^d v(x) e^{iz(t-x)}\, dx\, dz + \int\limits_{\Gamma^j_\rho} f(z) \int\limits_a^d v(x) e^{iz(t-x)}\, dx\, dz \right]$$

since for ρ satisfying $L^j_{\rho,\sigma} \subset D$ the expression in the square brackets is independent of ρ. Consequently the lemma is proved when it is shown that the second term has a vanishing limit for $\rho \to \infty$. Because of 2., this term may be written

$$I \underset{\text{def}}{=} \int\limits_{\Gamma^j_\rho} f(z) \left[\frac{v(d) e^{iz(t-d)} - v(a) e^{iz(t-a)}}{-iz} + \frac{1}{iz} \int\limits_a^d v'(x) e^{iz(t-x)}\, dx \right] dz. \tag{5. 6}$$

None of the exponentials in (5. 6) exceeds unity in absolute value — by the assumption $t \notin]a, d[$ and the t-dependent choice of j. With $M \underset{\text{def}}{=} \max\limits_{x\in[a,d]} (|v(x)| + |v'(x)|)$, therefore,

$$|I| \leqslant \pi\rho \left[\max_{z\in\Gamma^j_\rho} |f(z)| \cdot \frac{(2 + d - a)M}{\rho} \right] = \pi M (2 + d - a) \max_{z\in\Gamma^j_\rho} |f(z)|$$

and the lemma follows by 1.

§ 6. **Proof of theorems 1 and 2.** *Proof of Theorem* 1. 1. u_1 is obtained from $\int_{-r}^{r} f(z)\, U(v_{T,\varepsilon,1,\eta}, z)\, e^{izt}\, dz$ by application of two successive limit operations. It is easily established, first, that $\int_{-r}^{r} f(z)\, U(v_{T,\varepsilon,1,\eta}, z)\, e^{izt}\, dz$ exists and equals $\int_{\Delta r,\sigma} f(z)\, U(v_{T,\varepsilon,1,\eta}, z)\, e^{izt} dz$.

Define, second

$$(6.1) \qquad \nu(t, T) \underset{\text{def}}{=} \lim_{r\to\infty} \int_{\Delta r,\sigma} f(z)\, U(v_{T,\varepsilon,1,\eta}, z)\, e^{izt}\, dz.$$

The existence and continuity of $\nu(t, T)$ as well as the equality

$$(6.2) \qquad \nu(t, T) = \frac{1}{2\pi}\int_{-T}^{T} v_{T,\varepsilon,1,\eta}(\tau)\, J(f_1, t-\tau)\, d\tau + \frac{1}{2\pi}\int_{-T}^{T} v_{T,\varepsilon,1,\eta}(\tau)\, J(f_2, t-\gamma-\tau)\, d\tau$$

are obtained by direct application of Lemma 5.2 to each of the terms on the right-hand side of the identity

$$\int_{\Delta r,\sigma} f(z) U(v_{T,\varepsilon,1,\eta}, z)\, e^{izt}\, dz = \frac{1}{2\pi}\int_{\Delta r,\sigma} f_1(z) \int_{-T}^{T} v_{T,\varepsilon,1,\eta}(\tau)\, e^{iz(t-\tau)}\, d\tau dz + \frac{1}{2\pi}\int_{\Delta r,\sigma} f_2(z) \int_{-T}^{T} v_{T,\varepsilon,1,\eta}(\tau)\, e^{iz(t-\gamma-\tau)}\, d\tau\, dz.$$

The applicability of Lemma 5.2 follows from the assumption that f_1 and f_2 are holomorphic and satisfy 2. a).

2. To prove existence and continuity of $\lim_{T\to\infty} \nu(t, T)$ for a given $t = t_0$, $\nu(t, T)$ will be represented as a sum of three terms which depend on the value of t_0. Each of these terms will be shown to tend to a limit, continuous in t, for $T \to \infty$: Choose a fixed β satisfying $|t_0| < \beta$, define

$$(6.3) \qquad S = 2\beta + x_0 + \gamma + 1$$

and let t, T satisfy, henceforth

$$(6.4) \qquad |t| < \beta,\ T > S + \varepsilon.$$

Application of contour integration, taking account of condition 2.a), shows that $J(f_1, x) = J(f_2, x) = 0$ for $x < 0$. From (6.2), (6.4) and (6.3), therefore:

$$\nu(t, T) = \frac{1}{2\pi}\int_{-T}^{t} v_{T,\varepsilon,1,\eta}(\tau)\, J(f_1, t-\tau)\, d\tau +$$

$$(6.5) \qquad \frac{1}{2\pi}\int_{-T}^{t-\gamma} v_{T,\varepsilon,1,\eta}(\tau)\, J(f_2, t-\gamma-\tau)\, d\tau = \frac{1}{2\pi}\int_{-T}^{-S} v_{T,\varepsilon,1,\eta}(\tau)\, J(f, t-\tau)\, d\tau +$$

$$\frac{1}{2\pi}\int_{-S}^{t-\gamma} v_{T,\varepsilon,1,\eta}(\tau)\, J(f, t-\tau)\, d\tau + \frac{1}{2\pi}\int_{t-\gamma}^{t} v_{T,\varepsilon,1,\eta}(\tau) J(f_1, t-\tau) d\tau,$$

which is the desired representation of u.

3. Because of the inequalities

$$(6.6) \qquad \begin{gathered} -T + \varepsilon < -S = -2\beta - x_0 - \gamma - 1 < -|t| - \gamma - \\ -1 < t < 2\beta + x_0 + \gamma + 1 < T - \varepsilon \end{gathered}$$

the second term and the third term on the right-hand side of (6.5) are independent of T, by Definition 2.1. They obviously represent continuous functions of t. Existence of the limit of the first term and its continuity in t are shown by proving that $\int_{-T}^{-S} v_{T,\varepsilon,1,\eta}(\tau)\, J(f, t-\tau)\, d\tau$ and $\int_{-T}^{-S} v(\tau)\, J(f, t-\tau)\, d\tau$ tend, for $T \to \infty$, uniformly to equal limits. The proof rests on the following simple

Lemma 6.1. *Let $f \in C(]-\infty, -p])$, $g \in C([p, \infty[)$ — with $p > 0$ — and let $|f(x)| < b|x|^{c-\lambda}$, $|g(x)| < ax^{-c-1}$, $c > \lambda > 0$. Then, for $r > 0$, $q \geq \mathrm{Max}(r + p, 2r)$, $\int_{-T}^{-q} f(x)\, g(x_0 - x)\, dx$ and $\int_{-T}^{-q} f_{T,\varepsilon,0,\eta}(x)\, g(x_0 - x)\, dx$ converge, for $|x_0| < r$, $T \to \infty$, uniformly to equal limits.*

The proof of the lemma is suppressed here. It is based on the easily established estimate:

Lemma 6.2. *When $f \in C(]-\infty, -a[)$ and $|f(x)| < b|x|^c$ for $x < -a$, with positive a, b, c, then $|f_{T,\varepsilon,0,\eta}(x)| < b\left(1 + \frac{\eta}{ba^c}\right)|x|^c$ for $x < -a$, provided $T > a + \varepsilon$.*

The applicability of Lemma 6.1 stems from 1., Lemma 6.2 and the estimate,

(6.7) For $y > \gamma$ and μ_0 defined by (5.2)

$$|J(f, y)| < \mu_0 y^{-\epsilon-1}.$$

(6.7) follows directly from Lemma 5.1, having recourse to the contour-integral representation of $J(f,y)$, again obtainable because of condition 2.a).

