# ON THE PROLONGATIONS OF GEOMETRIC OBJECT FIELDS

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The purpose of the present paper is to describe a natural algorithm for the prolongations of geometric object fields. This algorithm is based on the works of É. Cartan, on some investigations by G. F. Laptev and on the idea of the canonical form on a prolongation of a principal fibre bundle introduced in our previous paper [6].

A natural setting for the definition of the prolongations of geometric object fields is in the theory of jets (by Ehresmann): a geometric object field being a cross section $\rho$ of an associated fibre bundle $E$, the prolongation of order $r$ of $\rho$ is the cross section $j^r\rho$ of $J^r E$. On the other hand, G. F. Laptev (see e. g. [7]) has established a „method of prolongations" (closely related with some algorithms by É. Cartan), which was successfully applied to many concrete investigations in differential geometry. As a rule, the prolongations of a geometric object field are not treated in a „pure" form in such a concrete situation, since some additional structure is usually given and this modifies the whole situation in a specific way. In the present paper, we are interested only in the „pure" form of the „corresponding" algorithm. (As far as we know, an attempt to pose the problem in its full generality was done only by Szybiak in the Appendix to [8]). The principal tool for this algorithm is the operation of exterior differentiation. Our main conclusion can be shortly expressed as follows: If the „initial" functions for this „pure" algorithm of prolongations are the coordinate functions of a geometric object field $\rho$, then the „resulting" functions after $r$ steps of this algorithm are the coordinate functions of the $r$-th prolongation $j^r\rho$ of $\rho$.

The standard terminology and notations of the theory of jets are used throughout the paper, see [1]. In addition, $j_r^s$ denotes the canonical projection of $r$-jets onto $s$-jets, $s < r$. Our considerations are in the category $C^\infty$.

**1. Relative images and coordinate functions.** Let $P(B, G)$ be a principal fibre bundle and let $E(B, F, G, P)$ be a fibre bundle associated with $P$. The elements of $E$ are the equivalence classes $\{(u, y)\}$, $u \in P$, $y \in F$, with respect to the equivalence relation $(u, y) \sim (ug, g^{-1}(y))$, $g \in G$. Hence every $u \in P_x$, $x \in B$, determines a diffeomorphism (denoted by the same symbol) $u : F \to E_x$, $y \to \{(u, y)\}$. In this connection, $u$ will be called a *frame* on $E$. If $z \in E_x$ and $u \in P_x$, then the point $u^{-1}(z) \in F$ will be said to be the *image* (or *relative image*) of $z$ by the frame $u$. Thus we have a mapping $\mu$ of $P \boxtimes E$ (=the fibre product over $B$ of $P$ and $E$) into $F$, $\mu(u, z) = u^{-1}(z)$, which can be called the *relative image map*. Let $K$ be a coordinate neighbourhood on $F$, let $\varkappa : K \to R^m$ be a coordinate diffeomorphism and let

$$y^p, \quad p, q, \ldots = 1, \ldots, m = \dim F,$$

be the corresponding coordinate functions on $K$. Put

$$\widetilde{K} = \{(u, z) \in P \boxtimes E; \quad \mu(u, z) \in K\}.$$

Then $A^p = y^p \circ \mu$ are some real-valued functions on $\widetilde{K}$, which will be said to be the *coordinate functions* on $P \boxtimes E$ *with respect to* $\varkappa$. (For the sake of simplicity, we say that $A^p$ are coordinate functions on $P \boxtimes E$, though they are defined only on $\widetilde{K} \subset P \boxtimes E$. Similarly, we shall not sometimes mention explicitly some natural restrictions of some mappings, provided there is no danger of confusion.) We remark that our coordinate functions are introduced analogously to [4].

A *geometric object field* (in direct form, cf. [5]) on $E$ is a cross section $\rho : B \to E$. The fibre product $P \boxtimes \rho \subset P \boxtimes E$ is canonically identified with $P$, so that the restriction of $\mu$ to $P \boxtimes \rho$ can be considered as a mapping of $P$ into $F$. This mapping will be denoted by $\mu_\rho$. (In [5], we have said that $\mu_\rho$ is the indirect form of $\rho$.) Put

$$\widetilde{K}(\rho) = \{u \in P; \quad \mu_\rho(u) \in K\}.$$

Then $a^p = y^p \circ \mu_\rho$ are some real-valued functions on $\widetilde{K}(\rho)$, which will be called the *coordinate functions of* $\rho$ *with respect to* $\varkappa$.

