ON THE MATRIX EQUATION $\mathbf{PAP}^T = \mathbf{B}$ IN THE TWO-ELEMENT BOOLEAN ALGEBRA AND SOME APPLICATIONS IN GRAPH THEORY

BY

CORINA REISCHER and DAN A. SIMOVICI

Introduction. Let $\mathcal{B}_2 = \{0, 1, \cup, \cdot, ^-\}$ the two-element Boolean algebra, $\mathcal{B}_2^n = \{\mathbf{A} \mid \mathbf{A} = (a_{ij})_{n \times n}$, with $a_{ij} \in \mathcal{B}_2\}$ the set of square Boolean, matrices whose entries belong to $\mathcal{B}_2$. It is clear that $|\mathcal{B}_2^n|^* = 2^{n^2}$. Wet shall denote by $\mathbf{I}$, $\mathbf{O}$, $\mathbf{E}$ the matrices having the elements: $i_{hk} = 1$, $o_{hk} = 0$ $e_{hk} = \delta_{hk}$ respectively, for every h and k. δ_{hk} is the well known Kronecker's symbol. We shall consider the Boolean matricial product. If $\mathbf{A}, \mathbf{B} \in \mathcal{B}_2^n$, the product of the matrices $\mathbf{A}$, $\mathbf{B}$ will be the matrix $\mathbf{C}$ with:

$$c_{hk} = \bigcup_{l=1}^{n} a_{hl} b_{lk}. \tag{1}$$

Let us denote by S_n the symmetrical group of order n. It is known tha $|S_n| = n!$.

Definition 1. $\mathbf{P} \in \mathcal{B}_2^n$ *will be a permutation matrix if it is obtained from* $\mathbf{E}$ *through a permutation of the rows.*

It is clear that $\mathbf{P}$ will have only one unit on whichever of its rows or columns.

In this Note we solve the equation $\mathbf{PAP}^T = \mathbf{B}$, with the unknown matrix $\mathbf{P}$, where $\mathbf{A}, \mathbf{B} \in \mathcal{B}_2^n$ and we give some applications into the graph theory. Namely we succed to solve through a Boolean approach an im-

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*) We denoted by $|M|$ the cardinality of the set M.

portant combinatorial problem: to answer either two graphs (whose incidence matrices are $\mathbf{A}$ and $\mathbf{B}$, respectively) are isomorphic or not.

§1. **The set of permutation matrices.** Let P be the set of permutation matrices, P is a finite group. Indeed, if $\mathbf{P}, \mathbf{Q} \in P$ then it follows immediately that $\mathbf{P}\mathbf{Q} \in P$, $\mathbf{P}^{-1}$ is the transposed matrix $\mathbf{P}^{T}$ a. s. o.

There exists an one-to one correspondence between P and S_n. Let $\mathbf{P} \in P$; if $\mathbf{P}$ has the unit of the j-th row in the i_j column $(j = \overline{1, n})$ we form the substitution S related to $\mathbf{P}$, namely:

$$S = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}.$$

Due to def. 1, S is indeed a substitution because $(i_1, \dots, i_n)$ is really a permutation of the numbers $1, 2, \dots, n$.

Conversely, let $S = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$ a substitution, its permutation matrix will be $\mathbf{P}_S = (p_{hk})$ where:

$$p_{hk} = \delta_{i_h k}$$

for $h, k \in (1, 2, \dots, n)$.

$\mathbf{P}$ is a permutation matrix because there is possible to obtain it by changing the rows of the matrix $\mathbf{E}$ in the manner indicated by the substitution.

Now, it follows immediately:

Lemma 1. *P and S_n are two isomorphic groups.*

Proof. We have to prove that the correspondence which was established earlier preserves the operations.

