On peut donc énoncer le

Théorème. *La métrique naturelle de l'espace projectif semiantiquaternionique $P_n(e, \varepsilon)$ peut être considérée comme la métrique d'une varieté non holonome V^{4n}_{4n+3} définie sur l'hypersphère S_{4n+3} de l'espace semieuclidien $E^{n+1,2n+2}_{4n+4}$ par un système de trois équations pfaffiennes.*

On obtient donc ce qui s'appelle [2] un plongement non holonome de l'espace projectif semiantiquaternionique $P_n(e, \varepsilon)$ dans l'espace $E^{n+1,2n+2}_{4n+4}$, obtenu à l'aide du système de Pfaff (11) et de l'équation completement intégrable (12).

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SCUFUNDAREA NATURALĂ ȘI NEOLONOMĂ A SPAȚIULUI PROIECTIV SEMIANTICUATERNIONIC

Rezumat

Se consideră problema scufundării naturale și neolonome a spațiului proiectiv semianticuaternionic cu n dimensiuni $P_n(e, \varepsilon)$ și se găsește metrica sa naturală, arătîndu-se că această metrică coincide cu metrica unei varietăți neolonome V^{4n}_{4n+3}, situată pe o hipersferă S_{4n+3} din spațiul semieuclidian $E^{n+1,2n+2}_{4n+4}$ și definită de un sistem de trei ecuații Pfaff.

ON Φ - TRANSFORMATIONS

BY

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Communicated at the Jubilee Scientific Session of the Al. I. Cuza-University Iași 20–25 august 1970

§ 0. Introduction. S a s a k i [5, p. 21–2] proved that the set of all automorphisms of an almost cocomplex (or almost contact) compact manifold is a Lie group. On the other hand, T a n n o [5, p. 25–1] showed that this property is valid for the group of all Φ-transformations over a contact manifold. But it seems still open the problem of finding the conditions under what a subset of all Φ-transformations over an almost cocomplex compact manifold is a Lie group. The Lemma 2.2 and the condition (I) gives an answer to this problem by the same method of proof as Sasaki's. The other results in § 2 and § 3 are consequences of this fact.

S a s a k i [5, p. 23–2] proved that every $(2m+1)$-dimensional Lie group admits a left invariant almost cocomplex structure. Analogously, in § 4 we prove that a $(4n+3)$-dimensional Lie group admits a left invariant almost coquaternion structure.

Throughout this paper, all differentiable manifolds and mappings are understood to be of class C^∞.

§ 1. Almost cocomplex structures and almost coquaternion structures. 1. Let M^{2m+1} be a differentiable manifold of dimension $2m+1$. *A* (Φ, ξ, η) *-structure* or an *almost contact structure* or an *almost cocomplex structure* on M^{2m+1} is, by definition, a triple (Φ, ξ, η) where Φ is a tensor field of type (1, 1), ξ is a vector field and η is a 1-form on M^{2m+1} such that

$$\eta(\xi) = 1, \quad \Phi^2 = -\mathrm{id} + \xi \otimes \eta. \tag{1}$$

The relations (1) imply

$$\Phi\xi = 0, \quad \eta\circ\Phi = 0, \text{ rank } \Phi = 2m. \tag{2}$$

M^{2m+1} is said to be an *almost contact manifold* or an *almost cocomplex manifold.*

Every differentiable manifold with an almost cocomplex structure is orientable and has a positive definite Riemannian metric h such that

$$\begin{aligned} &\eta(X) = h(\xi, X), \\ &h(\Phi X, \Phi Y) = h(X, Y) - \eta(X)\eta(Y), \ \forall X, Y \in \mathscr{X}(M^{2m+1}). \end{aligned} \tag{3}$$

h is called an *associated Riemannian metric* to the given almost cocomplex structure, (Φ, ξ, η, h) is called an *almost cocomplex metric structure* on M^{2m+1} and M^{2m+1} is said to be an *almost cocomplex Riemannian manifold.*

The tensor defined by $h(\Phi X, Y)$, $\forall X, Y$, is a 2-form.