Fix a point $x \in B$. Then $\mu_z : P_x \to F$, $\mu_z(u) = u^{-1}(z)$, is the relative image map of a point $z \in E_x$ and the real-valued functions $a^p = y^p \circ \mu_z$ on $\widetilde{K}(z) = \{u \in P_x; \; u^{-1}(z) \in K\}$ are the coordinate functions of $z$. Consider a basis $e_\alpha$ of $g$ (= the Lie algebra of $G$) and construct the fundamental vector fields $X_\alpha$ determined by $e_\alpha$ on $P_x$. Let $\omega^\alpha$ be the dual basis of $T^*(P_x)$. If $[e_\alpha, e_\beta] = - c^\gamma_{\alpha\beta}\, e_\gamma$, then



$$d\omega^\alpha = 1/2\, c^\alpha_{\beta\gamma}\, \omega^\beta \wedge \omega^\gamma. \tag{2}$$

Let $g_\alpha(t)$ be the one-parameter subgroup of $G$ corresponding to $e_\alpha$ and let $Y_\alpha$ be the vector field on $F$ determined by $(Y_\alpha)_y = j^1_0(g_\alpha(t)(y))$, $y \in F$. Since $G$ acts on $F$ on the left, the vector fields $Y_\alpha$ form a basis of a Lie algebra anti-isomorphic to $g$, i. e.

$$[Y_\alpha, Y_\beta] = c^\gamma_{\alpha\beta}\, Y_\gamma. \tag{3}$$

**Lemma 1.** *Let $X \in T_u(P_x)$ and let $z \in E_x$. Then*

$$(\mu_z)_*(X) = -(Y_\alpha)_y\, \omega^\alpha(X), \tag{4}$$

*where* $y = u^{-1}(z)$.

*Proof.* Let $\gamma$ be a curve on $G$ such that $X = j^1_0(u\gamma(t))$. Then $(d\gamma/dt)_0 = \omega^\alpha(X)e_\alpha$ by the definition of the fundamental vector fields on $P_x$. Using the well-known relation $(d\gamma^{-1}/dt)_0 = -(d\gamma/dt)_0$ we obtain $(\mu_z)_*(X) = = j^1_0(\gamma^{-1}(t)(u^{-1}(z))) = -\omega^\alpha(X)(Y_\alpha)_y$, QED.

For every $(u, y) \in P_x \times F$, we denote by $\Delta_{(u, y)}$ the tangent space at $(u, y)$ to the graph of the mapping (of $P_x$ into $F$) $v \to v^{-1}(u(y))$, $v \in P_x$. Thus we obtain a distribution $\Delta$ on $P_x \times F$, which will be called the *fundamental distribution* on $P_x \times F$. By the very definition of $\Delta$, a mapping of $P_x$ into $F$ represents the relative images of a point of $E_x$ if and only if its graph is a solution of $\Delta$.

Let the restriction of $Y_\alpha$ to $K$ be given by $\eta^p_\alpha(y^q)\, \partial/\partial y^p$. Then (4) implies directly that the equations of the fundamental distribution on $P_x \times F$ with respect to $\varkappa$ are

$$dy^p + \eta^p_\alpha(y^q)\, \omega^\alpha = 0. \tag{5}$$

By another words, the coordinate functions $a^p$ of any $z \in E_x$ satisfy

$$da^p + \eta^p_\alpha(a^q)\, \omega^\alpha = 0. \tag{6}$$

Furthermore, the coordinate form of (3) is

$$\frac{\partial \eta^p_\beta}{\partial y^q}\, \eta^q_\alpha - \eta^q_\beta\, \frac{\partial \eta^p_\alpha}{\partial y^q} = c^\gamma_{\alpha\beta}\, \eta^p_\gamma. \tag{7}$$

**Lemma 2.** *Let $P(B, G)$ be a principal fibre bundle such that the fundamental vector fields $X_\alpha$ determined by $e_\alpha$ on $P$ can be completed by some vector fields $X_i$ to a basis of $T(P)$. Let $\tilde{\omega}^\alpha$, $\tilde{\omega}^i$ be the dual basis of $T^*(P)$. If $a^p$ are the coordinate functions of a geometric object field on $E$, then it holds*

$$da^p + \eta^p_\alpha(a^q)\, \tilde{\omega}^\alpha = 0 \pmod{\tilde{\omega}^i}. \tag{8}$$

*Proof.* If we consider a fibre $P_x$ of $P$, then the restrictions of $\tilde{\omega}^\alpha$ to $P_x$ are the forms $\omega^\alpha$ used in Lemma 1. Hence (8) is a simple consequence of (6).