Let S, $\mathbf{P}_S$, be two related elements belonging to S_n and P respectively. Let Q be the substitution related to $\mathbf{P}_{S^{-1}} = \mathbf{P}^{T}_{S}$. We have

$$Q = \begin{pmatrix} i_1 & i_2 & \dots & i_n \\ 1 & 2 & \dots & n \end{pmatrix}.$$

Hence, it is clear that $Q = S^{-1}$ and $\mathbf{P}_S^{-1} = \mathbf{P}_{S^{-1}}$.
Moreover, it can be seen immediately that

$$\mathbf{P}_S \mathbf{P}_Q = \mathbf{P}_{SQ}.$$

We note that the matrix $\mathbf{P}_S\mathbf{A}$ ($\mathbf{P}_S \in P$, $\mathbf{A} \in \mathcal{B}_2^n$) is obtained from $\mathbf{A}$ by rearranging the rows in accordance with the substitution S, while the matrix $\mathbf{AP}_S$ is obtained through the substitution S^{-1} effected on the columns of the matrix $\mathbf{A}$.

§2. **The equation $\mathbf{PAP}^{T} = \mathbf{A}$.** Let $\mathbf{A} \in \mathcal{B}_2^n$ and

$$M_A = \{\mathbf{P} \mid \mathbf{P} \in P,\ \mathbf{PAP}^{T} = \mathbf{A}\}.$$

Theorem 1. *M_A is a subgroup of the permutation group P.*

Proof. Firstly, we note that $M_A \neq \emptyset$, because $\mathbf{E} \in M_A$. Let $\mathbf{P} \in M_A$. It follows immediately that $\mathbf{P}^{-1} = \mathbf{P}^{T} \in M_A$. Aftewards, if $\mathbf{P}, \mathbf{Q} \in M_A$, from $\mathbf{PAP}^{T} = \mathbf{QAQ}^{T} = \mathbf{A}$ we can obtain $(\mathbf{PQ})\,\mathbf{A}\,(\mathbf{PQ})^{T} = \mathbf{P}\,(\mathbf{QAQ}^{T})\,\mathbf{P}^{T} = \mathbf{A}$. Hence $M_A M_A \subseteq M_A$, which we had to prove.

Corollary. Let $\mathbf{H} \in P - M_A$ and the left coset $\mathbf{H}M_A$ of the element $\mathbf{H}$. Every $\mathbf{Q} \in \mathbf{H}M_A$ carries the matrix $\mathbf{A}$ into the same matrix.

Indeed, $\mathbf{QAQ}^{T} = \mathbf{HPA}\,(\mathbf{HP})^{T} = \mathbf{HPAP}^{T}\mathbf{H}^{T} = \mathbf{HAH}^{T}$, where $\mathbf{P} \in M_A$.

Corollary. There exists $\left|\dfrac{P}{M_A}\right|$ distinct matrices $\mathbf{B}$, for which the equation $\mathbf{PAP}^{T} = \mathbf{B}$ has a solution.

Corollary. Let $M_{AB} = \{\mathbf{P} \mid \mathbf{P} \in P,\ \mathbf{PAP}^{T} = \mathbf{B}\}$. Then $|M_{AB}| = |M_A|$, because every left coset has the same number of elements as the subgroup.

Moreover, we have:

$$|M_{AB}| = |M_A| = |M_B|.$$

Corollary. Let $C_A = \{\mathbf{B} \mid \mathbf{B} \in \mathcal{B}_2^n,\ \exists \mathbf{P},\ \mathbf{PAP}^{T} = \mathbf{B}\}$. Then $C_A = \left|\dfrac{P}{M_A}\right|$.

If $|M_A| < n!$ there exists some matrices $\mathbf{B} \neq \mathbf{A}$ with $\mathbf{B} \in C_A$.

§3. **The equation $\mathbf{PAP}^{T} = \mathbf{B}$.** Let us solve the equation

$$\mathbf{PAP}^{T} = \mathbf{B}, \tag{2}$$

where $\mathbf{A}, \mathbf{B} \in \mathcal{B}_2^n$, $\mathbf{P} \in P$ where $\mathbf{P}$ is unknown.

We shall give a necessary condition of compatibility.