(6.5) now entails

$$\lim_{T\to\infty} \nu(t, T) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{t-\gamma} v_{T,\varepsilon,1,\gamma}(\tau) J(f, t-\tau)\, d\tau +$$

$$+ \frac{1}{2\pi} \lim_{T\to\infty} \int_{t-\gamma}^{t} v_{T,\varepsilon,1,\gamma}(\tau) J(f_1, t-\tau)\, d\tau = \tag{6.8}$$

$$= \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{t-\gamma} v(\tau) J(f, t-\tau)\, d\tau + \frac{1}{2\pi} \int_{t-\gamma}^{t} v(\tau) J(f_1, t-\tau)\, d\tau.$$

For $|t| < \beta$ these limits are uniform in t. Consequently u_1 is continuous in $]-\beta, \beta[$.

4. The proof of Theorem 1 will be concluded by showing that $u_1 = u_2 = u_3$. The first equality is proved by adding $\lim\limits_{T\to\infty} \int_{t-\gamma}^{t} v_{T,\varepsilon,1,\gamma}(\tau) J(f_2, t-\gamma-\tau)\, d\tau$ and $\lim\limits_{T\to\infty} \int_{t}^{t} v_{T,\varepsilon,1,\gamma_1}(\tau) J(f_1, t-\tau)\, d\tau$ to the first representation (6.8) of $\lim\limits_{T\to\infty} \nu(t,T)$, both vanishing since, again, $J(f_i,y) = 0$ for $i = 1,2$, $y < 0$. The second equality is proved similarly.

Proof of theorem 2. 1. To obtain the proof for given $t = t_0$, $x = \xi$, u will again be represented as a sum of three terms which depend on the values of t_0 and ξ: Choose, first, a positive β for which $|t_0| < \beta$ and a $(\xi_1, \xi_2) \in X$ for which $\xi_1 < \xi < \xi_2$ when $\xi \neq x_1$, $\xi_1 = \xi$ when $\xi = x_1$. Let (6.4) and $\xi_1 \leqslant x \leqslant \xi_2$ hold henceforth.

With the notation $\tilde{J}(x, y) \underset{\text{def}}{=} J[f(\cdot, x), y]$ Theorem 1 provides, second, the representation

$$u(x, t) = \frac{1}{2\pi} \lim_{T^*\to\infty} \Bigg\{ \int_{-T^*}^{-T} v_{T^*,\varepsilon,N+2,\gamma}(\tau) \tilde{J}(x, t-\tau)\, d\tau +$$

$$+ \int_{-T}^{t} v_{t,\varepsilon,N+2,\gamma}(\tau) \tilde{J}(x, t-\tau)\, d\tau + \tag{6.9}$$

$$\int_{-t}^{t} [v_{T^*,\varepsilon,N+2,\gamma}(\tau) - v_{T,\varepsilon,N+2,\gamma}(\tau)] \tilde{J}(x, t-\tau) d\tau \Bigg\},$$

again holding because $\tilde{J}(x,y) = 0$ for $y < 0$ and because of 3. and 4. By applying the limit operation in (6.9) to each term separately the desired representation of u is obtained.

The first term: In the proof of Theorem 1 the limit of this term has been shown to equal $\frac{1}{2\pi} \int_{-\infty}^{-T} v(\tau) \tilde{J}(x, t-\tau) d\tau$.

The second term: Being independent of T^*, the limit of this term may be put into the form $\lim\limits_{r\to\infty} \int_{\Delta r,\sigma} f(z, x) U(v_{T,\varepsilon,N+2,\gamma}, z) e^{izt}\, dz$, by (6.1) and (6.2).

The third term: By Definition 2.1 the limit of this term equals

$$\frac{1}{2\pi} \int_{-t}^{-T+\varepsilon} [v(\tau) - v_{T,\varepsilon,N+2,\gamma}(\tau)] \tilde{J}(x, t-\tau) d\tau.$$

Consequently

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{-T} v(\tau)\, \tilde{J}(x, t-\tau) d\tau + \lim_{r\to\infty} \int_{\Delta r\, \sigma} f(z, x)\, U(v_{T,\varepsilon,N+2,\gamma}, z) e^{izt}\, dz +$$

$$+ \frac{1}{2\pi} \int_{-T}^{-T+\varepsilon} [v(\tau) - v_{T,\varepsilon,N+2,\gamma}(\tau)] \tilde{J}(x, t-\tau) d\tau \tag{6.10}$$

and this is the representation desired.

2. It will now be shown that the operator $D_{m',n'} \underset{\text{def}}{=} \dfrac{\partial^{m'+n'}}{\partial t^{m'} \partial x^{n'}}$ may be applied to each of the terms in (6.10) and a corresponding additive re-

presentation for $D_{m',n'}u$ will thus be obtained. Estimates for the values produced when $D_{m',n'}$ is applied to the integrands of the first term and the third term in (6.10) are needed in the process of proof. The following two statements, (6.11) and (6.12), are instrumental in obtaining these:

(6.11) For $y > \gamma$, $0 \leq m' \leq m$, $0 \leq n' \leq n$ $D_{m',n'}\widetilde{J}(x\ y)$ exists, is continuous and satisfies

$$|D_{m',n'}\widetilde{J}(x, y)| < \mu^{\xi_1, \xi_2}_{m', n'} y^{-m'-c-1},$$

where $\mu^{\xi_1, \xi_2}_{m', n'}$ is introduced by the following successive definitions:

$$L_2 \underset{\text{def}}{=} (pi,\ \rho + pi) \cup \Gamma^1_{p,\sigma} \cup (-\rho + pi,\ pi),$$

$$\Phi^{\xi_1, \xi_2}_{m', n'} \underset{\text{def}}{=} \operatorname*{Max}_{z \in L_2,\ x \in [\xi_1, \xi_2]} |f^{(n')}(z, x) z^{m'}|,$$

$$\mu^{\xi_1, \xi_2}_{m', n'} \underset{\text{def}}{=} 2s^*_{n'}(\xi_1, \xi_2)\,\Gamma(m'+c+1) + \rho(\pi+2)\,\Phi^{\xi_1, \xi_2}_{m', n'}\left(\frac{m'+c+1}{p}\right)^{m'+c+1} e^{-m'-c-1}$$

(6.12) *When f and g are complex functions from $]-\infty, -r]$, $[r, \infty[$, respectively, satisfying $|f(x)| < a|x|^b$, $|g(x)| < cx^{-d}$ in their respective domains and a, b, c, d, r are positive, then for any real $\bar{x}$ and for $x \leq$ Min $(\bar{x} - r, -2|\bar{x}|, -r)$ the inequality $|f(x)\,g(\bar{x} - x)| < 2^d ac|x|^{b-d}$ holds.*

(6.11) is a direct consequence of Lemma 5. 1: As in the proof of Theorem 1, for ρ satisfying $L^1_{\rho,\sigma} \subset D$ and for $y > \gamma$ $\widetilde{J}(x, y) = \int_{L^1_{\rho,\sigma}} f(z, x)\, e^{iyz}\, dz$. Therefore $\dfrac{\partial^{n'}\widetilde{J}(x, y)}{\partial x^{n'}}$ exists, is continuous and equals $\int_{L^1_{\rho,\sigma}} f^{(n')}(z, x) e^{iyz}\, dz$, since f possesses the necessary differentiability and continuity properties. Thus all the conditions of Lemma 5.1 are satisfied. $\Phi^{\xi_1, \xi_2}_{m', n'}$ exists because of 5.

The first term: From 1. and (6.11), applying (6.12) with $r = x_0 + \gamma$, $\bar{x} = t$, the estimate

$$(6.13)\quad |v(\tau)\, D_{m',n'}\widetilde{J}(x,\ t - \tau)| < 2^{m'+c+1}\alpha\mu^{\xi_1, \xi_2}_{m', n'}|\tau|^{-m'-\lambda-1}, \text{ valid for } \tau \leq -T,$$

is immediately derived since Min $(\bar{x} - r,\ -2|\bar{x}|,\ -r) > -2\beta - x_0 - \gamma - 1 = -S > -T$. Since $t - \tau > \gamma$, $D_{m',n'}\widetilde{J}(x,\ t - \tau)$ turns out to be continuous in (x, t, τ). $\int_{-\infty}^{-T} v(\tau) D_{m',n'}\widetilde{J}(x, t - \tau)d\tau$ and $D_{m',n'}\int_{-\infty}^{-T} v(\tau)\,\widetilde{J}(x,\ t - \tau)d\tau$

both exist, are equal and represent a continuous function of (x, t). Finally

$$(6.14)\quad \left|\int_{-\infty}^{-T} v(\tau) D_{m',n'}\widetilde{J}(x,\ t - \tau)d\tau\right| \leq 2^{m'+c+1}\alpha\mu^{\xi_1, \xi_2}_{m', n'}(m' + \lambda)^{-1}\, T^{-m'-\lambda}.$$

The second term: The existence of

$$D_{m',n'} \lim_{r\to\infty}\int_{\Delta r,\sigma} f(z, x)\, U(v_{T,\varepsilon,N+2,\eta}, z)\, e^{izt}\, dz,$$

of $\lim_{r\to\infty}\int_{\Delta r,\sigma} D_{m',n'}[f(z, x)\, U(v_{T,\varepsilon,N+2,\eta}, z)e^{izt}]\,dz$, the equality of these expressions and their continuity easily follow from 2., 7. and the estimate

$$(6.15)\qquad |U(v_{T,\varepsilon,N+2,\eta}, z)| < \mu(v_{T,\varepsilon,N+2,\eta})\,|z|^{-N-2},$$

valid for $z \in R - \{0\}$, and a suitable function $\mu(v_{T,\varepsilon,N+2,\eta})$. (6. 15) is obtainable in the usual way by an application of partial integration to the expression defining $U(v_{T,\varepsilon,N+2,\eta}, z)$, having recourse to Definition 2. 1.

The third term: From 1., Lemma 6. 2 and (6.11), applying (6.12) again with $r = x_0 + \gamma$, $\bar{x} = t$, the estimate

$$(6.16)\qquad \begin{aligned} &|[v(\tau) - v_{T,\varepsilon,N+2,\eta}(\tau)]D_{m',n'}\widetilde{J}(x,\ t - \tau)| < \\ &< 2^{m'+c+1}\alpha\mu^{\xi_1, \xi_2}_{m', n'}\left(2 + \frac{\eta}{\alpha x_0^{c-\lambda}}\right)|\tau|^{-m'-\lambda-1} \end{aligned}$$

follows for $\tau \leq -T + \varepsilon$. The continuity of $[v(\tau) - v_{T,\varepsilon,N+2,\eta}(\tau)]D_{m',n'}\widetilde{J}(x,\ t - \tau)$ in (x, t, τ), the existence of $\int_{-T}^{-T+\varepsilon}[v(\tau) - v_{T,\varepsilon,N+2,\eta}(\tau)]\,D_{m',n'}\widetilde{J}(x,\ t - \tau)d\tau$, of $D_{m',n'}\int_{-T}^{-T+\varepsilon}[v(\tau) - v_{T,\varepsilon,N+2,\eta}(\tau)]\,\widetilde{J}(x,\ t - \tau)d\tau$, the equality of these expressions and their continuity in (x,t) are readily shown. Finally

$$(6.17)\qquad \begin{aligned} &\left|\int_{-T}^{-T+\varepsilon}[v(\tau) - v_{T,\varepsilon,N+2,\eta}(\tau)]D_{m',n'}\widetilde{J}(x,\ t - \tau)d\tau\right| < \\ &< 2^{m'+c+1}\alpha\mu^{\xi_1, \xi_2}_{m', n'}\left(2 + \frac{\eta}{\alpha x_0^{c-\lambda}}\right)(T - \varepsilon)^{-m'-\lambda-1}. \end{aligned}$$

The observations made with respect to each of the three terms in (6.10) show that $u_{m',n'}$ exists, that it is continuous and that

$$u_{m',n'}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{-T} v(\tau) D_{m',n'} \tilde{J}(x, t-\tau)\,d\tau +$$

$$+ \lim_{r\to\infty} \int_{\Delta_{r,a}} D_{m',n'}[f(z,x)\, U(v_{T,\varepsilon,N+2,\eta}, z) e^{izt}]\,dz + \tag{6.18}$$

$$+ \frac{1}{2\pi}\int_{-T}^{T+\varepsilon} [v(\tau) - v_{T,\varepsilon,N+2,\eta}(\tau)] D_{m',n'} \tilde{J}(x, t-\tau)\,d\tau.$$

[3] $u_{m',n'}(x,t)$ equals $u^*_{m',n'}(x,t)$ since, firstly, $u_{m',n'}(x,t)$ is independent of T and equals, therefore, $\lim_{T\to\infty} u_{m',n'}(x,t)$ and since, second, the first term and the last term in (6.18) tend to zero for $T\to\infty$, by (6.14) and (6.17).