**2. The prolongations of geometric object fields.** Let $P(B, G, \pi)$ and $\overline{P}(\overline{B}, G, \overline{\pi})$ be two principal fibre bundles with the same structure group $G$, let $(\psi, \varphi): P(B, G) \to \overline{P}(\overline{B}, G)$ be a principal fibre bundle isomorphism and let $E(B, F, G, P)$ and $\overline{E}(\overline{B}, F, G, \overline{P})$ be the corresponding associated fibre bundles with the same standard fibre $F$. Then $\psi$ induces a mapping $\hat{\psi}: E \to \overline{E}$ defined by

$$\hat{\psi}(\{(u, y)\}) = \{(\psi(u), y)\}, \quad u \in P, \quad y \in F. \tag{9}$$

In [6], we have introduced the $r$-th prolongation $W^r(P)$ of $P$ as the manifold of all $r$-jets with source $(0, e)$ of the allowable charts on $P$, $e$ being the unit of $G$. (An allowable chart on $P$ is a principal fibre bundle isomorphism $(\psi, \varphi): U \times G \to \pi^{-1}(V)$, where $U$ is a neighbourhood of $O \in R^n$, $n = \dim B$, and $V \subset B$. Such an allowable chart is uniquely determined by the diffeomorphism $\varphi: U \to V$ and by the cross section $\sigma: V \to \pi^{-1}(V)$, $\sigma(\varphi(x)) = \psi(x, e)$). $W^r(P)$ has a natural structure of a principal fibre bundle over $B$ with structure group $G^r_n =$ the group of all $r$-jets at $(0, e)$ of the local automorphisms $(\psi, \varphi)$ of $R^n \times G$ satisfying $\varphi(0) = 0$. According to [6], every $h \in G^r_n$ can be uniquely expressed as $h = (X, S)$, $X \in L^r_n$, $S \in T^r_n(G)$.

Let $J^r E$ be the fibred manifold (over $B$) of all $r$-jets of the local cross sections of $E$. We shall show that $J^r E$ has a natural structure of an associated fibre bundle of the symbol $(B, T^r_n(F), G^r_n, W^r(P))$. We first remark that the fibre bundle associated with the trivial principal fibre bundle $\mathbf{R}^n \times G$ with standard fibre $F$ is canonically identified with $\mathbf{R}^n \times F$ provided one identifies $\{((x, e), y)\}$ with $(x, y)$, $x \in \mathbf{R}^n$, $y \in F$. Further we can identify $T^r_n(F)$ with the fibre over $O \in \mathbf{R}^n$ of $J^r(\mathbf{R}^n \times F)$ by means of the canonical projection of $\mathbf{R}^n \times F$ onto the second factor. Moreover, every $h \in G^r_n$ is an $r$-jet of a local automorphism $(\psi, \varphi)$ of $\mathbf{R}^n \times G$ satisfying $\varphi(0) = 0$. Hence $\hat{\psi}$ is a local mapping of $\mathbf{R}^n \times F$ onto itself such that $\hat{\psi}(\{0\} \times F) = \{0\} \times F$ and we can introduce a left action $(h, Y) \to h(Y)$ of $G^r_n$ on $T^r_n(F)$ as follows: Let $h = (X, S)$, $X \in L^r_n$, $S \in T^r_n(G)$, $X = j^r_0 \varphi$, $S = j^r_0 \sigma$ and let $Y \in T^r_n(F)$, $Y = j^r_0 \rho$. Then $x \to (x, \rho(x))$ is the corresponding local cross section of $\mathbf{R}^n \times F$ and it is

$$\begin{aligned}\hat{\psi}(x \to (x, \rho(x))) &= \varphi(x) \to (\varphi(x), \sigma(x)(\rho(x))) = \\ &= x \to (x, \sigma(\varphi^{-1}(x))(\rho(\varphi^{-1}(x)))).\end{aligned} \tag{10}$$

Thus, we can define $h(Y) = j^r_0 [\sigma(\varphi^{-1}(x))(\rho(\varphi^{-1}(x)))]$. From (10) we deduce immediately that this action of $G^r_n$ on $T^r_n(F)$ can also be written as

$$(X, S)(Y) = (S(Y)) X^{-1}, \tag{11}$$



where $(S, Y) \to S(Y)$ is the natural action of $T^r_n(G)$ on $T^r_n(F)$ (see [1]) and $(S(Y)) X^{-1}$ means the composition of jets. (This formula was deduced in a different way by V i r s í k, [9]; see also [2]). Furthermore, if $u \in W^r(P)$, $u = j^r_{(0,e)} \psi$, $\psi = (\varphi, \sigma)$ and if $Y \in T^r_n(F)$, $Y = j^r_0 \rho$, then $\hat{\psi}(x \to (x, \rho(x)))$ is a cross section of $E$. Setting $u(Y) = j^r_{\varphi(0)} [\hat{\psi}(x \to (x, \rho(x)))]$, we establish $u$ as a frame on $J^r_{\varphi(0)} E$. Let $\mu^r$ denote the corresponding relative image map of $W^r(P) \boxtimes J^r E$ into $T^r_n(F)$.