Theorem 2. *If there exists a solution of the equation (2) we have:*

α) $$|M_A| = |M_B|;$$

β) *let us consider the vectors:*

$$\mathbf{l}_A = (l_1^A, \dots, l_n^A), \text{ the out-degree-vector,}$$

$$\mathbf{c}_A = (c_1^A, \dots, c_n^A), \text{ the in-degree-vector,}$$

with

$$l_i^A = \sum_{j=1}^{n} a_{ij},\ c_j^A = \sum_{i=1}^{n} a_{ij}. \tag{3}$$

We must have the equality of the sets of components both for $\mathbf{l}_A$ and $\mathbf{c}_A$, namely:

$$\{l_1^A, \dots, l_n^A\} = \{l_1^B, \dots, l_n^B\},\ \{c_1^A, \dots, c_n^A\} = \{c_1^B, \dots, c_n^B\}. \tag{4}$$

Here, the brackets indicate that the equalities (4) are interpreted as set-equalities (and not vector-equalities).

From the previous considerations it is clear that, if $\mathbf{H}$ is a particular solution of the equations (2), then either $\mathbf{H}M_{A}$ or $M_{B}\mathbf{H}$ gives us the whole set of solutions of the equation (2).

Hence, to solve the equation (2) comes to find out a particular solution of this equation, and to solve completely the equation $\mathbf{PAP}^{T} = \mathbf{A}$ or $\mathbf{PBP}^{T} = \mathbf{B}$.

§ 4. An algorithm which gives a particular solution of the equation $\mathbf{PAP}^{T} = \mathbf{B}$. Further, we suppose that the conditions which are included into theorem 2 are fulfiled and we shall consider the matrices $\mathbf{A}$ and $\mathbf{B}$ as being adjacency matrices of two graphs.

Let us consider two graphs $\mathcal{A} = (A, \Gamma_{A})$ and $\mathcal{B} = (B, \Gamma_{B})$, with $|A| = |B| = n$.

We shall introduce a graph related to the graphs $\mathcal{A}$ and $\mathcal{B}$, *„the permutational product“*, denoted by $\mathcal{T} = \mathcal{A} \square \mathcal{B}$ and constructed as follows:

Let $T = \{t_1, \ldots, t_n\}$, $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$.

$$\Gamma_{T} t_i = \{t_j \mid j \in \mathcal{J}_i\}, \text{ where} \tag{5}$$

$$\mathcal{J}_i = \{j \mid \text{od } b_j = \text{od } a_i\} \cap \{j \mid \text{id } b_j = \text{id } a_i\}, \tag{6}$$

where od b and in b mean respectively the out-degree and the in-degree of the vertex b.

Example. Let $\mathcal{A} = (A, \Gamma_{A}), \mathcal{B} = (B, \Gamma_{B})$ two graphs given by their adjacency matrices:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The out-degree and the in-degree vectors are respectively:

$$\mathbf{l}_{A} = (2, 2, 2, 3, 2), \quad \mathbf{l}_{B} = (3, 2, 2, 2, 2),$$

$$\mathbf{c}_{A} = (3, 2, 2, 1, 3), \quad \mathbf{c}_{B} = (1, 3, 3, 2, 2).$$

Regarding the permutational product $\mathcal{A} \square \mathcal{B} = (T, \Gamma_{T})$, we have:

$$\Gamma_{T} t_1 = \{t_2, t_3, t_4, t_5\} \cap \{t_2, t_3\} = \{t_2, t_3\},$$

$$\Gamma_{T} t_2 = \{t_2, t_3, t_4, t_5\} \cap \{t_4, t_5\} = \{t_4, t_5\},$$

$$\Gamma_{T} t_3 = \{t_2, t_3, t_4, t_5\} \cap \{t_4, t_5\} = \{t_4, t_5\},$$

$$\Gamma_{T} t_4 = \{t_1\} \cap \{t_1, t_5\} = \{t_1\},$$

$$\Gamma_{T} t_5 = \{t_2, t_3, t_4, t_5\} \cap \{t_2, t_3\} = \{t_2, t_3\}.$$

This graph is described in Fig. 1.

To every factor of a graph (G, Γ_{G}) we can adjoin a permutation matrix. Indeed, let us consider the vertex $g_i \in G$.

