AN INEQUALITY FOR THE POINT-ARBORICITY OF A GRAPH

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The point-arboricity $\rho(G)$ of a graph G is defined [2] as the minimum number of subsets into which the point set of G may be partitioned so that each subset induces an acyclic subgraph. A subgraph H of G is referred to as an induced or section subgraph if every line of G which joins two points of H is a line of H and we use the notation $H < G$ to indicate that H is an induced subgraph of G.

We denote by X the point set of G ($|X| = n$) and by $d(x_i)$ the degree of point $x_i \in X$. We suppose that $d(x_1) \geqslant d(x_2) \geqslant \ldots \geqslant d(x_n)$ and $\sum_{i=1}^{n} d(x_i) \neq 0$. Furthermore, for any real number r, $[r]$ and $\{r\}$ denote the greatest integer not exceeding r and the least integer not less than r, respectively.

Theorem. *If $\Delta = \{X_k\}_{1 \leq k \leq q}$ is a partition of the point set of G so that each X_k induces an acyclic subgraph, then we have:*

$$\rho(G) \leqslant \max_{k=2,\ldots,q} (\min (k, 1 + \max_{x_p \in X_k} [d(x_p)/2])). \tag{1}$$

Proof. For $H < G$ we denote by Y the point set of H. We define a subset $Z \subset Y$, which induces an acyclic subgraph of H to be maximal wit respect to H, if for every point $v \in Y \setminus Z$, the set $Z \cup \{v\}$ does not induce an acyclic subgraph of H. If set X_1 is not maximal with respect to G, then we can obtain a new partition $X_1^1, X_2^1, \ldots, X_q^1$ ($X_1^1 \supset X_1$, $X_2^1 \subseteq X_2, \ldots, X_q^1 \subseteq X_q$) so that X_1^1 is maximal with respect to G, by removing some points of $X_2, \ldots, X_q$. In this way a new partition $\Delta' = \{X'_k\}_{1 \leq k \leq q'}$ of X is obtained, having the following properties:

1) $q' \leqslant q$; 2) X'_k is maximal with respect to the subgraph induced by point set $X \setminus \bigcup_{i=1}^{k-1} X'_i$ and $X_k \subseteq \bigcup_{i=1}^{\min(k,q')} X'_i$ for every $k = 1, \ldots, q$.

If $x_m \in X \setminus \bigcup_{i=1}^{k} X'_i$ we have $d(x_m) \geqslant 2k$, hence $x_t \in \bigcup_{i=1}^{\varphi(t)} X'_i$ where $\varphi(t) = \min(q', 1 + [d(x_t)/2]) = \min(q', \{[1 + d(x_t)]/2\})$ for every $x_t \in X$. Suppose $x_t \in X_k$, hence $x_t \in \bigcup_{i=1}^{\min(k,q')} X'_i$. Furthermore $x_t \in \bigcup_{i=1}^{\varphi(t)} X'_i$ implies that $x_t \in \bigcup_{i=1}^{\psi(k)} X'_i$ with $\psi(k) = \min(q', \max_{x_p \in X_k} (\varphi(p))) = \min(q', \max_{x_p \in X_k} (1 + [d(x_p)/2])) = \min(q', 1 + \max_{x_p \in X_k} [d(x_p)/2])$.

This implies that $x_t \in \bigcup_{i=1}^{\eta(k)} X'_i$ for every $x_t \in X_k$, where $\eta(k) = \min(\min(k, q'), \psi(k)) = \min(q', \min(k, 1 + \max_{x_p \in X_k} [d(x_p)/2]))$. In conclusion $x_t \in \bigcup_{i=1}^{\min(q', \gamma_\Delta(G))} X'_i$ for any $x_t \in X$, if we denote $\gamma_\Delta(G) = \max_{k=2,\ldots,q} (\min(k, 1 + \max_{x_p \in X_k} [d(x_p)/2]))$, because $\min(q', \max_{k=2,\ldots,q} \eta(k)) = \min(q', \gamma_\Delta(G))$ and $X = \bigcup_{i=1}^{\min(q', \gamma_\Delta(G))} X'_i = \bigcup_{i=1}^{q'} X'_i$, hence $\min(q', \gamma_\Delta(G)) = q'$ from which it follows that $\gamma_\Delta(G) \geqslant q' \geqslant \rho(G)$. q.e.d.