The canonical coordinate functions

$$b^p, b^p_i, \dots, b^p_{i_1 \dots i_r}, \quad i, j, \dots = 1, \dots, n = \dim B \tag{12}$$

on $T^r_n(\mathbf{R}^m)$ are defined as follows. Every $Y \in T^r_n(\mathbf{R}^m)$ can be uniquely expressed as an $r$-jet of a mapping of the form

$$a^p + a^p_i x^i + \dots + \frac{1}{r!} a^p_{i_1 \dots i_r} x^{i_1} \dots x^{i_r};$$

then we put $b^p(Y) = a^p$, $b^p_i(Y) = a^p_i, \dots, b^p_{i_1 \dots i_r}(Y) = a^p_{i_1 \dots i_r}$. The coordinate diffeomorphism $\varkappa: K \to \mathbf{R}^m$ is naturally extended to a coordinate diffeomorphism $\varkappa^r_n: T^r_n(K) \to T^r_n(\mathbf{R}^m)$; we shall denote by $y^p, y^p_i, \dots, y^p_{i_1 \dots i_r}$ the corresponding coordinate functions on $T^r_n(K)$. Let

$$\tilde{K}^r = \{(u, z) \in W^r(P) \boxtimes J^r E;\ u^{-1}(z) \in T^r_n(K)\}. \tag{13}$$

Then $A^p = y^p \circ \mu^r$, $A^p_i = y^p_i \circ \mu^r, \dots, A^p_{i_1 \dots i_r} = y^p_{i_1 \dots i_r} \circ \mu^r$ are some real valued functions on $\tilde{K}^r$, which will be called the *coordinate functions on* $W^r(P) \boxtimes J^r E$ *with respect to* $\varkappa$. Obviously, if $\bar{A}^p, \dots, \bar{A}^p_{i_1 \dots i_{r-1}}$ are the coordinate functions on $W^{r-1}(P) \boxtimes J^{r-1} E$ with respect to $\varkappa$, then $A^p = (j^{r-1}_r)^* \bar{A}^p, \dots, A^p_{i_1 \dots i_{r-1}} = (j^{r-1}_r)^* \bar{A}^p_{i_1 \dots i_{r-1}}$.

Let $\rho: B \to E$ be a geometric object field on $E$. Then the $r$-jets of $\rho$ determine a cross section $j^r \rho: B \to J^r E$, which is said to be the $r$-th *prolongation* of $\rho$. The restriction of $\mu^r$ to $W^r(P) \boxtimes j^r \rho$ can be considered as a mapping of $W^r(P)$ into $T^r_n(F)$. This mapping will be denoted by $\mu^r_\rho$. (One can also say that $\mu^r_\rho$ is the indirect form of $j^r \rho$, cf. [5].) The real-valued functions $a^p = y^p \circ \mu^r_\rho, \dots, a^p_{i_1 \dots i_r} = y^p_{i_1 \dots i_r} \circ \mu^r_\rho$ on

$$\tilde{K}^r(\rho) = \{u \in W^r(P);\ u^{-1}(j^r_x \rho) \in T^r_n(K),\ x = \pi(\beta u)\}$$

will be called the *coordinate functions of* $j^r\rho$ *with respect to* $\varkappa$. Obviously, if $\bar{a}^p, \dots, \bar{a}^p_{i_1 \dots i_{r-1}}$ are the coordinate functions of $j^{r-1}\rho$ with respect to $\varkappa$, then $a^p = (j^{r-1}_r)^* \bar{a}^p, \dots, a^p_{i_1\dots i_{r-1}} = (j^{r-1}_r)^* \bar{a}^p_{i_1 \dots i_{r-1}}$.

**3. Coordinate form of the first prolongation.** In [6], we have introduced the canonical form $\Theta_r$ of $W^r(P)$. This is a $w^{r-1}_n(G)$-valued 1-form on $W^r(P)$, where $w^{r-1}_n(G)$ means the tangent space of $W^{r-1}(\mathbf{R}^n \times G)$ at its distinguished element $e^{r-1} = j^{r-1}_{(0,e)} \mathrm{id}_{\mathbf{R}^n \times G}$. The basis

$$e_\alpha, \qquad \alpha, \beta, \dots = n+1, \dots, n + \dim G,$$

of $\mathfrak{g}$ together with the canonical basis $e_i$ of $\mathbf{R}^n$ induces a basis

$$e_\lambda, e^i_\lambda, \dots, e^{i_1 \dots i_{r-1}}_\lambda, \qquad \lambda = 1, \dots, n + \dim G,$$

of $w^{r-1}_n(G) = \mathbf{R}^n \oplus g^{r-1}_n$.