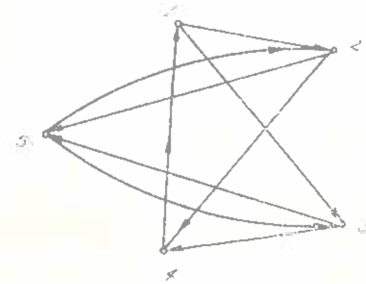

Fig. 1

Due to the definition of the factor, g_i belongs to exactly one elementary circuit C. It follows immediately that the set $\Gamma g_i \cap C$ contains only one element, say g_{k_i}. Then the permutation matrix related to the factor is defined by:

$$\mathbf{K} = (k_{ij}) = (\delta_{jk_i}).$$

We have:

Theorem 3. *The solutions of the equation (2), if any, are among the matrices related to the factors of the permutational product $\mathcal{A} \square \mathcal{B}$.*

Proof. Let $\mathbf{P}$ a solution of the equation $\mathbf{PAP}^{T} = \mathbf{B}$. The permutation matrix $\mathbf{P}$ is related to a substitution in the manner which was indicated in the first paragraph.

By performing the substitution S under the set of vertices of the graph $\mathcal{A}$ the vertex g_i is carried out in another vertex g_j.

There are many possibilities for this transport which are summarized by the permutational product of the graphs $\mathcal{A}$ and $\mathcal{B}$. By renumbering the vertices, the out-degree and the in-degree of a certain vertex remain unchanged, and this justifies the previous statement.

Let us consider a renumbering of the vertices described by a permutation matrix $\mathbf{P}$. We can obtain immediately a circuit of the graph $\mathcal{A} \square \mathcal{B}$ by decomposing the substitution S into disjoint parts.

Every cycle contained by the substitution S gives one of the elementary circuits which enter into the factor.

For us, it suffices to find out only one factor, whose related matrix is a particular solution of the equation (2).

We shall give also, a direct, combinatorial approach of the same problem. Let us suppose that the 2-nd condition from Theorem 2 is fulfiled and let $l_i^{\mathbf{A}} = l_j^{\mathbf{B}}$. We shall change the order of the columns in the matrix $\mathbf{A}$ in order to obtain in the i-th row of this matrix the j-th row of the matrix $\mathbf{B}$ by the substitution S. In this way, we obtain a new matrix $\mathbf{A}_1$. If $c_{\mathbf{B}} = c_{\mathbf{A}_1}$ we shall perform the same permutation over the rows of matrix $\mathbf{A}_1$ and, if the result is equal to $\mathbf{B}$, then the permutation matrix related to the substitution S^{-1} will be a solution.

If $\mathbf{c_B} \neq \mathbf{c_{A_1}}$ we shall try with another pair of components $(l_r^{\mathbf{A}}, l_s^{\mathbf{B}})$.

§5. The solutions of the equation $\mathbf{PAP^T = A}$. On the set $N = \{1, \ldots, n\}$ we shall introduce a binary relation „$\sim$" as follows $i \sim j$, iff it holds:

$$a_{ik} = a_{jk}, \quad a_{ki} = a_{kj}, \text{ for every } k \in N - \{i, j\}, \tag{7}$$

and

$$a_{ii} = a_{jj}, \quad a_{ij} = a_{ji}. \tag{8}$$

It is obvious that the relation „$\sim$" is symmetrical and reflexive. We have:

Lemma 2. *The relation „$\sim$" is an equivalence.*

We have to prove only the transitivity of „$\sim$". Let $i \sim j$, $j \sim s$, $i < j < s$. In accordance with the definition of the relation „$\sim$" we have:

$$\begin{aligned} &a_{ik} = a_{jk}, \quad a_{ki} = a_{kj}, \quad \text{for } k \in N - \{i, j\}, \\ &a_{jh} = a_{sh}, \quad a_{hj} = a_{hs}, \quad \text{for } h \in N - \{j, s\}, \\ &a_{ii} = a_{jj}, \quad a_{ij} = a_{ji}, \\ &a_{jj} = a_{ss}, \quad a_{js} = a_{sj}. \end{aligned} \tag{9}$$

It follows immediately that:

$$a_{ik} = a_{sk}, \quad a_{ki} = a_{ks} \text{ for } k \in N - \{i, j, s\}.$$

From $i \sim j$ we obtain

$$a_{is} = a_{js}, \quad a_{si} = a_{sj}, \quad a_{ij} = a_{ji}$$

and from $j \sim s$:

$$a_{ij} = a_{is}, \quad a_{ji} = a_{si} \quad a_{js} = a_{sj}.$$

Hence

$$a_{is} = a_{js} = a_{sj} = a_{si}, \quad a_{ij} = a_{ji} = a_{si} = a_{sj},$$

$$a_{ji} = a_{si} = a_{sj} = a_{js},$$

and we can conclude $i \sim s$.