Since m' and n' are arbitrary non-negative integers not exceeding m and n, respectively, the theorem is proved.

Remarks concerning theorems 1 and 2: 1. The results of Theorem 2 may be generalized to the effect that the partial differentiations with respect to x and t may be carried out in any order. Obviously, a theorem analogous to Theorem 2 may be formulated with the scalar parameter x replaced by a vector parameter.

2. The function f of Theorems 1 and 2 may be replaced by a function regular in $\mathfrak{S}$ except on (i) a finite number of sections each of which joins a point of the real axis with a point lying above it, and ($2i$) an additional bounded point set $A \in \mathfrak{S}$ satisfying $\liminf_{z\in A} \operatorname{Im}(z) > 0$. The location of singularities of this kind in the upper half-plane only is characteristic of dissipative phenomena.

3. (6.18), (6.14) and (6.17) provide a simple estimate for the rest term $u^*_{m,n} - \lim_{r\to\infty}\int_{-r}^{r} D_{m,n}[f(z,x)\; U\;(v_{T,\varepsilon,N+2,\eta},\; z)e^{izt}]\,dz$. An estimate for the rest term which appertains to Lemma 3.1 also does not present any difficulties.

4. The estimates mentioned in 3. entail the existence of important stability properties of the solution of certain boundary value problems. Let the solution fixed by a function $v(t)$ — satisfying the conditions of the Theorems — be given by $u(x,t)$. Replacement of v by $v_{T,\varepsilon,N+2,\eta}$ then produces a solution $u^T(x,t)$ of the problem which, for finite time intervals, tends uniformly to $u(x,t)$ when $T\to\infty$.

§ 7. Application of theorems 1 and 2: Flow of an incompressible Maxwell-Fluid about a rigid sphere

1. In [1] the problem of flow of an incompressible Maxwell-Fluid about a rigid sphere is discussed. Improper integral expressions for the local velocity **v** of the fluid, the pressure p and the drag generated, W, are presented. These expressions involve the values of the velocity $V(t)$ of the centre of the sphere at time t on infinite time intervals and become meaningless for some physically important functions V. Application of the limit process delineated in Theorems 1 and 2 and in Lemma 3.1, however, permit an extension of the domain of substitution of V in the expressions mentioned, thereby greatly enhancing the range of physical applicability of the results obtained in [1]. The procedure is described in the sequel.

Let the radius of the sphere, the density, the Young modulus, and the coefficient of viscosity of the fluid be denoted by a, ρ, E and η, respectively. Analysis of the problem presented in [1] then yielded

$$\Delta \mathbf{v} - \frac{3\rho}{E}\mathbf{v}_{tt} - \frac{\rho}{\eta}\mathbf{v} = \nabla\left(\frac{3}{E}p_t + \frac{1}{\eta}p\right) \tag{7.1}$$

as a linearized version of the equation of motion of the Maxwell-Fluid, to be solved simultaneously with the equation of continuity $\nabla\mathbf{v} = 0$ and the boundary conditions, reading

$$\mathbf{v}(\infty,\theta,\varphi,t) = -V(t) \tag{7.2}$$

$$\mathbf{v}(a,\theta,\varphi,t) = 0 \tag{7.3}$$

in polar coordinates (r,θ,φ). When $v_1 \underset{\text{def}}{=} v_r, v_2 \underset{\text{def}}{=} v_\theta,\; v_3 \underset{\text{def}}{=} p$, when the x, r and θ coordinates are related in accordance with the equation $x = r\cos\theta$ and when $\mathbf{1}_x$ is a unit vector in the x-direction, then, for the axially symmetric problem — characterized by $\mathbf{V} = \mathbf{V}_x \times \mathbf{1}_x$ — it was shown in [1] that

$$v_j = \int_{-\infty}^{\infty} U(V_x;\omega)\nu_j(r,\theta,\omega)e^{i\omega t}\,d\omega, \tag{7.4}$$

with

$$\nu_2(r,\theta,\omega) = \frac{\sin\theta}{2}\left[2 + \frac{3a}{k^2r^3} + \frac{3a^2}{kr^3} + \frac{a^3}{r^3} - e^{k(a-r)}\left(\frac{3a}{k^2r^3} + \right.\right.$$

$$\left.\left. + \frac{3a}{kr^2} + \frac{3a}{r}\right)\right], \tag{7.5}$$

(7.6) $$\nu_3(r,\theta,\omega) = \rho i \cos\theta \cdot \omega \left(r + \frac{a^3}{2r^2} + \frac{3a^2}{2r^2k} + \frac{3a}{2r^2k^2}\right),$$

(7.7) $$k = \sqrt{\alpha}, \quad \alpha = \frac{\rho\omega i}{\eta} - \frac{3\rho}{E}\omega^2,$$

(7.8) $$-\frac{\pi}{2} < \arg k < \frac{\pi}{2}.$$

$v_2 = 0$. Since the subsequent treatment of v_2 covers, in principle, the treatment of v_1, the latter is not recorded here.

To warrant validity of (7. 4), (2. 1) must converge for $\varphi = V_x$. The most commonly employed sufficient criterion guaranteeing this assumes, inter alia, the condition

(7.9) $$\int_{-\infty}^{\infty} |V_x(\tau)|\, d\tau < \infty,$$

implying $\lim\limits_{T\to\infty} V_x(t) = 0$, a limitation not complied with in many physical applications. Some other sufficient criteria, as e. g. [2], are impracticable too from a physical point of view in being too special.

(7. 9) could not, be applied e. g., to the Brownian Motion of a supposedly rigid spherical particle in a Maxwell-Fluid. The well-known statistical theory shows the probability distribution of $\mathbf{V}(t)$, for $t\to\infty$, to be Maxwellian. Otherwise stated, finite intervals of positive values for $V_x(t)$ are allotted positive probabilities for t large enough, a condition incompatible with $\lim\limits_{t\to\infty} V_x(t) = 0$.