**Propositon 1.** *Let* $\rho : B \to E$ *be a geometric object field, let* $\bar{a}^p$ *be the coordinate functions of* $\rho$ *with respect to* $\varkappa$ *and let* $a^p = (j^0_1)^* \bar{a}^p$, $a^p_i$ *be the coordinate functions of* $j^1\rho$ *with respect to* $\varkappa$. *Let* $\Theta_1 = \Theta^i e_i + \Theta^\alpha e_\alpha$ *be the canonical form of* $W^1(P)$. *Then it holds*

$$da^p + \eta^p_\alpha(a^q)\,\Theta^\alpha = a^p_i\,\Theta^i. \tag{14}$$

*Proof.* We have to show

$$da^p(X) + \eta^p_\alpha(a^q(u))\,\Theta^\alpha(X) = a^p_i(u)\,\Theta^i(X) \tag{15}$$

for every $u \in W^1(P)$ and every $X \in T_u(W^1(P))$. By [6], $u$ determines a linear isomorphism $\tilde{u} : w^0_n(G) \to T_{\bar{u}}(P)$, $\bar{u} = j^0_1 u$. By definition, $\Theta_1(X) = \tilde{u}^{-1}(\bar{X})$, where $\bar{X} = (j^0_1)_* X$. Put $\tilde{u}^{-1}(\bar{X}) = \xi^i e_i + \xi^\alpha e_\alpha$. Then we have a decomposition $\bar{X} = \bar{X}_1 + \bar{X}_2$ such that $\tilde{u}(\xi^i e_i) = \bar{X}_1$ and $\tilde{u}(\xi^\alpha e_\alpha) = \bar{X}_2$. Since $\bar{X}_2 \in T_{\bar{u}}(P_x)$, we deduce directly by Lemma 1 that $da^p(X_2) = -\eta^p_\alpha(a^q(\bar{u}))\,\xi^\alpha$. Further assume $u = j^1_{(0,e)}\psi$, $\psi = (\varphi, \sigma)$. Let $\gamma_i$ be the curve $x^j = \delta^j_i t$ on $\mathbf{R}^n$. By definition [illegible] is $a^p_i(u) = j^1_0 y^p(\sigma^{-1}(\varphi(\gamma_i(t)))(\rho(\varphi(\gamma_i(t))))$, where $y^p$ are the functions (1). Since the vector $\bar{X}_1$ is tangent to $\sigma$, it can be written as $\bar{X}_1 = j^1_0\sigma(\varphi(\gamma(t)))$, where $\gamma$ is the curve $x^i = \xi^i t$ on $\mathbf{R}^n$. Then $da^p(X_1) = j^1_0 a^p(\sigma(\varphi(\gamma(t)))) = j^1_0 y^p(\sigma^{-1}(\varphi(\gamma(t)))(\rho(\gamma(t))) = a^p_i(u)\xi^i$. Hence

$$da^p(X) = -\eta^p_\alpha(a^q(\bar{u}))\,\xi^\alpha + a^p_i(u)\,\xi^i,$$

which is equivalent to (15), QED.

*Remark* 1. If a geometric object field $\rho$ is given, then $\bar{a}^p$ are some well determined functions. On the other hand, the components $\Theta^i$, $\Theta^\alpha$ of the canonical form $\Theta_1$ of $W^1(P)$ can also be considered as known. Since $a^p = (j^0_1)^* \bar{a}^p$, we may say that *the coordinate functions* $a^p_i$ of $j^1\rho$ *are determined by* (14). Furthermore, considering the coordinate functions $A^p$, $A^p_i$ on $\tilde{K}^1$, one can say in an analogous sense that $A^p_i$ are determined by means of the relations $dA^p + \eta^p_\alpha(A^q)\,\Theta^\alpha = A^p_i\,\Theta^i$.



Fix a point $x \in B$ and consider the fundamental distribution on $W^1_x(P) \times T^1_n(F)$. To deduce the equations of this distribution with respect to $\varkappa^1_n$, we shall apply Lemma 2. We first recall that the structure equations of $\Theta_1$ are

$$
\begin{aligned}
d\Theta^i &= \Theta^j \wedge \Theta'^i_j, \\
d\Theta^\alpha &= \frac{1}{2} c^\alpha_{\beta\gamma}\Theta^\beta \wedge \Theta^\gamma + \Theta^j \wedge \Theta'^\alpha_j,
\end{aligned}
\tag{16}
$$

where $\Theta'_1 = \Theta_1 + \Theta'^\lambda_j e^j_\lambda$ is an admissible extension of $\Theta_1$. By [6], the components of $\Theta'_1$ satisfy the assumptions of Lemma 2. By virtue of (16) and (7), the exterior differentiation of (14) yields

$$\left[ da^p_i + \frac{\partial \eta^p_\alpha}{\partial y^q} a^q_i\,\Theta^\alpha + \eta^p_\alpha\,\Theta'^\alpha_i - a^p_j\,\Theta'^j_i \right] \wedge \Theta^i = 0. \tag{17}$$