Remark. We note that the existence of a pair $i \sim j$ implies that the transposition $\mathrm{T}_{ij} = (i, j)$ performed simultaneously on the rows and columns of the matrix $\mathbf{A}$ carries this matrix into itself.

Hence $\mathbf{P}_{\mathrm{T}_{ij}}$ is a solution of the equation

$$\mathbf{PAP^T = A}. \tag{10}$$

Remark. It is clear that Theorem 3 allows also the study of the equation, by considering the permutational second „power" of the matrix $\mathbf{A}$, namely $\mathbf{A} \square \mathbf{A} = \mathbf{A}^{\square 2}$.

Nevertheless, we shall give here a combinatorial approach of this problem by means of the results which was obtained earlier (in the paragraphs 2 and 3).

Let $\mathbf{A}$ be a given matrix. We shall construct the table given below

$$(1, i_1) \ldots (1, i_{n_1-1}) \quad \text{with } 1 \sim i_s.$$

$$(2, j_1) \ldots (2, j_{n_2-1}) \quad \text{with } 2 \sim j_s.$$

This table could be *minimized* by using the symmetry and the transitivity of the relation „$\sim$". We note that any two pairs belonging to different lines of the minimized table are disjoint.

We shall suppose further that every pair (i, j) of the table generates a transposition T_{ij}. Let us consider the substitution group S generated by all the transpositions from the table.

If $\mathrm{S} \in S$, $\mathrm{S} = \mathrm{S}_1 \mathrm{S}_2 \ldots$ where S_i belongs to the group generated by the transpositions contained in the i-th line because transpositions from different lines being disjoint are commutative.

Every line from the table generates a permutation group with $n_i!$ elements (where n_i is the number of pairs situated on this line). It follows that the order of the group

$$|S| = \prod_i n_i!. \tag{11}$$

We note that the group generated by the transpositions from the i-th row is isomorphic with the substitution group S_{n_i} (with the correspondence $i \to 1$, $k_1 \to 2, \ldots, k_{n_i-1} \to n_i$). Every $\mathrm{S} \in S_{n_i}$ could be afterwards extended till a substitution with n objects maintaining fixed all the elements which do not appear among the given n_i numbers.

We can also see that:

$$\{\mathbf{S}, \mathrm{S} \in S\} \subset M_{\mathrm{A}}. \tag{12}$$

We used here the same letter for the permutation matrix and for the related substitution.

Indeed, if $\mathrm{S} \in S$, $\mathbf{SAS^T = A}$, because S is obtained as a product of transpositions which transform $\mathbf{A}$ into $\mathbf{A}$.

Conversely, let $\mathbf{T} \in M_{\mathbf{A}}$. We shall decompose it into a product of transpositions (which is allways possible), but we do not know if these transpositions belong to $M_{\mathbf{A}}$.

Remark. An algorithm which allows to find out all the factors of a graph (namely, the graph $\mathcal{A}^{[\,]2}$) will give all the solutions of the eq. 10. For such an algorithm, see the paragraph 7.

§ **6. Some applications in graph theory**. Let **A**, **B** the adjacency matrices of two graphs with n vertices, without multiple edges.

These two graphs are isomorphic iff there exists a permutation matrix **P**, which enjoys the property

$$\mathbf{P}\mathbf{A}\mathbf{P}^{\mathrm{T}} = \mathbf{B}.$$

There are solved the following problems:

1) We find out a superior bound of the number of graphs which are isomorphic with a given graph $\mathcal{A}$. This number is $|P/S|$, the exact number being $|P/M_{\mathbf{A}}|$.