It is for these resons that the limit-process of §2 is invoked to provide modified expressions for v_1, v_2, v_3 and $\mathbf{W}$. The result is summed up in

Theorem 7.1 : 1. *Let l, β, ε, λ and ξ be positive and let $\lambda < \frac{1}{2}$.*

2. *Let V_x be a $C^{(4)}$ (R) mapping into R, and let $|V_x(x)| < l(-x)^{\frac{1}{2}-\lambda}$ for $x < -\beta$.*

Then

(*i*) for t, θ, ε, R, $r \geq a$, $j = 1, 2, 3$

(7.10) $$v_j(r,\theta,t) \underset{\text{def}}{=} \lim_{t\to\infty}\lim_{s\to\infty} \int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega]\, \nu_j(r,\theta,\omega) e^{i\omega t} d\omega$$

exists,

(*ii*) *v_j possesses all partial derivatives up to the second order with respect to r, θ and t;*

(*iii*) *the corresponding differentiations may be carried out by differentiating under the limit and integral signs.*

For the general hydrodynamic problem (in which $\mathbf{V}$ does not necessarily point in the x-direction) analogous conditions apply.

It is of interest to observe that the magnitude of $|V_x(t)|$ for $t\to\infty$ plays no role in the theorem.

Finally, the fourth remark made after the proof of Theorem 2 applies, thus proving stability of v_j in the sense mentioned : The functions v_j which correspond to the velocity $V_{T,\varepsilon,4,\xi}$ of the sphere tend, for $T\to\infty$, to the respective values of these functions corresponding to the velocity V of the sphere. More general stability properties may also easily be proved. The drag $\mathbf{W}$, too, can be approximated with any desired degree of accuracy by the drag generated by a sphere moving with velocity $V_{T,\varepsilon,4,\xi}$, provided T is large enough. Thus, for a given accuracy of the measuring apparatus a time of length $T(t)$ can be provided for every system of the type treated, to the effect that the special character of the movement preceding the time $-T$ does not measurably affect v_j at time t. $T(t)$ may be chosen as an appropriate constant for bounded time intervals.

Proof of theorem 7.1 : With $P \underset{\text{def}}{=} \frac{1}{3\eta}$, $Q \underset{\text{def}}{=} \frac{1}{E}$, $j = 2,3$ let

(7.11) $$\nu_j = \Psi_j + F_j$$

where

(7.12) $$\Psi_2(r,\theta,\omega) = \frac{\sin\theta}{2}\left[2 + \frac{a^3}{r^3} - \frac{3a}{r} e^{(a-r)\sqrt{3\rho}\left(\frac{P}{2Q^{1/2}} + iQ^{1/2}\omega\right)}\right],$$

(7.13) $$F_2(r,\theta,\omega) = \frac{\sin\theta}{2}\left[\frac{3a}{k^2r^3} + \frac{3a^2}{kr^3} - e^{k(a-r)}\left(\frac{3a}{k^2r^3} + \frac{3a}{kr^2} + \frac{3a}{r}\right) + \frac{3a}{r} e^{(a-r)\sqrt{3\rho}\left(\frac{P}{2Q^{1/2}} + iQ^{1/2}\omega\right)}\right].$$

(7.14) $$\Psi_3(r,\theta,\omega) = \left|-\frac{a^2 i}{2r^2}\sqrt{\frac{3}{\rho Q}} + \left(r + \frac{a^3}{2r^2}\right)\omega\right| \rho i \cos\theta.$$

(7.15) $$F_3(r,\theta,\omega) = \left[\frac{3a^2}{2r^2}\left(\frac{\omega}{k} + \frac{i}{\sqrt{3\rho Q}}\right) - \frac{ai}{2\rho r^2}\frac{1}{P + iQ\omega}\right] \rho i \cos\theta.$$

Theorem 7.1 will be proved by establishing that both

$$(7.16)\qquad \Sigma_j(r,\theta,t) \underset{\text{def}}{=} \lim_{T\to\infty}\lim_{s\to\infty}\int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega]\Psi_j(r,\theta,\omega)e^{i\omega t}d\omega$$

and

$$(7.17)\qquad \Phi_j(r,\theta,t) \underset{\text{def}}{=} \lim_{T\to\infty}\lim_{s\to\infty}\int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega]F_j(r,\theta,\omega)e^{i\omega t}d\omega$$

exist and possess the differentiation properties mentioned in Theorem 7.1.
For Σ_j this follows from the particular forms of Ψ_2 and Ψ_3 on the one hand and from the relations

$$(7.18)\qquad V_x(t) = \lim_{T\to\infty}\lim_{s\to\infty}\int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega]e^{i\omega t}d\omega.$$

$$(7.19)\qquad V_x'(t) = \lim_{T\to\infty}\lim_{s\to\infty}\int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega]e^{i\omega t}\omega\, d\omega$$

on the other hand. The last two formulae are directly derived from

$$(7.20)\qquad (V_x)_{T,\varepsilon,4,\xi}(t) = \lim_{s\to\infty}\int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega]e^{i\omega t}d\omega,$$

and

$$(7.21)\qquad (V_x)'_{T,\varepsilon,4,\xi}(t) = \lim_{s\to\infty}\int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega]\omega e^{i\omega t}d\omega,$$

the first of which constitutes an application of Fourier's Integral Theorem to $(V_x)_{T,\varepsilon,4,\xi}(t)$, while the second is obtained from an application of the same theorem to $(V_x)'_{T,\varepsilon,4,\xi}(t)$, subsequent partial integration, bearing in mind Definition 2.1.