Remark. If (Φ, ξ, η, h) is an almost cocomplex metric structure, then $(\Phi, \xi/\alpha, \alpha\eta, \alpha h + (\alpha^2 - \alpha)\eta\otimes\eta)$, $\forall \alpha > 0$, $\alpha \in \mathscr{F}(M^{2m+1})$, is again an almost cocomplex metric structure on M^{2m+1}.

For other results on this subject, see [2], [5] and the references contained therein.

2. We consider $m = 2n + 1$. In the papers [6], [7], we defined the almost coquaternion structure on a $(4n + 3)$-dimensional manifold M^{4n+3} and we studied some properties of this structure.

An *almost coquaternion structure* on M^{4n+3} is given by three almost cocomplex structures $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, satisfying the following conditions

$$\begin{aligned} &\underset{a}{\Phi}\circ\underset{b}{\Phi} - \underset{a}{\xi}\otimes\underset{b}{\eta} = -\underset{b}{\Phi}\circ\underset{a}{\Phi} + \underset{b}{\xi}\otimes\underset{a}{\eta} = \underset{c}{\Phi}, \\ &\underset{a}{\Phi}\,\underset{b}{\xi} = -\underset{b}{\Phi}\,\underset{a}{\xi} = \underset{c}{\xi}, \\ &\underset{a}{\eta}\circ\underset{b}{\Phi} = -\underset{b}{\eta}\circ\underset{a}{\Phi} = \underset{c}{\eta}, \\ &\underset{a}{\eta}(\underset{b}{\xi}) = \underset{b}{\eta}(\underset{a}{\xi}) = 0, \end{aligned} \tag{4}$$

where $\{a, b, c\}$ is an even permutation of $\{1, 2, 3\}$.

Remark. To have an almost coquaternion structure, it suffices to give on M^{4n+3} two almost cocomplex structures $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2$, which satisfy

$$\begin{aligned} &\underset{1}{\Phi}\circ\underset{2}{\Phi} + \underset{2}{\Phi}\circ\underset{1}{\Phi} = \underset{1}{\xi}\otimes\underset{2}{\eta} + \underset{2}{\xi}\otimes\underset{1}{\eta}, \\ &\underset{1}{\eta}(\underset{2}{\xi}) = 0, \end{aligned} \tag{5}$$

since these relations imply $\underset{2}{\eta}(\underset{1}{\xi}) = 0$, $\underset{1}{\Phi}\,\underset{2}{\xi} = -\underset{2}{\Phi}\,\underset{1}{\xi}$, $\underset{1}{\eta}\circ\underset{2}{\Phi} = -\underset{2}{\eta}\circ\underset{1}{\Phi}$ and M^{4n+3} admits a third almost cocomplex structure defined by

$$\begin{aligned} &\underset{3}{\Phi} = \underset{1}{\Phi}\circ\underset{2}{\Phi} - \underset{1}{\xi}\otimes\underset{2}{\eta} = -\underset{2}{\Phi}\circ\underset{1}{\Phi} + \underset{2}{\xi}\otimes\underset{1}{\eta}, \\ &\underset{3}{\xi} = \underset{1}{\Phi}\,\underset{2}{\xi} = -\underset{2}{\Phi}\,\underset{1}{\xi}, \quad \underset{3}{\eta} = \underset{1}{\eta}\circ\underset{2}{\Phi} = -\underset{2}{\eta}\circ\underset{1}{\Phi} \end{aligned} \tag{6}$$

so that the relations (4) are valid [7].

M^{4n+3} is said to be an *almost coquaternion manifold.*

Every almost coquaternion manifold has a positive definite Riemannian metric g such that

$$\begin{aligned} &\underset{a}{\eta}(X) = g(\underset{a}{\xi}, X), \\ &g(\underset{a}{\Phi}X, \underset{a}{\Phi}Y) = g(X, Y) - \underset{a}{\eta}(X)\underset{a}{\eta}(Y), \ a = 1, 2, 3, \ \forall X, Y \in \mathscr{X}(M^{4n+3}). \end{aligned} \tag{7}$$

g is called an *associated Riemannian metric* to the given almost coquaternion structure, $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta}, g)$, $a = 1, 2, 3$, is called an *almost coquaternion metric, structure* and M^{4n+3} is said to be an *almost coquaternion Riemannian, manifold.*

The tensors defined by $g(\underset{a}{\Phi}X, Y)$, $\forall X, Y$, are 2-forms.