Put $\Phi^p_i(a^q, a^q_i, \Theta^\alpha, \Theta'^\lambda_i) = \dfrac{\partial \eta^p_\alpha}{\partial y^q} a^q_i\,\Theta^\alpha + \eta^p_\alpha\,\Theta'^\alpha_i - a^p_j\,\Theta'^j_i$ and denote by $\omega^\alpha$, $\omega^\lambda_i$ the restrictions of $\Theta^\alpha$, $\Theta'^\lambda_i$ to $W^1_x(P)$. (These restrictions do not depend on the choice of an admissible extension of $\Theta_1$, see [6], § 4.) By Lemma 2 and by (17), the equations of the fundamental distribution on $W^1_x(P) \times T^1_n(F)$ with respect to $\varkappa^1_n$ are

$$dy^p + \eta^p_\alpha(y^q)\,\omega^\alpha = 0,$$

$$dy^p_i + \Phi^p_i(y^q, y^q_i, \omega^\alpha, \omega^\lambda_i) = 0.$$

4. **Algorithm of prolongations.** The prolongations of higher order of a geometric object field can be treated by a recurrent algorithm whose first step we have just described. Assume by induction that we have applied this algorithm $(r-1)$-times, $r > 1$, and that we have deduced the equations of the fundamental distribution on $W^{r-1}_x(P) \times T^{r-1}_n(F)$ with respect to $\varkappa^{r-1}_n$ in the form

$$
\begin{aligned}
&d\bar{y}^p + \eta^p(\bar{y}^q)\,\bar{\omega}^\alpha = 0, \\
&d\bar{y}^p_{i_1 \dots i_{r-1}} + \Phi^p_{i_1 \dots i_{r-1}}(\bar{y}^q, \dots, \bar{y}^q_{i_1 \dots i_{r-1}}, \bar{\omega}^\alpha, \dots, \bar{\omega}^\lambda_{i_1 \dots i_{r-1}}) = 0,
\end{aligned}
\tag{18}
$$

where the $\bar{y}$'s are the coordinate functions on $T^{r-1}_n(K)$, the $\bar{\omega}$'s are the restrictions of an admissible extension of the canonical form of $W^{r-1}(P)$ to a fibre $W_x^{r-1}(P)$ of $W^{r-1}(P)$ and $\Phi^p_{i_1\dots i_k}$ is a linear combination of $\bar\omega^\alpha, \dots, \bar\omega^\lambda_{i_1\dots i_k}$ with some coefficients in $\bar y^q, \dots, \bar y^q_{i_1\dots i_k}$, $k = 1, \dots, r-1$. Further assume that the coordinate functions

$$\bar a^p, \dots, \bar a^p_{i_1\dots i_{r-1}} \tag{19}$$

of the $(r-1)$-st prolongation $j^{r-1}\rho$ of a geometric object field $\rho$ with respect to $\varkappa$ satisfy

$$\begin{aligned} &d\bar a^p + \eta^p_\alpha(\bar a^q)\,\bar\Theta^\alpha = \bar a^p_i\,\bar\Theta^i, \\ &\vdots \\ &d\bar a^p_{i_1\dots i_{r-2}} + \Phi^p_{i_1\dots i_{r-2}}(\bar a^q, \dots, \bar a^q_{i_1\dots i_{r-2}}, \bar\Theta^\alpha, \dots, \bar\Theta^\lambda_{i_1\dots i_{r-2}}) = \bar a^p_{i_1\dots i_{r-1}}\bar\Theta^{i_{r-1}}, \end{aligned} \tag{20}$$

where the $\bar\Theta$'s are the components of the canonical form of $W^{r-1}(P)$. Let $\Theta_r = \Theta^\lambda e_\lambda + \dots + \Theta^\lambda_{i_1\dots i_{r-1}} e_\lambda^{i_1\dots i_{r-1}}$ be the canonical form of $W^r(P)$.