To obtain proof of the corresponding properties of Φ_j, $(j = 2,3)$, Theorems 1 and 2 will be invoked. The parameters employed in the formulation of Theorem 7.1 are carried over to the application of Theorems 1 and 2 and the following additional substitutions in the formulation of these theorems are made: Let $c = 1/2$, $q = E/3\tau_i$, $N = 2$, $\gamma = (r-a)\sqrt{3\rho Q}$. Let

b be any positive number, $0 < p < E/3\tau_i$, let τ_i of Theorems 1 and 2 equal ξ. The boundedness of the corresponding functions in some disk about the origin will prove to be a consequence of other results. The obvious extension of Theorem 2 to the case of two parameters θ,r, with $r > a$, will be applied. Let, finally, $f_1 = f_{1j}$, $f_2 = f_{2j}$, $f = F_j$ $(j = 2,3)$ in turn, with

$$(7.22)\qquad f_{12}(r,\theta,\omega) \underset{\text{def}}{=} \frac{\sin\theta}{2}\left(\frac{3a}{k^2r^3} + \frac{3a^2}{kr^3}\right),$$

$$(7.23)\qquad f_{22}(r,\theta,\omega) \underset{\text{def}}{=} -\frac{\sin\theta}{2}\, e^{\frac{P(a-r)}{2}\sqrt{\frac{3\rho}{Q}}}\left\{e^{(a-r)\left[k-\sqrt{3\rho}\left(\frac{P}{2Q^{1/2}}+iQ^{1/2}\omega\right)\right]}\left(\frac{3a}{k^2r^3} + \frac{3a}{kr^2}\right) + \frac{3a}{r}\left[e^{(a-r)\left[k-\sqrt{3\rho}\left(\frac{P}{2Q^{1/2}}+iQ^{1/2}\omega\right)\right]} - 1\right]\right\},$$

$$(7.24)\qquad f_{13}(r,\theta,\omega) = F_3(r,\theta,\omega),\ f_{23}(r,\theta,\omega) = 0$$

where k denotes the analytic continuation into D of the function k, previously defined on the real axis only.

To prove that Theorems 1 and 2 may indeed be applied, compliance with the non-obviously satisfied conditions only is demonstrated. All of these conditions constitute properties of some particular elementary functions and the corresponding statements are presented in the form of Lemmas 7.1 7.7. Definitions 7.1 and 7.2, in conjunction with Lemma 7.1, serve to fix $\arg k(\omega)$ on the appropriate Riemann surface.

2. **Definition 7.1**: *Let the principal square root,* $\sqrt[p]{z}$ *of* $z \in \mathfrak{C}$ *be defined by* $\sqrt[p]{z} \in \mathfrak{C}$, $\left(\sqrt[p]{z}\right)^2 = z$ *and* $\operatorname{Im}\left(\sqrt[p]{z}\right) > 0$ *for* $z \notin R_+ \cup \{0\}$, $\sqrt[p]{z} \geqslant 0$ *otherwise.*

The non-principal root, $\sqrt[np]{z}$, *of* z *is defined by* $\sqrt[np]{z} = -\sqrt[p]{z}$.

Definition 7.2: *Let* $*$ *in* $\sqrt{i\omega(P+iQ\omega)}^{\,*}$ *denote the principal root when either* $\operatorname{Re}(\omega)>0$ *or* $\operatorname{Re}(\omega)=0$, $\operatorname{Im}(\omega)\leqslant 0$ *and the non-principal root when either* $\operatorname{Re}(\omega) < 0$ *or* $\operatorname{Re}(\omega) = 0$, $\operatorname{Im}(\omega) \geqslant \frac{P}{Q}\cdot$ $\sqrt{i\omega(P+iQ\omega)}^{\,*}$ *is not defined for other values of* ω.

The proofs of most of the lemmas in the sequel are trivial and will not be carried out in detail. Some proofs are sketched cursorily.

Lemma 7.1 : $k(\omega) = \sqrt{3\rho}\ \overset{*}{\sqrt{i\omega(P+iQ\omega)}}$.

Lemma 7.2 : *For* $|\omega| > 4\,\dfrac{P}{Q}$

$$\frac{3}{2}Q^{1/2}|\omega| \geqslant \left|\overset{*}{\sqrt{i\omega(P+iQ\omega)}}\right| \geqslant \left(\frac{Q}{2}\right)^{1/2}|\omega|, \tag{7.25}$$

$$\left|\overset{*}{\sqrt{i\omega(P+iQ\omega)}} + iQ^{1/2}\omega\right| > \frac{1}{2}Q^{1/2}|\omega|, \tag{7.26}$$

$$\left|\overset{*}{\sqrt{i\omega(P+iQ\omega)}} - iQ^{1/2}\omega - \frac{P}{2Q^{1/2}}\right| < \frac{2P^2}{Q^{3/2}}\frac{1}{|\omega|}, \tag{7.27}$$

$$\left|\frac{\omega}{k(\omega)} + \frac{i}{(3\rho Q)^{1/2}}\right| < P\left(\frac{2}{3\rho Q^3}\right)^{1/2}\frac{1}{|\omega|}. \tag{7.28}$$

Proof: The proof of (7.26) is based on (7.25), by applying simple estimates, dependent upon the values of $\arg(\omega)$, to $\arg\left[\overset{*}{\sqrt{i\omega(P+iQ\omega)}}\right]$. (7.27) follows from the indentity

$$\overset{*}{\sqrt{i\omega(P+iQ\omega)}} - iQ^{1/2}\omega - \frac{P}{2Q^{1/2}} = -\frac{iP^2\omega}{2Q^{1/2}}\,\frac{1}{\left[\overset{*}{\sqrt{i\omega(P+iQ\omega)}} + iQ^{1/2}\omega\right]^2},$$

valid for $\omega \notin \left[0, \dfrac{P}{Q}i\right]$, by applying (7.26). (7.28): from (7.26) follows, for $|\omega| > 4\,\dfrac{P}{Q}$,

$$\left|\overset{*}{\sqrt{i\omega(P+iQ\omega)}} - iQ^{1/2}\omega\right| < \frac{P}{2Q^{1/2}} + \frac{2P^2}{Q^{3/2}}\frac{1}{|\omega|} < \frac{P}{Q^{1/2}}.$$

Therefore and by (7.25)

$$\left|\frac{\omega}{k(\omega)} + \frac{i}{\sqrt{3\rho Q}}\right| = \left|\frac{\overset{*}{\sqrt{i\omega(P+iQ\omega)}} - iQ^{1/2}\omega}{\sqrt{3\rho Q}\ \overset{*}{\sqrt{i\omega(P+iQ\omega)}}}\right| < \frac{P\cdot 2^{1/2}}{Q^{1/2}\sqrt{3\rho Q}\,Q^{1/2}|\omega|} =$$

$$= P\sqrt{\frac{2}{3\rho Q^3}}\,\frac{1}{|\omega|}$$

Lemma 7.3 : *For* $|\omega| < \dfrac{P}{2Q}$, $\left|\overset{*}{\sqrt{i\omega(P+iQ\omega)}}\right| > \left(\dfrac{P}{2}\right)^{1/2}|\omega|^{1/2}$.