Remark. If $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta}, g)$ is an almost coquaternion metric structure, then $\forall \alpha > 0$, $\alpha \in \mathscr{F}(M^{4n+3})$, $\forall (C_b^a) \in SO(3)$, $a, b = 1, 2, 3$,

$$(c_b^a \underset{a}{\Phi}, 1/\alpha\, c_b^a \underset{a}{\xi}, \alpha c_b^a \underset{a}{\eta}, \alpha g + (\alpha^2 - \alpha)\sum_a \underset{a}{\eta}\otimes\underset{a}{\eta}),$$

is again an almost coquaternion metric structure on M^{4n+3}.

§ 2. **Φ- transformations.** 1. Let M^{2m+1} be an almost cocomplex manifold with the almost cocomplex structure (Φ, ξ, η) and f a transformation of M^{2m+1}, i. e., a diffeomorphism of M^{2m+1} onto itself.

We shall study those transformations of M^{2m+1} which leave Φ invariant.

Definition 2.1. *A transformation f of M^{2m+1} which leaves Φ invariant, i.e,*

$$f_* \circ \Phi = \Phi \circ f_* \tag{8}$$

is called a Φ-transformation [5, p. 241].

Lemma 2.2. *If f is a Φ-transformation of M^{2m+1}, then there exists a function $\rho : M^{2m+1} \to R$ such that :*

$$f_*\xi = \rho\xi, \quad f^*\eta = \rho\circ f\eta. \tag{9}$$

Proof. From (2) and

$$\Phi\circ f_*\xi = f_*\circ\Phi\xi$$

it follows that $\exists \rho : M^{2m+1} \to R$ such that $f_* \xi = \rho \xi$. Analogously, $\exists \beta : M^{2m+1} \to R$ *such that* $f^* \eta = \beta \eta$. On the other hand, from

$$(f^* \eta)_x (\xi_x) = \eta_{f(x)} (f_* \xi_x)$$

we get

$$\beta(x) \eta(\xi)_x = \rho(f(x)) \eta(\xi)_{f(x)}.$$

This relations and (1) imply

$$\beta(x) = \rho(f(x)).$$

2. Let X be a vector field on a manifold M. Generally, X generates a local 1-parameter group of local transformations. If it generates a global 1-parameter group of transformations of M, we say that X is a *complete vector field*. On a compact manifold, every vector field X is complete [3, p. 14].

When X is a complete vector field, elements of the 1-parameter group of transformations generated by X can be expressed as

$$\varphi_t = \exp tX, \; t \in R.$$

Let G be a certain group of transformations of a differentiable manifold M and S be the set of all complete vector fields X over M such that $\varphi_t \in G$. Then we have [4, p. 103]:

Theorem 2.3. *If S generates a finite dimensional Lie algebra, then G is a Lie transformation group and S is the Lie algebra of G.*

Let M be a compact manifold. We have [1].

Theorem 2.4. *If S is a space of vector fields X over M such that. $\forall x \in M$, there is a system of elliptic differential equations*

$$h^{jk}(x) \frac{\partial^2 X^i}{\partial x^j \partial x^k} + h^{ik}_j(x) \frac{\partial X^j}{\partial x^k} + h^i_j(x) X^j = h^i(x).$$

defined in a neighborhood of x satisfied by components X^i of X, then the dimension of S is finite.

3. Let M^{2m+1} be a manifold with the almost cocomplex structure (Φ, ξ, η) and $\mathfrak{F}(M^{2m+1}; \Phi)$ be the set of all Φ-transformations.