If we choose $X_1 = \{x_1\}, X_2 = \{x_2\}, \ldots, X_n = \{x_n\}$, we obtain the following upper bound:

(2) $$\rho(G) \leqslant \max_{k=2,\ldots,n} (\min(k, 1 + [d(x_k)/2])).$$

and if we choose $X_1 = \{x_1, x_2\}$, $X_2 = \{x_3, x_4\}, \ldots$ we obtain $\rho(G) \leqslant \max_{k=2,\ldots,\{n/2\}} (\min(k, 1 + [d(x_{2k-1})/2]))$.

In order to obtain a good upper bound for $\rho(G)$ we must consider a partition Δ of the point set of G such that $\max_{x_p \in X_1} (d(x_p)) \geqslant \max_{x_p \in X_2} (d(x_p)) \geqslant \ldots \geqslant \max_{x_p \in X_q} (d(x_p))$.

In this case we have $d(x_1) = \max_{x_p \in X_1} (d(x_p))$; $d(x_2) \geqslant \max_{x_p \in X_2} (d(x_p))$; $\ldots$; $d(x_q) \geqslant \max_{x_p \in X_q} (d(x_p))$, hence $\min(k, 1 + \max_{x_p \in X_k} [d(x_p)/2]) \leqslant \min(k, 1 + [d(x_k)/2])$ for $k = 2, \ldots, q$.

Clearly, $\gamma_\Delta(G) \leqslant \max_{k=2,\ldots,q} (\min(k, 1 + [d(x_k)/2])) \leqslant \max_{k=2,\ldots,n} (\min(k, 1 + [d(x_k)/2]))$.

If $\Delta = \{X_k\}_{1 \leqslant k \leqslant q}$ is a partition of X with $q = \rho(G)$, so that each X_k induces an acyclic subgraph, then $\gamma_\Delta(G) \leqslant \max_{k=2,\ldots,q} (k) = q = \rho(G)$; thus $\rho(G) = \min_\Delta (\gamma_\Delta(G))$. An analogue of this theorem concerning the chromatic number is given in [3].

In this way we can obtain an upper bound for the minimum number of subsets into which the point set of G may be partitioned so that each subset does not contain a special configuration.

For example, if every subset of the partition does not contain K_m (complete subgraph having m points), then the upper bound takes the form:

$$\max_{k=2,\ldots,q} (\min(k, 1 + \max_{x_p \in X_k} [d(x_p)/(m-1)]))$$

where $\Delta = \{X_k\}_{1 \leqslant k \leqslant l}$ is a partition of the point set of G so that each X_k does not contain K_m.

In particular we have at most $\max_{k=2,\ldots,\{n/(m-1)\}} (\min(k, 1 + [d(x_{1+(k-1)(m-1)})/(m-1)]))$ classes of a minimal partition of the point set of G so that each subset does not contain K_m.

REFERENCES

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O INEGALITATE PENTRU NUMĂRUL DE ARBORICITATE AL VÎRFURILOR UNUI GRAF

Rezumat

Numărul de arboricitate al vîrfurilor unui graf este numărul minim de submulţimi de vîrfuri în care poate fi împărţită mulţimea vîrfurilor grafului, astfel ca fiecare submulţime să inducă un graf aciclic. În lucrare se dă o limitare superioară pentru numărul de arboricitate, analogă unei limitări superioare pentru numărul cromatic, dată într-o lucrare anterioară.