Lemma 7.4 : *For* $z \in \mathcal{C}$, $|z| < \dfrac{1}{2}$, $m = 0,1,2\ldots$, $\Lambda_m(z) \underset{\text{def}}{=} e^z - \displaystyle\sum_{n=0}^{m}\frac{z^n}{n!}$

$$|\Lambda_m(z)| < \frac{2|z|^{m+1}}{(m+1)!}. \tag{7.29}$$

Lemma 7.5 : *For* $r > a$, $|\omega| > \operatorname{Max}\left|\dfrac{4P}{Q},\ 4P^2(r-a)\sqrt{\dfrac{3\rho}{Q^3}}\right|$,

$$\left|e^{(a-r)\left[k-\sqrt{3\rho}\left(\frac{P}{2Q^{1/2}}+iQ^{1/2}\omega\right)\right]} - 1\right| < 4P^2(r-a)\left(\frac{3\rho}{Q^3}\right)^{1/2}\frac{1}{|\omega|}. \tag{7.30}$$

Proof: (7.27) and Lemma 7.4.

Lemma 7.6 : *Let* $\omega \in D$, $r > 0$, $\theta \in R$. *Then*

$$\text{(i)}\ F_2(r,\theta,\omega) + \frac{\sin\theta}{2}\left[\frac{3a^3}{2r^3} + \frac{3a}{2r} - \frac{3a}{r}e^{(a-r)\sqrt{3\rho}\left(\frac{P}{2Q^{1/2}}+iQ^{1/2}\omega\right)}\right] = \tag{7.31}$$

$$= -\frac{\sin\theta}{2}\left\{\frac{3a(a^2-r^2)}{2r^2}k + \frac{3a(a-r)^2k^2}{2r} + \Lambda_2[k(a-r)]\left(\frac{3a}{k^2r^3} + \frac{3a}{kr^2} + \frac{3a}{r}\right)\right\}.$$

(ii) *Let* $r_o > a$, $w = \operatorname{Min}\left[\dfrac{E}{3\eta}, \dfrac{\eta}{8\rho(r_o-a)^2}\right]$. *Then, for* $r \in [a, r_o]$,

$$\omega \in \mathcal{C},\ |\omega| < w,\ \omega \notin\,]0, w[$$

$$\left|F_2(r,\theta,\omega) + \frac{\sin\theta}{2}\left[\frac{3a}{2r} + \frac{3a^3}{2r^3} - \frac{3a}{r}e^{(a-r)\sqrt{3\rho}\left(\frac{P}{2Q^{1/2}}+iQ^{1/2}\omega\right)}\right]\right| \leqslant \tag{7.32}$$

$$\leqslant \Pi(a,\rho,\eta,r,E)\,|\omega|^{1/2}$$

with

$$\Pi(a,\rho,\eta,r,E) = \left(1-\frac{a}{r}\right)\left(4-\frac{a}{r}+\frac{a^2}{r^2}\right)a\sqrt{\frac{2\rho}{\eta}} < 4a\sqrt{\frac{2\rho}{\eta}}.$$

(iii) For $j = 1,2$ $r \geqslant a$, $\omega \in D$ functions $\Phi_j(a, \rho, \eta, E, r, \theta, \omega)$ — regular in D, positive functions $\lambda_j(a, \rho, \eta)$ and neighbourhoods $N_j(a, \rho, \eta, E, r_o)$ of $\omega = 0$ exist so that for $\omega \in N_j$, $r \in [a, r_o]$

$$\left| \frac{\partial^j F_2(r,\theta,\omega)}{\partial r^j} - \Phi_j(a,\rho,\eta, E, r, \theta, \omega) \right| < \lambda_j(a,\rho,\eta) \, |\omega|^{1/2}. \tag{7.33}$$

Proof: (*i*) Substitute in (7. 13) $1 + k(a - r) + \frac{1}{2} k^2(a - r)^2 + \Lambda_2[k(a - r)]$ for $e^{k(a-r)}$ and expand.

(*ii*) By (*i*), taking account of Lemma 7.4 and the value of w and by having recourse to the inequalities $|k(\omega)| < \left[\frac{2\rho}{\eta}|\omega|\right]^{1/2}$, $|k(\omega)|(r - a) < \frac{1}{2}$, valid for $|\omega| < w$.

(*iii*) By differentiation of (7. 31) and a subsequent estimate of the left-hand member — in a fashion similar to the one described in (*ii*).

Lemma 7.7 : *For $j = 0, 1, 2$, $r_o > a$ positive functions $g_j(a, \rho, \eta, E, r_o)$, $c_j(a, \rho, \eta, E, r_o)$ exist so that for $\omega \in R$, $|\omega| > g_j(a, \rho, \eta, E, r_o)$, $r \in [a, r_o]$*

$$\left| \frac{\partial^j F_2(r,\theta,\omega)}{\partial r^j} \right| < c_j(a,\rho,\eta,E,r_o)\,|\omega|^j. \tag{7.34}$$

Proof: From (7.13) and the corresponding expressions for the first two partial derivatives of F_2 with respect to r, having recourse to (7.25) and (7.8) — the latter being valid for real nonvanishing ω.

3. Verification of the conditions of Theorem 2 can now be brought to a conclusion. Since the formulation of Theorem 2 draws on the conditions of Theorem 1, some of these conditions are also mentioned in the sequel.

(*i*) Condition 6. of Theorem 2 is satisfied with respect to F_2, by Lemma 7. 6.

(*ii*) 2a) of Theorem 1 is satisfied, by (7. 22), Lemma 7.1 on the one hand and (7. 23), (7. 25) and Lemma 7. 5 on the other. 4 of Theorem 2 follows from Lemma 7. 6.

(*iii*) 7 is proved in Lemma 7. 7.

(*iv*) For F_3 2b) of Theorem 1 follows from (7. 15), by Lemmas 7.1 and 7. 3. 2a) of Theorem 1 follows from (7. 24) and (7. 28).

Verification of the remaining conditions of Theorem 2 is straightforward.