An *infinitesimal Φ-transformation* of an almost cocomplex structure (Φ, ξ, η) on M^{2m+1} is a vector field X such that (see Lemma 2.2. and [3, p. 33])

$$L_X \Phi = 0. \tag{10}$$

and consequently

$$L_X \eta = \sigma \eta, \tag{11}$$

where L_X denotes the Lie differentiation with respect to X and σ is a function defined over M^{2m+1} which depends upon X. From $(1)_1$ it follows

$$(L_X \eta)(\xi) + \eta(L_X \xi) = 0$$

and hence σ is given by $\sigma = -\eta(L_X \xi)$. Let S_0 be a subset of infinitesimal Φ-transformations such that there is a system of differential equations

$$\eta_k \xi^i \frac{\partial^2 X^k}{\partial x^i \partial x^j} = l^i_{kj}(x) \frac{\partial X^k}{\partial x^i} + l_{jk}(x) X^k + l_j(x), \tag{I}$$

defined in a neighborhood of $x \in M^{2m+1}$ and satisfied by components X^i of X. The subset of $\mathcal{T}(M^{2m+1}; \Phi)$ which correspond to S_0 will be denoted by $\mathcal{T}_0(M^{2m+1}; \Phi)$.

Theorem 2.5. *If M^{2m+1} is compact, then $\mathcal{T}_0(M^{2m+1}; \Phi)$ is a Lie group of transformations.*

Proof. ([5]) In a coordinte neighborhood, we have,

$$\frac{\partial \Phi^i_j}{\partial x^h} X^h - \frac{\partial X^i}{\partial x^h} \Phi^h_j + \frac{\partial X^h}{\partial x^j} \Phi^i_h = 0, \tag{10'}$$

$$\frac{\partial \eta_j}{\partial x^h} X^h + \frac{\partial X^h}{\partial x^j} \eta_h = \sigma \eta_j. \tag{11'}$$

Differentiating (10') and (11') with respect to x^k, and using (1), we get

$$-\frac{\partial^2 X^i}{\partial x^h \partial x^k} \Phi^h_j + \frac{\partial^2 X^h}{\partial x^j \partial x^k} \Phi^i_h + \ldots = 0, \tag{12}$$

$$\frac{\partial^2 X^h}{\partial x^j \partial x^k} \eta_h + \ldots + l_k \eta_j = 0, \tag{13}$$

where the unwritten terms are linear with respect to X^i and $\dfrac{\partial X^i}{\partial x^j}$.

Let h be an associated Riemannian metric to the almost cocomplex structure (Φ, ξ, η). Then $\Phi^{ik} = \Phi^k_j h^{ij}$ is skew symmetric with respect to i and k.

If we contract the relations (12) and (13) with h^{jk}, then we find respectively

$$h^{jk} \frac{\partial^2 X^h}{\partial x^j \partial x^k} \Phi^i_h + \ldots = 0,$$

$$h^{jk} \frac{\partial^2 X^h}{\partial x^j \partial x^k} \eta_h + \ldots + l_k \xi^k = 0.$$

The last relations imply

$$h^{jk} \frac{\partial^2 X^h}{\partial x^j \partial x^k} (\Phi^i_h + \xi^i \eta_h) + \ldots + l_k \xi^k \xi^i = 0.$$

from which, contracting with $-\Phi^l_i + \xi^l \eta_i$, we get

$$h^{jk}\frac{\partial^2 X^h}{\partial x^j \partial x^k} + \ldots + l_k \xi^k \xi^h = 0.$$

This shows that each infinitesimal Φ-transformation $X \in S_0$ satisfies a system of elliptic differential equations. By Theorem 2.4 we conclude that S_0 is a finite dimensional Lie algebra.

On the other hand, M^{2m+1} is compact by hypothesis and hence each element $X \in S_0$ is complete. By Theorem 2.3 the group of Φ-transformations $\mathcal{T}_0(M^{2m+1}; \Phi)$ is a Lie group and the Lie algebra of infinitesimal Φ-transformations S_0 is the Lie algebra of $\mathcal{T}_0(M^{2m+1}; \Phi)$.

4. Let M^{4n+3} be an almost coquaternion manifold with the structure tensors $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$.

Definition 2.6. *A transformation f of M^{4n+3} which leaves $\underset{1}{\Phi}$ and $\underset{2}{\Phi}$ invariant, i. e.,*

$$f_* \circ \underset{a}{\Phi} = \underset{a}{\Phi} \circ f_*, \quad a = 1, 2,$$

is called a $(\underset{1}{\Phi}, \underset{2}{\Phi})$-transformation.