2) We find out actually the graphs isomorphic with a given graph $\mathcal{A}$. Firstly, we find out the group S, we decompose P into cosets related to the subgroup S. By taking only one representative from every coset we obtain the family

$$L = \{\mathbf{P}_i \mid i \in I\}.$$

The distinct matrices **B** which are obtained with the equality

$$\mathbf{B} = \mathbf{P}\mathbf{A}\mathbf{P}^{\mathrm{T}}, \quad \text{where } \mathbf{P} \in L,$$

give all the graphs isomorphic with the graph $\mathcal{A}$.

3) If there are given two graphs by means of their adjacency matrices **A** and **B** we wish to find out if these graphs are isomorphic.

In the 3-rd paragraph, we obtained sufficient conditions for nonisomorphism. The isomorphism could be tasted by using again the previous conclusion.

If, using Theorem 3, we find a particular solution of the equation (2) we are able to find out all the solutions of the equation (2) by multiplying this particular solution with the elements of the group $M_{\mathbf{A}}$ (or $M_{\mathbf{B}}$).

We note that the group $M_{\mathbf{A}}$ could by obtained as a meet of left cosets of the subgroup S, constructed as it was pointed out earlier. We shall include into the group $M_{\mathbf{A}}$ the cosets of the group S whose representatives are solutions of the equation $\mathbf{P}\mathbf{A}\mathbf{P}^{\mathrm{T}} = \mathbf{A}$.

§ **7. An algorithm which allows to find out all the factors of a graph.** We shall give an algorithm which performs the enumeration of all the elementary circuits of a graph [1, pp. 185—186]. This method was obtained through a slight modification of the well known *method of Kaufmann, the algorithm of the latin multiplication*.

Let $\{t_1, \ldots, t_n\}$ the set of vertices of the graph (T, Γ_T). We shall form the latin matrix of the graph. For the graph which was considered earlier (namely $\mathcal{T} = \mathcal{A}[\,]\mathcal{B}$), wehave:

$$\mathbf{M}_0 = \begin{pmatrix} 0 & t_2 & t_3 & 0 & 0 \\ 0 & 0 & 0 & t_4 & t_5 \\ 0 & 0 & 0 & t_4 & t_5 \\ t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & t_3 & 0 & 0 \end{pmatrix}.$$

t_4

The multiplication of the latin matrices will be performed observing the same rules as usual: we shall delate every word which contains two identical symbols. In the first step we shall perform the second latin power of the matrix $\mathbf{M}_0$.

$$\mathbf{M}_1 = \mathbf{M}_0^2 = \begin{pmatrix} 0 & 0 & 0 & \begin{matrix} t_2 t_4 \\ t_3 t_4 \end{matrix} & \begin{matrix} t_2 t_5 \\ t_3 t_5 \end{matrix} \\ t_4 t_1 & t_5 t_2 & t_5 t_3 & 0 & 0 \\ t_4 t_1 & t_5 t_2 & t_5 t_3 & 0 & 0 \\ 0 & t_1 t_2 & t_1 t_3 & 0 & 0 \\ 0 & 0 & 0 & \begin{matrix} t_2 t_4 \\ t_3 t_4 \end{matrix} & \begin{matrix} t_2 t_5 \\ t_3 t_5 \end{matrix} \end{pmatrix}.$$

From this matrix we shall extract all the diagonal entries. It is very easy to see that by adjoining to left of these words their last symbols, we obtain all the elementary circuits with length 2, namely: $t_2 t_5 t_2$, $t_3 t_5 t_3$, $t_5 t_2 t_5$, $t_5 t_3 t_5$.

Let $\widehat{\mathbf{M}}_1$ be the matrix which is obtained from $\mathbf{M}_1$ by delating all its diagonal entries. In the next step of the algorithm we shall obtain the matrix $\mathbf{M}_2$ by

$$\mathbf{M}_2 = \widehat{\mathbf{M}}_1 \mathbf{M}_0 .$$

We have

$$\mathbf{M}_2 = \begin{pmatrix} 0 & 0 & 0 & \begin{matrix} t_2 t_4 \\ t_3 t_4 \end{matrix} & \begin{matrix} t_2 t_5 \\ t_3 t_5 \end{matrix} \\ t_4 t_1 & 0 & t_5 t_3 & 0 & 0 \\ t_4 t_1 & t_5 t_2 & 0 & 0 & 0 \\ 0 & t_1 t_2 & t_1 t_3 & 0 & 0 \\ 0 & 0 & 0 & \begin{matrix} t_2 t_4 \\ t_3 t_4 \end{matrix} & 0 \end{pmatrix} \neq M_0 =$$