4. It is shown in [1] that when the (r, r) and (r, θ) components of the tensor of stress in the Maxwell-Fluid are denoted by p_{rr} and $p_{r\theta}$, respectively, and the surface of the immersed sphere by S, then the drag, **W**, is related to **v** by means of

$$\mathbf{W} = \iint_S (p_{rr} \cos\theta - p_{r\theta} \sin\theta)\, dS \cdot \mathbf{1}_x,$$

$$p_{rr} = -p + \frac{2E}{3} e^{-\frac{E}{3\eta}t} \int_0^t \frac{\partial v_r}{\partial r} e^{\frac{E}{3\eta}\tau} d\tau \text{ and } p_{r\theta} = \frac{E}{3} e^{-\frac{E}{3\eta}t} \int_0^t \frac{\partial v_\theta}{\partial r} e^{\frac{E}{3\eta}\tau} d\tau.$$

Consequently it becomes clear that the only new question of principle arising in connection with the modified value (7. 10) of **v** is whether, for $j = 1, 2$,

$$\int_0^t \lim_{T\to\infty} \lim_{s\to\infty} \int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega] \frac{\partial \mathrm{v}_j(r,\theta,\omega)}{\partial r} e^{\left(i\omega + \frac{E}{3\eta}\right)\tau} d\omega d\tau$$

and

$$\lim_{T\to\infty} \lim_{s\to\infty} \int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega] \frac{\partial \mathrm{v}_j(r,\theta,\omega)}{\partial r} \frac{e^{\left(i\omega + \frac{E}{3\eta}\right)t} - 1}{i\omega + \frac{E}{3\eta}} d\omega$$

exist and whether these expressions are equal. Again, for the sake of brevity, only the θ-component ($j = 2$) is dealt with in the sequel. The answer to the question posed turns out in the affirmative and — on the basis of the decomposition (7. 11) — is laid down in the following two lemmas:

Lemma 7.8 : *For $t \in R$*

$$Y(r,\theta,t) \underset{\text{def}}{=} \int_0^t \lim_{T\to\infty} \lim_{s\to\infty} \int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega] \frac{\partial \Psi_2(r,\theta,\omega)}{\partial r} e^{\left(i\omega + \frac{E}{3\eta}\right)\tau} d\omega d\tau$$

and

$$Z(r,\theta,t) \underset{\text{def}}{=} \lim_{T\to\infty} \lim_{s\to\infty} \int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega] \frac{\partial \Psi_2(r,\theta,\omega)}{\partial r} \frac{e^{\left(i\omega + \frac{E}{3\eta}\right)t} - 1}{i\omega + \frac{E}{3\eta}} d\omega$$

both exist and are equal.

Proof: Since $\partial\Psi_2/\partial r$ is a linear combination of 1, $e^{-i\gamma\omega}$ and $\omega e^{-i\gamma\omega}$ (with coefficients independent of ω) the inner integral in the definition of Y is, in turn, a linear combination of $(V_x)_{T,\varepsilon,4,\xi}(\tau)$, $(V_x)_{T,\varepsilon,4,\xi}(\tau - \gamma)$ and $(V_x)'_{T,\varepsilon,4,\xi}(\tau - \gamma)$, independent of T for $T > |t| + \gamma + \varepsilon$. For such T the $\lim_{T\to\infty}$-sign in the definition of Y may consequently be removed, thus proving the existence of Y, as well as

$$Y(r,0,t) = \int_0^t \lim_{s\to\infty} \int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega] \frac{\partial \Psi_2(r,0,\omega)}{\partial r} e^{\left(i\omega + \frac{E}{3\eta}\right)\tau} d\omega d\tau.$$

Since $U[(V_x)_{T,\varepsilon,4,\xi},\omega]$ is $O(|\omega|^{-4})$ for ω real, $\omega \to \infty$ and since $\partial\Psi_2(r,0,\omega)/\partial r$ is $O(|\omega|)$, the order of integration may be reversed. Since, for $T > |t| + \gamma + \varepsilon$, the integral is independent of T, a $\lim_{T\to\infty}$ - sign may again be prefixed.

Lemma 7.9: $\displaystyle\int_0^t e^{\frac{E}{3\eta}\tau} \frac{\partial}{\partial r} \left\{\lim_{T\to\infty} \lim_{s\to\infty} \int_{-s}^{s} U[(V_x)_{T,\varepsilon,4,\xi},\omega] e^{i\omega\tau} F_2(r,0,\omega)\right\} d\omega d\tau$

and

$$\lim_{T\to\infty} \lim_{s\to\infty} \int_{-s}^{s} \frac{\partial}{\partial r} [F_2(r,0,\omega)\, U[(V_x)_{T,\varepsilon,4,\xi},\omega]] \frac{e^{\left(i\omega + \frac{E}{3\eta}\right)t} - 1}{i\omega + \frac{E}{3\eta}} d\omega$$

exist and are equal.

Proof: Apply the representation (6. 18) to the partial derivative in the first integral, substituting F_2 for the function f from Theorem 2, r for x, V_x for v, $m' = 0$, $n' = 1$. By [2] from the proof of Theorem 2, Lemma 3.1 may now be applied,. The application concludes the proof of the lemma.

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ASUPRA UNEI GENERALIZĂRI A INTEGRALEI DE TIP FOURIER ȘI APLICAȚIILE ACESTEIA LA UNELE PROBLEME PE CONTUR

Rezumat

De multe ori soluțiile unor ecuații cu derivate parțiale sînt date sub forma (1.1). Pentru a evita unele dificultăți ce se ivesc datorită condiției ca (1.3) să conveargă, se studiază următoarele probleme: a) Cînd există limita (1.7) cu $Y(X,t,T)$ dat de (1.6) și U_T dat de (1.5); b) Cînd este această funcție derivabilă; c) Cînd se pot obține aceste derivate derivînd funcția Y.

După ce se dau condiții în care răspunsurile la aceste întrebări sînt afirmative, se studiază în § 7 problema curgerii unui fluid Maxwell, în jurul unei sfere rigide, cu ajutorul ecuației (7.1).

STOCHASTIC INTEGRAL EQUATIONS IN ENGINEERING AND BIOLOGICAL SCIENCES

BY

CHRIS P. TSOKOS*

1.0. Introduction. In mathematical models of phenomena occurring in the general areas of engineering, biological and physical sciences, random or stochastic equations appear frequently, [4], [10] [11]. The aim of this talk is to formulate stochastic models in fluid mechanics and chemotherapy which lead to stochastic integral equations of the form

$$v(t;\omega) = \int_0^t \psi(v(\tau;\omega), t, ;\tau;\omega) d\tau, \tag{1.1}$$

and

$$x(t;\omega) = h(t;\omega) + \int_0^t k(s, x(s;\omega);\omega) ds. \tag{1.2}$$

We shall also be concerned with the existence of a random solution and its physical interpretation in the aforementioned models.

2.0. On a Stochastic Model for Fluid Mechanics. Consider a tagged material point in a continuous fluid in turbulent motion. The location of the designated point at time $t>0$, is a vector-valued stochastic variable, $v(t;\omega)$, $\omega \in \Omega$ [where Ω is the supporting set of the underlying probability

* A talk presented at the mathematics conference at the University „Al. I. Cuza" Iassy, Romania, August 20—25, 1970, on the occasion of the 60th anniversary of the „A. Myller" Mathematical Seminar.