Lemma 2.7. *If f is a $(\underset{1}{\Phi}, \underset{2}{\Phi})$-transformation of M^{4n+3}, then $f_* \circ \underset{3}{\Phi} = \underset{3}{\Phi} \circ f_*$ and there exists a function $\rho : M^{4n+3} \to R$ such that*

$$f_* \underset{a}{\xi} = \rho \underset{a}{\xi}, \quad f^* \underset{a}{\eta} = \rho \circ f \underset{a}{\eta}, \quad a = 1, 2, 3.$$

Proof. From

$$\underset{a}{\Phi} \circ f_* \underset{a}{\xi} = f_* \circ \underset{a}{\Phi} \underset{a}{\xi}, \quad a = 1, 2,$$

it follows that $\exists \underset{a}{\alpha} : M^{4n+3} \to R$ such that

$$f_* \underset{a}{\xi} = \underset{a}{\alpha} \underset{a}{\xi}.$$

From

$$\underset{1}{\Phi}\underset{2}{\xi} = -\underset{2}{\Phi}\underset{1}{\xi} = \underset{3}{\xi},$$

we get

$$f_* \underset{1}{\Phi} \underset{2}{\xi} = -f_* \underset{2}{\Phi} \underset{1}{\xi} = f_* \underset{3}{\xi} \Rightarrow \underset{1}{\Phi} f_* \underset{2}{\xi} = -\underset{2}{\Phi} f_* \underset{1}{\xi} = f_* \underset{3}{\xi}.$$

$$\Rightarrow \underset{1}{\Phi}\underset{2}{\alpha}\underset{2}{\xi} = -\underset{2}{\Phi}\underset{1}{\alpha}\underset{1}{\xi} = f_* \underset{3}{\xi} \Rightarrow \underset{1}{\alpha} = \underset{2}{\alpha} = \rho \text{ and } f_* \underset{3}{\xi} = \rho \underset{3}{\xi}.$$

Analogously, $\exists \beta : M^{4n+3} \to R$ such that

$$f^* \underset{a}{\eta} = \beta \underset{a}{\eta}, \quad a = 1, 2, 3.$$

On the other hand, from

$$(f^* \underset{a}{\eta})_x (\underset{a}{\xi})_x = \underset{a}{\eta}_{f(x)} (f_* \underset{a}{\xi}_x),$$

we get

$$\beta(x) \underset{a}{\eta}(\underset{a}{\xi})\, x = \rho(f(x)) \underset{a}{\eta}(\underset{a}{\xi})_{f(x)},$$

and hence (see § 1) $\beta = \rho \circ f$. The relation $f_* \circ \underset{3}{\Phi} = \underset{3}{\Phi} \circ f_*$ follows from (6).

Let $\mathcal{T}_0(M^{4n+3} : \underset{1}{\Phi}, \underset{2}{\Phi})$ be the subset of all $(\underset{1}{\Phi}, \underset{2}{\Phi})$-transformations given by the condition (1). We have

$$\mathcal{T}_0(M^{4n+3} ; \underset{1}{\Phi}, \underset{2}{\Phi}) \subset \mathcal{T}_0(M^{4n+3} ; \underset{1}{\Phi})$$

and hence

Corollary 2.8. *If M^{4n+3} is a compact manifold with the almost coquaternion structure $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, then $\mathcal{T}_0(M^{4n+3} ; \underset{1}{\Phi}, \underset{2}{\Phi})$ is a Lie group of transformations.*

More precisely, $\mathcal{T}_0(M^{4n+3} ; \underset{1}{\Phi}, \underset{2}{\Phi})$ is a Lie subgroup of the Lie group $\mathcal{T}_0(M^{4n+3} ; \underset{1}{\Phi})$.

§3. Automorphisms. 1. Let M^{2m+1} be an almost cocomplex manifold with the almost cocomplex structure (Φ, ξ, η).

Definition 3.1. *A transformation f of M^{2m+1} which leaves the structure (Φ, ξ, η) invariant,* i. e.,

$$f_* \circ \Phi = \Phi \circ f_*, \; f_* \xi = \xi (\Rightarrow f^* \eta = \eta),$$

is called an automorphism of M^{2m+1} [5, p. 21—1].

Let $\mathcal{A}(M^{2m+1} ; \Phi, \xi, \eta)$ be the set of all automorphisms of M^{2m+1}; We have

$$\mathcal{A}(M^{2m+1} ; \Phi, \xi, \eta) \subset \mathcal{T}_0(M^{2m+1} ; \Phi),$$

and hence

Corollary 3.2. *If M^{2m+1} is a compact manifold with the almost cocomplex structure (Φ, ξ, η), then $\mathcal{A}(M^{2m+1} ; \Phi, \xi, \eta)$ is a Lie group of transformations* [5, p. 21—2].

More precisely, $\mathcal{A}(M^{2m+1} ; \Phi, \xi, \eta)$ is a Lie subgroup of the Lie group $\mathcal{T}_0(M^{2m+1} ; \Phi)$.

2. Let M^{4n+3} be an almost coquaternion manifold with the almost coquaternion structure $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$.

Definition 3.3. *A transformation f of M^{4n+3} which leaves the structure $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$ invariant, i. e.,*

$$f_* \circ \underset{1}{\Phi} = \underset{1}{\Phi} \circ f_*, \; f_* \circ \underset{2}{\Phi} = \underset{2}{\Phi} \circ f_*, \; f_* \underset{1}{\xi} = \underset{1}{\xi},$$

$$(\Rightarrow f_* \circ \underset{3}{\Phi} = \underset{3}{\Phi} \circ f_*, \; f_* \underset{2}{\xi} = \underset{2}{\xi}, \; f_* \underset{3}{\xi} = \underset{3}{\xi}, \; f^* \underset{a}{\eta} = \underset{a}{\eta}, \; a = 1, 2, 3),$$

is called an automorphism of M^{4n+3}.

Let $\mathcal{A}(M^{4n+3}; \underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$ be the set of all automorphisms of M^{4n+3}. We have

$$\mathcal{A}(M^{4n+3}; \underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta}) \subset \mathcal{A}(M^{4n+3}; \underset{1}{\Phi}, \underset{1}{\xi}, \underset{1}{\eta})$$

and hence

Corollary 3.4. *If* M^{4n+3} *is a compact manifold with the almost coquaternion structure* $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, *then* $\mathcal{A}(M^{4n+3}; \underset{a}{\Phi}, \underset{a}{\zeta}, \underset{a}{\eta})$ *is a Lie group of transformations.*

More precisely, $\mathcal{A}(M^{4n+3}; \underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$ is a Lie subgroup of the Lie group $\mathcal{A}(M^{4n+3}; \underset{1}{\Phi}, \underset{1}{\xi}, \underset{1}{\eta})$.

Remark. On a 3-dimensional manifold M^3 the almost coquaternion structure is equivalent with an absolute parallelism [7] and $\forall f \in \mathcal{A}(M^3; \underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, f is an automorphism of the absolute parallelism. Kobayashi [3, Note 9] proved that the group of all automorphisms of an absolute parallelism is a Lie group.

§4. **The almost coquaternion structure of the $(4n+3)$-dimensional Lie groups.** In [5, p. 23—2] it is proved that a $(2n+1)$-dimensional Lie group admits a left invariant almost cocomplex structure. Analogously, we shall prove that a $(4n+3)$-dimensional Lie group admits a left invariant almost coquaternion structure.

Let G be a $(4n+3)$-dimensional Lie group and $\mathfrak{G}$ be the Lie algebra of G regarded as the set of all left invariant vector fields on G with the usual addition, scalar multiplication and bracket operation. As a vector space, $\mathfrak{G}$ is isomorphic with the tangent space $T_e G$ at the identity. Let $\mathfrak{G}^*$ be the dual space of the Lie algebra $\mathfrak{G}$ regarded as the set of all left invariant 1-forms on G. If $X \in \mathfrak{G}$ and $\omega \in \mathfrak{G}^*$, then the function $\omega(X)$ is constant on G [3, p. 41].

Let $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, be an almost coquaternion structure on G.

Definition 4.1. *An almost coquaternion structure* $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$ *on G is called left invariant if all left translations of G are automorphisms of* $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, i.e.,

$$\underset{a}{\Phi} \circ L_t = L_t \circ \underset{a}{\Phi}, \; L_t \underset{a}{\xi} = \underset{a}{\xi}, \; L_t^* \underset{a}{\eta} = \underset{a}{\eta}, \; \forall t \in G. \tag{14}$$

Let $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, be a left invariant almost coquaternion structure on G. $\forall X \in \mathfrak{G}$ and $\forall t \in G$, we have

$$L_t \underset{a}{\Phi} X = \underset{a}{\Phi} L_t X = \underset{a}{\Phi} X, \tag{15}$$

i.e., $\underset{a}{\Phi} X \in \mathfrak{G}$. Let $\underset{a}{\Phi_0}$ be the restriction of $\underset{a}{\Phi}$ to $T_e G$. From (15) it follows that $\underset{a}{\Phi_0}$ maps $T_e G$ into $T_e G$. By (14) $\underset{a}{\xi}$ are left invariant vector fields. We denote their values at e by $\underset{a}{\xi_0}$. By (14), we have $\underset{a}{\eta} \in \mathfrak{G}^*$. $\forall X \in \mathfrak{G}$, the constants $\underset{a}{\eta}(X)$ can be determined on $T_e G$ by using the restrictions $\underset{a}{\eta_0}$ of $\underset{a}{\eta}$ to $T_e G$. It is clear that $(\underset{a}{\Phi_0}, \underset{a}{\xi_0}, \underset{a}{\eta_0})$ satisfies the relations (4).

We may identify $T_e G$ with $\mathfrak{G}$ and hence $\mathfrak{G}$ admits an almost coquaternion structure $(\underset{a}{\Phi_0}, \underset{a}{\xi_0}, \underset{a}{\eta_0})$. We now prove the converse

Theorem 4.2. *Every $(4n+3)$-dimensional Lie group G admits a left invariant almost coquaternion structure.*

Proof. Let $\mathfrak{G}$ be the Lie algebra of G. $\mathfrak{G}$ has the dimension $4n+3$ and hence it admits an almost coquaternion structure $(\underset{a}{\Phi_0}, \underset{a}{\xi_0}, \underset{a}{\eta})$, $a = 1, 2, 3$. Let X_ε, $\varepsilon = 1, 2, \dots, 4n+3$, be left invariant vector fields which determine a basis of $\mathfrak{G}$. Then we can construct the constant functions over G, $\underset{a}{\eta_{0\varepsilon}} = \underset{a}{\eta_0}(X_\varepsilon) = \underset{a}{\eta_0}((X_\varepsilon)_e)$ and the left invariant vector fields $\underset{a}{\Phi_0} X_\varepsilon$ whose values at e are $\underset{a}{\Phi_0}(X_\varepsilon)_e$. Every vector field X can be uniquely expressed by $X = X^\varepsilon X_\varepsilon$, where X^ε are $4n+3$ functions over G.

Let X be an arbitrary left invariant vector field. Then X_ε must be constant functions and the relations

$$\underset{a}{\Phi} X = X^\varepsilon \underset{a}{\Phi_0} X_\varepsilon, \; \underset{a}{\eta}(X) = X^\varepsilon \underset{a}{\eta_0}(X_\varepsilon)$$

define the left invariant tensor fields $\underset{a}{\Phi}$, $\underset{a}{\eta}$ over G. Defining $\underset{a}{\xi} = \underset{a}{\xi^\varepsilon} X_\varepsilon$ as the left invariant vector fields such that $\underset{a}{\xi_e} = \underset{a}{\xi_0}$, we only have to show that $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$ has the required properties.

The relations

$$\underset{a}{\eta_0}(\underset{b}{\xi_0}) = \delta_{ab}, \quad a, b = 1, 2, 3,$$

imply

$$\underset{a}{\eta}_{0\varepsilon}\underset{b}{\xi}^{\varepsilon} = \delta_{ab}.$$

This being said, we have

$$\underset{a}{\eta}(\underset{b}{\xi}) = \underset{b}{\xi}^{\varepsilon}\underset{a}{\eta}_0(X_\varepsilon) = \underset{a}{\eta}_{0\varepsilon}\underset{b}{\xi}^{\varepsilon} = \delta_{ab}.$$

On the other hand, we get

$$\underset{a}{\Phi}^2 X = \underset{a}{\Phi}(X^\varepsilon \underset{a}{\Phi}_0 X_\varepsilon) = X^\varepsilon \underset{a}{\Phi}{}_0^2 X_\varepsilon = X^\varepsilon(-X_\varepsilon + \underset{a}{\eta}_0(X_\varepsilon)\underset{a}{\xi}_0) = -X + \underset{a}{\eta}(X)\underset{a}{\xi}.$$

$$\underset{a}{\Phi}\underset{b}{\Phi}X - \underset{b}{\eta}(X)\underset{a}{\xi} = \underset{a}{\Phi}(X^\varepsilon \underset{b}{\Phi}_0 X_\varepsilon) - X^\varepsilon \underset{b}{\eta}_0(X_\varepsilon)\underset{a}{\xi} =$$

$$= X^\varepsilon(\underset{a}{\Phi}_0\underset{b}{\Phi}_0 X_\varepsilon) - X^\varepsilon \underset{b}{\eta}_0(X_\varepsilon)\underset{a}{\xi} = X^\varepsilon(\underset{a}{\Phi}_0\underset{b}{\Phi}_0 - \underset{a}{\xi}_0 \otimes \underset{b}{\eta}_0)(X_\varepsilon) = \underset{c}{\Phi}X,$$

$$-\underset{b}{\Phi}\underset{a}{\Phi}X + \underset{a}{\eta}(X)\underset{b}{\xi} = \underset{c}{\Phi}X,$$

for any cyclic permutation $\{a, b, c\}$ of $\{1, 2, 3\}$.
From the construction $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, is left invariant.

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ASUPRA Φ-TRANSFORMĂRILOR

Rezumat

In §1 se dau unele definiţii şi rezultate referitoare la structurile aproape cocomplexă şi respectiv aproape cocuaternionică. În § 2 se stabileşte că o anumită submulţime a tuturor Φ-transformărilor peste o varietate compactă aproape cocomplexă M^{2m+1} este un grup Lie şi deci o anumită submulţime a tuturor $(\underset{1}{\Phi}, \underset{2}{\Phi})$-transformărilor peste o varietate compactă aproape cocuaternionică M^{4n+3} este un grup Lie. Prin urmare, automorfismele lui M^{2m+1} [5, p. 21—2] şi ale lui $M^4{}_n{}^{+3}$ formează respectiv grupuri Lie (§3). În §4 se arată că un grup Lie cu $4n + 3$ dimensiuni admite o structură aproape cocuaternionică invariantă la stînga.

QUELQUES REMARQUES SUR L'ESPACE FIBRÉ TANGENT D'UNE VARIÉTÉ COSYMPLECTIQUE

PAR

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Communication présentée à la session jubilaire „A. Myller" Août 20—25, 1970

Les structures cosymplectiques ont été considérées par P. Libermann [7], A. Lichnerowicz [8], G. Reeb [11] et C. Barros [1]. Au sens des structures presque de contact, elles ont été étudiées par D. Blair [2], S. Goldberg et K. Yano [5], K. Ogiue [10] et d'autres. Les connexions compatibles aux structures cosymplectiques ont été déterminées par R. Miron et V. Oproiu [9].

1. Soit une variété différentiables [1]) a $2n + 1$ dimensions. Si le groupe structurel de l'espace fibré tangent TM de la variété M est réductible au groupe $U(n)\times 1$ on dit que la variété M est presque de contact (au sens de Sasaki [13]). Sur une variété presque de contact il existe un champ de tenseurs φ de type (1, 1), un champ de vecteurs ξ et une 1-forme η tels que

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{1.1}$$
$$\varphi\xi = 0, \qquad \eta\varphi = 0.$$

Nous pouvons trouver d'une manière naturelle une métrique de Riemann sur M telle que

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\,\eta(Y), \qquad X, Y \in \mathfrak{X}(M), \tag{1.2}$$

où $\mathfrak{X}(M)$ est le module des champs de vecteurs de la variété M. Alors, on

[1]) Les variétés différentiables envisagées seront toujours supposées de classe C^∞, ainsi que les tenseurs introduits sur elles.