$$= \begin{pmatrix} \substack{t_2 t_4 t_1 \\ t_3 t_4 t_1} & t_3 t_5 t_2 & t_2 t_5 t_3 & 0 & 0 \\ 0 & t_4 t_1 t_2 & t_4 t_1 t_3 & t_5 t_3 t_4 & 0 \\ 0 & t_4 t_1 t_2 & t_4 t_1 t_3 & t_5 t_2 t_4 & 0 \\ 0 & 0 & 0 & \substack{t_1 t_2 t_4 \\ t_1 t_3 t_4} & \substack{t_1 t_2 t_5 \\ t_1 t_3 t_5} \\ \substack{t_2 t_4 t_1 \\ t_3 t_4 t_1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this matrix we obtain the 3-length elementary circuits: $t_1 t_2 t_4 t_1$, $t_1 t_3 t_4 t_1$, $t_2 t_4 t_1 t_2$, $t_3 t_4 t_1 t_3$, $t_4 t_1 t_2 t_4$, $t_4 t_1 t_3 t_4$. Now, the matrix $\widehat{\mathbf{M}}_2$ is

$$\widehat{\mathbf{M}}_2 = \begin{pmatrix} 0 & t_3 t_5 t_2 & t_2 t_5 t_3 & 0 & 0 \\ 0 & 0 & t_4 t_1 t_3 & t_5 t_3 t_4 & 0 \\ 0 & t_4 t_1 t_2 & 0 & t_5 t_1 t_4 & 0 \\ 0 & 0 & 0 & 0 & \substack{t_1 t_2 t_5 \\ t_1 t_3 t_5} \\ \substack{t_2 t_4 t_1 \\ t_3 t_4 t_1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\mathbf{M}_3$ is computed in the same manner:

$$\mathbf{M}_3 = \begin{pmatrix} 0 & 0 & 0 & \substack{t_3 t_5 t_2 t_4 \\ t_2 t_5 t_3 t_4} & t_4 t_1 t_3 t_5 \\ t_5 t_3 t_4 t_1 & 0 & 0 & 0 & t_4 t_1 t_3 t_5 \\ t_5 t_2 t_4 t_1 & 0 & 0 & 0 & t_4 t_1 t_2 t \\ 0 & t_1 t_3 t_5 t_2 & t_1 t_2 t_5 t_3 & 0 & 0 \\ 0 & t_3 t_4 t_1 t_2 & t_2 t_4 t_1 t_3 & 0 & 0 \end{pmatrix}.$$

By observing that $\widehat{\mathbf{M}}_3 = \mathbf{M}_3$, we have:

$$\mathbf{M}_4 = \widehat{\mathbf{M}}_3 \mathbf{M}_0 = \begin{pmatrix} \substack{t_3 t_5 t_2 t_4 t_1 \\ t_2 t_5 t_3 t_4 t_1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \substack{t_5 t_3 t_4 t_1 t_2 \\ t_4 t_1 t_3 t_5 t_2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \substack{t_5 t_2 t_4 t_1 t_3 \\ t_4 t_1 t_2 t_5 t_3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \substack{t_1 t_3 t_5 t_2 t_4 \\ t_1 t_2 t_5 t_3 t_4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \substack{t_3 t_4 t_1 t_2 t_5 \\ t_2 t_4 t_1 t_3 t_5} \end{pmatrix}.$$

Hence the factors of our graph are: $(t_1 t_3 t_5 t_2 t_4 t_1)$, $(t_1 t_2 t_5 t_3 t_4 t_1)$, $(t_2 t_5 t_3 t_4 t_1 t_2)$, $(t_2 t_4 t_1 t_3 t_5 t_2)$, $(t_3 t_5 t_2 t_4 t_1 t_3)$ $(t_3 t_4 t_1 t_2 t_5 t_3)$, $(t_4 t_1 t_3 t_5 t_2 t_4)$, $(t_4 t_1 t_2 t_5 t_3 t_4)$, $(t_5 t_3 t_4 t_1 t_2 t_5)$, $(t_5 t_2 t_4 t_1 t_3 t_5)$.