**Proposition 2.** *The coordinate functions* $a^p = (j_r^{r-1})^* \bar a^p, \dots, a^p_{i_1\dots i_{r-1}} = (j_r^{r-1})^* \bar a^p_{i_1\dots i_{r-1}}$, $a^p_{i_1\dots i_r}$ *of the r-th prolongation* $j^r\rho$ *of* $\rho$ *satisfy*

$$\begin{cases} da^p + \eta^p_\alpha(a^q)\,\Theta^\alpha = a^p_i\Theta^i, \\ \vdots \\ da^p_{i_1\dots i_{r-2}} + \Phi^p_{i_1\dots i_{r-2}}(a^q, \dots, a^q_{i_1\dots i_{r-2}}, \Theta^\alpha, \dots, \Theta^\lambda_{i_1\dots i_{r-2}}) = a^p_{i_1\dots i_{r-1}}\Theta^{i_{r-1}}, \end{cases} \tag{21}$$

$$da^p_{i_1\dots i_{r-1}} + \Phi^p_{i_1\dots i_{r-1}}(a^q, \dots, a^q_{i_1\dots i_{r-1}}, \Theta^\alpha, \dots, \Theta^\lambda_{i_1\dots i_{r-1}}) = a^p_{i_1\dots i_r}\Theta^{i_r}. \tag{22}$$

*Proof.* By Proposition 1 of [6], it is

$$\Theta^\lambda = (j_r^{r-1})^*\,\bar\Theta^\lambda, \dots, \Theta^\lambda_{i_1\dots i_{r-2}} = (j_r^{r-1})^*\,\bar\Theta^\lambda_{i_1\dots i_{r-2}}. \tag{23}$$

Hence (21) is a direct consequence of (20) and (23). Further, let $u \in W^r(P)$, $u = j^r_{(0,e)}\psi$, $\psi = (\varphi, \sigma)$ and let $X \in T_u(W^r(P))$. Then $u$ determines a linear isomorphism $\tilde u : w_n^{r-1}(G) \to T_{\bar u}(W^{r-1}(P))$, $\bar u = j_r^{r-1}u$, see [6]. By definition, $\Theta_r(X) = \tilde u^{-1}(\bar X)$, where $\bar X = (j_r^{r-1})_* X$. Put $\Theta_r(X) = \xi^\lambda e_\lambda + \dots + \xi^\lambda_{i_1\dots i_{r-1}} e_\lambda^{i_1\dots i_{r-1}}$. Then we have a decomposition $\bar X = \bar X_1 + \bar X_2$ such that $\tilde u(\xi^i e_i) = \bar X_1$. Since $\bar X_2 \in T_{\bar u}(W_x^{r-1}(P))$, $x = \varphi(0)$, we obtain directly by Lemma 1 that $d\bar a^p_{i_1\dots i_{r-1}}(\bar X_2) = -\Phi^p_{i_1\dots i_{r-1}}(\bar a^q(\bar u), \dots, \bar a^q_{i_1\dots i_{r-1}}(\bar u), \xi^\alpha, \xi^\lambda_{i_1\dots i_{r-1}})$. On the other hand, let $\gamma_i$ be the curve $x^j = \delta^j_i t$ on $\mathbf{R}^n$. By definition, it is $a^p_{i_1\dots i_{r-1}i}(u) = \left(\frac{d}{dt}\right)_0 \bar y^p_{i_1\dots i_{r-1}}(j_0^{r-1}[\sigma^{-1}(\varphi(x+\gamma_i(t)))\,(\rho(\varphi(x+\gamma_i(t))))])$. Furthermore, the vector $\bar X_1$ can be written as $\bar X_1 = j_0^1[\psi j_0^{r-1}\tau_{\gamma(t)}]$, where $\gamma$ is the curve



$x^i = \xi^i t$ on $\mathbf{R}^n$ and $\tau_{\gamma(t)}$ means the mapping $(x, g) \to (x + \gamma(t), g)$, $x \in \mathbf{R}^n$, $g \in G$. Then we find $d\bar a^p_{i_1\dots i_{r-1}}(\bar X_1) = \left(\frac{d}{dt}\right)_0 \bar a^p_{i_1\dots i_{r-1}}(\psi j_0^{r-1}\tau_{\gamma(t)}) = \left(\frac{d}{dt}\right)_0 \bar y^p_{i_1\dots i_{r-1}}(j_0^{r-1}[\sigma^{-1}(\varphi(x+\gamma(t)))\,(\rho(\varphi(x+\gamma(t))))]) = a^p_{i_1\dots i_r}(u)\xi^{i_r}$, which completes the proof of Proposition 2.

*Remark* 2. Similarly to Remark 1, we may assert that the coordinate functions $a^p_{i_1\dots i_r}$ of $j^r\rho$ are determined by (22). Analogously, one can say that the coordinate functions $A^p_i, \dots, A^p_{i_1\dots i_r}$ on $W^r(P) \boxtimes J^rE$ with respect to $\varkappa$ are determined by means of (21) and (22), where the $a$'s are replaced by the corresponding $A$'s.

For the next step of the algorithm we shall need the equations of the fundamental distribution on $W_x^r(P) \times T_n^r(F)$ with respect to $\varkappa_n^r$. Applying the structure equations of $\Theta_r$, see [6] (these equations can also be found in [3]), and using the fact that (18) is completely integrable, one finds easily that the exterior differentiation of (22) yields

$$[da^p_{i_1\dots i_r} + \Phi^p_{i_1\dots i_r}(a^q, \dots, a^q_{i_1\dots i_r}, \Theta^\alpha, \dots, \Theta'^\lambda_{i_1\dots i_r})] \wedge \Theta^{i_r} = 0, \tag{24}$$

where $\Theta'_r = \Theta_r + \Theta^\lambda_{i_1\dots i_r} e_\lambda^{i_1\dots i_r}$ is an admissible extension of $\Theta_r$ and $\Phi^p_{i_1\dots i_r}$ is a linear combination of $\Theta^\alpha, \dots, \Theta^\lambda_{i_1\dots i_{r-1}}, \Theta'^\lambda_{i_1\dots i_r}$ with some coefficients in $a^q, \dots, a^q_{i_1\dots i_r}$. Hence we conclude by Lemma 2 that the equations of the fundamental distribution on $W_x^r(P) \times T_n^r(F)$ with respect to $\varkappa_n^r$ are

$$\begin{aligned} &dy^p + \eta^p_\alpha(y^q)\,\omega^\alpha = 0, \\ &\vdots \\ &dy^p_{i_1\dots i_{r-1}} + \Phi^p_{i_1\dots i_{r-1}}(y^q, \dots, y^q_{i_1\dots i_{r-1}}, \omega^\alpha, \dots, \omega^\lambda_{i_1\dots i_{r-1}}) = 0, \\ &dy^p_{i_1\dots i_r} + \Phi^p_{i_1\dots i_r}(y^q, \dots, y^q_{i_1\dots i_r}, \omega^\alpha, \dots, \omega^\lambda_{i_1\dots i_r}) = 0, \end{aligned} \tag{25}$$

where the $\omega$'s are the restrictions of the corresponding $\Theta$'s to $W_x^r(P)$ and the $y$'s are the coordinate functions on $T_n^r(K)$.

*Remark* 3. We have established only a recurrent algorithm for finding the functions $\Phi^p_{i_1\dots i_k}$. Of course, it might be of some interest to derive the resulting formulae by this algorithm, but we do not intend to deduce them here.

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## DESPRE PRELUNGIRILE CÎMPURILOR DE OBIECTE GEOMETRICE

Rezumat

Scopul lucrării este de a descrie un algoritm natural pentru prelungirile cîmpurilor de obiecte geometrice. Acest algoritm este bazat pe lucrările lui Cartan, pe unele cercetări ale lui G. F. Laptev şi pe noţiunea de formă canonică definită pe prelungirea unui spaţiu fibrat principal.

# LE PLONGEMENT NATUREL ET NON HOLONOME DE L'ESPACE PROJECTIF SEMIANTIQUATERNIONIQUE

PAR

EFTIMIE GRECU

On considère le problème du plongement naturel et non holonome de l'espace projectif semiantiquaternionique à $n$ dimensions $P_n(e, \varepsilon)$ et on trouve sa métrique naturelle en montrant que cette métrique coïncide avec la métrique d'une variété non holonome $V^{4n}_{4n+3}$ située sur une hypersphère $S_{4n+3}$ de l'espace semieuclidien $E^{n+1,\,2n+2}_{4n+4}$ et définie par un système de trois équations pfaffiennes.

**Introduction.** Selon un résultat de M a n n o u r y [1], on sait que l'espace projectif complexe à $n$ dimensions $P_n(i)$ peut être considéré comme une variété d'un espace euclidien à $(n+1)^2$ dimensions. D'autre part [2] on sait que la métrique naturelle de $P_n(i)$ coïncide avec la métrique d'un espace non holonome $V^{2n}_{2n+1}$ situé sur une hypersphère $S_{2n+1}$ de l'espace euclidien $E_{2n+2}$. On obtient ainsi ce qu'on appelle ([3], p. 367) un plongement non holonome de l'espace $P_n(i)$, obtenu à l'aide d'une équation de Pfaff et d'une équation complètement intégrable.

Des propriétés analogues sont aussi valables pour l'espace projectif quaternionique $P_n(i, j)$. Dans ce cas, le plongement naturel a lieu dans un espace euclidien à $(n+1)(2n+1)$ dimensions et le plongement non holonome est obtenu à l'aide d'un système de quatre équations pfaffiennes dont une complètement intégrable, la métrique de l'espace $P_n(i, j)$ étant égale à la métrique d'une variété non holonome $V^{4n}_{4n+3}$ située sur une hypersphère $S_{4n+3}$ de l'espace euclidien $E_{4n+4}$ [4].

Le but de cet article est de voir de quelle manière ces considérations peuvent être élargies au cas de l'espace projectif sur l'algèbre des semiantiquaternions.