We have also the following factors:

$$\{(t_2 t_5 t_2),\quad (t_1 t_3 t_4 t_1)\},\quad \{(t_3 t_5 t_3),\quad (t_1 t_2 t_4 t_1\}.$$

We must now, seek by a cut-and-trial process if among the matrices related to these factors there exists a solution of our problem.

For instance, to the factor $(t_1 t_2 t_5 t_3 t_4 t_1)$ we can relate the matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

which is a solution of the equation $\mathbf{PAP^T = B}$.

Now, it is possible to conclude that the graphs (A, Γ_A), (B, Γ_B) from the example which was given in the fourth paragraph, are isomorphic.

REFERENCES

1. Ivănescu P., Rudeanu S. — *Boolean Methods in Operational Research*. Springer-Verlag, Berlin 1968.

ASUPRA ECUAȚIEI MATRICEALE $\mathbf{PAP^T = B}$ ÎN ALGEBRA BOOLE $\mathfrak{B}_2$ ȘI UNELE APLICAȚII ÎN TEORIA GRAFURILOR

Rezumat

În această Notă este rezolvată ecuația $\mathbf{PAP^T = B}$, cu matricea necunoscută $\mathbf{P}$, unde $\mathbf{A}$, $\mathbf{B}$ sînt matrice cu elemente din $\mathfrak{B}_2$, iar $\mathbf{P}$ este o matrice de permutări. Abordarea acestei chestiuni, în esență combinatorială, se face prin procedee booleene, rezolvîndu-se simultan problema izomorfismului grafurilor.

EINE ANWENDUNG DER α-ZERLEGBAREN GRAPHEN IN DER INFORMATIONSTHEORIE

VON

E. OLARU

1. **Graphentheoretische Betrachtungen.** In der vorliegenden Arbeit betrachten wir endliche ungerichtete Graphen $G = (X, U)$, die weder Schlingen noch mehrfache Kanten enthalten. Dabei bezeichnet X die Knotenpunktmenge und U die Kantenmenge von G. Verbindet die Kante u die Knotenpunkte x und y, so schreiben wir $u = (x, y)$ $(= (y, x))$. Wir wollen die Tatsache, dass die Knotenpunkte x und y durch eine Kante in G verbunden (bzw. nicht verbunden) sind, auch durch die Bezeichnung $x \rho y$ (bzw. $x \bar{\rho} y$) ausdrücken.

Ein Graph G' heisst *Teilgraph* des Graphen G, wenn jeder Knoetenpunkt von G' auch Knotenpunkt von G ist und zwei Knotenpunkte von G' genau dann in G' durch eine Kante verbunden sind, wenn sie es auch in G sind. Ist $X' \subset X$ die Knotenpunktmenge von G', so nennen wir G' auch den durch die Knotenpunktmenge X' *aufgespannten Teilgraphen* von G, und wir bezeichnen ihn mit $[X']_G$. Ein vollständiger Teilgraph von G heisst eine *Clique* von G.

Eine Teilmenge $S' \subset X$ heisst eine Menge *unabhängiger Knotenpunkte* (in Bezug auf G), wenn gilt: $[S']_G = (S', \emptyset)$, d. h. wenn für je zwei Knotenpunkte $x', y' \in S'$ gilt: $x' \bar{\rho} y'$. Die maximale Anzahl der unabhängigen Knotenpunkte des Graphen G heisst die (innere) Stabilitätszahl von G, und wir bezeichnen sie mit $\alpha(G)$. Eine Menge S von in G unabhängigen Knotenpunkten mit genau $\alpha(G)$ Elementen wollen wir ein Stabilitätssystem von G nennen.

Eine Zerlegung $\{X_1, X_2, \ldots, X_k\}$ der Knotenpunktmenge X des Graphen G mit folgenden Eigenschaften: