CONTRIBUTIONS TO THE STUDY OF PROXIMAL-CONTINUITY

Summary

Although the notion of proximal continuity is situated between simple continuity and uniform continuity and for the latest there are known many criteria, for the proximal-continuity we do not know any necessary and sufficient condition. In the first part of this paper we give some criteria for proximal-continuity, using various operators introduced by B. Banaschewschi and J. Maranda in [2] for the definion of proximity-spaces. In the second part we give two conditions, characterizing proximal-continuity in metric spaces.

NETS OF FUNCTIONS

BY

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1. Introduction. Considering sequences of continuous functions E. Borel [1], E.W. Hobson [2], P.S. Alexandrov [3] and C. Arzelà [4] give necessary and sufficient conditions for convergence to a continuous function, by using the concept of quasi-uniform convergence. M. Nicolescu [5] shows the criticism made by S. Marcus to this concept in an unpublished communication. Later I. Muntean [6] generalizes the concept of quasi-uniform convergence to nets of functions defined on a topological space, having values in an uniform space in Weil's [7] acception. He succeeded to give a necessary and sufficient condition for a net of continuous functions to converge to a continuous function.

In the first part of this paper, we shall try to find necessary and sufficient conditions for a net of functions, not necessarily continuous, to converge to a continuous function.

K. Bögel [8], M. Nicolescu [9] E. Dobrescu and I. Sălăgeanu [10] define a concept of bidimensional continuity for the functions defined on a cartesian product of topological spaces.

In the last part of our paper, we shall study the convergence of a net of functions to a bidimensional continuous function.

Notations. Definitions. Let X, Y be two topological spaces, Z a uniform space, E a linear topological space and $(A, \leqslant)$ a directed set. If $x_0 \in X$, $y_0 \in Y$, $(x_0, y_0) \in X \times Y$, we shall denote by $\mathcal{V}(x_0)$, $\mathcal{V}(y_0)$, $\mathcal{V}(x_0, y_0)$ respectively the neighborhoods filters of $x_0, y_0, (x_0, y_0)$ in the topologies of X, Y or in the product topology of $X \times Y$. We shall denote by $\mathcal{U}$ a fundamental system of symmetrical neighborhoods in the uniform space Z. We shall denote by 0, the origin of the linear topological space E and by $\mathcal{V}(0)$ the neighborhoods filter of 0 in the topology of E, compatible with its linearity.

We shall denote nets of functions by

$$\pi = \{f_a : X \to Z; a \in A\},$$

$$\tilde{\pi} = \{f_a ; X \times Y \to E; a \in A\}.$$

In order to define the concept of subnet, J. Kelley [11] introduces a mapping between two directed sets with a specific property. We shall call such a mapping a "K-mapping"

Definition 1. *Let $(A \leqslant)$, $(B, \leqslant')$ be two directed sets. The function*

$$h: B \to A$$

is a „K-mapping" iff

$$(\pi_1) \quad \underset{a \in A}{\forall} \; \underset{b \in B}{\exists} \; \underset{\bar{b} \in B}{\forall} \; b \leqslant' \bar{b} \Rightarrow a \leqslant h(\bar{b}).$$

Definition 2. *The net π, converges pointwise to*

$$f: X \to Z,$$

iff

$$(\pi_2) \quad \underset{x_0 \in X}{\forall} \; \underset{U \in \mathcal{U}}{\forall} \; \underset{a_{x_0} \in A}{\exists} \; \underset{a \in A}{\forall} \; a_{x_0} \leqslant a \Rightarrow (f_a(x_0), f(x_0)) \in U.$$

Definition 3. *The net π, converges uniformly to*

$$f: X \to Z,$$

iff

$$(\pi_3) \quad \underset{U \in \mathcal{U}}{\forall} \; \underset{a_U \in A}{\exists} \; \underset{a \in A}{\forall} \; a_U \leqslant a \; \underset{x \in X}{\forall} \Rightarrow (f_a(x), f(x)) \in U.$$

Definition 4. *The net π, converges quasi-uniformly to*

$$f: X \to Z,$$

iff it converges pointwise to f, and we have

$$(\pi_4) \quad \underset{U \in \mathcal{U}}{\forall} \; \underset{(B, \leqslant')}{\exists} \; \underset{h: B \to A}{\exists} \; \underset{\tilde{\mathcal{G}} = \{G_b\}_{b \in B}}{\exists} \; \bigcup_{b \in B} G_b = X \; \underset{x \in G_b}{\forall} \Rightarrow (f(x), f_{h(b)}(x)) \in U,$$

where $(B, \leqslant')$ is a directed set, h a K-mapping, $\tilde{\mathcal{G}}$ a family of open sets in the topology of X

Remark. The definition of quasi-uniform convergence given here is equivalent to that one given by I. Muntean [6].

Definition 5. *The net $\tilde{\pi}$, converges pointwise to.*

$$f: X \times Y \to E,$$

iff

$$(\tilde{\pi}_5) \quad \underset{(x_0, y_0) \in X \times Y}{\forall} \; \underset{V \in \mathcal{V}(0)}{\forall} \; \underset{a_{x_0, y_0} \in A}{\exists} \; \underset{a \in A}{\forall} \; a_{x_0, y_0} \leqslant a \Rightarrow f_a(x_0, y_0) - f(x_0, y_0) \in V,$$

Definition 6. *The net $\tilde{\pi}$, converges uniformly to*

$$f: X \times Y \to E,$$

iff

$$(\tilde{\pi}_6) \quad \underset{V \in \mathcal{V}(0)}{\forall} \; \underset{a_V \in A}{\exists} \; \underset{a \in A}{\forall} \; a_V \leqslant a \; \underset{(x, y) \in X \times Y}{\forall} \Rightarrow f_a(x, y) - f(x, y) \in V.$$

Definition 7. *The net $\tilde{\pi}$, converges quasi-uniformly to*

$$f: X \times Y \to E,$$

iff it converges pointwise, and we have

$$(\tilde{\pi}_7) \quad \underset{V \in \mathcal{V}(0)}{\forall} \; \underset{(B, \leqslant')}{\exists} \; \underset{h: B \to A}{\exists} \; \underset{\tilde{\mathcal{G}} = \{G_b\}_{b \in B}}{\exists} \; \bigcup_{b \in B} G_b = X \times Y \; \underset{(x, y) \in G_b}{\forall} \Rightarrow$$

$$f_{h(b)}(x, y) - f(x, y) \in V,$$

where $(B, \leqslant')$ is a directed set, h a K-mapping and $\tilde{\mathcal{G}}$ is a family of open sets in the product topology of $X \times Y$.

Definition 8. *The function*

$$f: X \to Z$$

is continuous in $x_0 \in X$, iff

$$(\pi_8) \quad \underset{U \in \mathcal{U}}{\forall} \; \underset{V \in \mathcal{V}(x_0)}{\exists} \; \underset{x \in V}{\forall} \Rightarrow (f(x), f(x_0)) \in U.$$

Definition 9. *The function*

$$f: X \times Y \to E$$

is bidimensionally continuous in $(x_0, y_0) \in X \times Y$, iff

$$(\tilde{\pi}_9) \quad \underset{V \in \mathcal{V}(0)}{\forall} \; \underset{U \in \mathcal{V}(x_0, y_0)}{\exists} \; \underset{(x, y) \in U}{\forall} \Rightarrow f(x_0, y) + f(x, y_0) - f(x, y) - f(x_0, y_0) \in V.$$

3. The nets π. It is well known the

Theorem 1. *A net of continuous functions which is uniformly convergent, has as limit a continuous function.*

I. Muntean [6] proved

Theorem 2. *In order a pointwise limit*

$$f\colon X \to Z$$

of a net π, *of continuous functions, be continuous, the necessary and sufficient condition is that the net* π *be quasi-uniform convergent to* f.

Example 1. Let be

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in Q, \\ 0 & \text{if } x \in R \setminus Q, \end{cases}$$

where Q, R, are the sets of rational, respectively real numbers.

Let be π the net defined by

$$f_a(x) = \frac{1}{a}\,\alpha(x)$$

where a runs over the directed set $(R, \leqslant)$.

It is easily seen that although the functions of the net π are not continuous, π converges uniformly to

$$f(x) \equiv 0,$$

which is a continuous function. Therefore it is quite natural to look for the necessary and sufficient condition that a net of functions not necessarily continuous, uniform convergent, have as a limit a continuous function.

Theorem 3. *Let* π *be a net uniformly convergent to*

$$f\colon X \to Z;$$

the necessary and sufficient condition for f *to be continuous is*

$$\underset{x_0 \in X}{\forall}\ \underset{U \in \mathfrak{U}}{\forall}\ \underset{a \in A}{\forall}\ \underset{V \in \mathcal{V}(x_0)}{\exists}\ \underset{a_1 \in A}{\exists}\ a \leqslant a_1\ \underset{x \in V}{\forall} \Rightarrow (f_{a_1}(x), f_{a_1}(x_0)) \in U.$$

Proof. Let us suppose that π is a net, uniformly convergent to f and f continuous in x_0. From the properties of uniform spaces, (N. Bourbaki [12]), it follows that

$$\underset{U \in \mathfrak{U}}{\forall}\ \underset{U_1 \in \mathfrak{U}}{\exists} \Rightarrow U_1 \circ U_1 \subset U,$$

$$U_1 \in \mathfrak{U}\ \underset{U_2 \in \mathfrak{U}}{\exists} \Rightarrow U_2 \circ U_2 \subset U_1.$$

Let be

$$U_3 = U_1 \cap U_2 \in \mathfrak{U};$$

we have

$$(U_3 \circ U_3) \circ U_3 \subset (U_2 \circ U_2) \circ U_1 \subset U_1 \circ U_1 \subset U.$$

Therefore

$$\underset{U \in \mathfrak{U}}{\forall}\ \underset{U_3 \in \mathfrak{U}}{\exists} \Rightarrow (U_3 \circ U_3) \circ U_3 \subset U. \tag{1}$$

From the uniform convergence of the net π, it results π_3

$$U_3 \in \mathfrak{U}\ \underset{a_0 \in A}{\exists}\ \underset{a \in A}{\forall}\ a_0 \leqslant a\ \underset{x \in X}{\forall} \Rightarrow (f_a(x), f(x)) \in U_3.$$

or

$$U_3 \in \mathfrak{U}\ \underset{x \in X}{\forall} \Rightarrow (f_{\bar{a}}(x), f(x)) \in U_3\ \&\ (f_{\bar{a}}(x_0), f(x_0)) \in U_3. \tag{2}$$

From the properties of a directed set $(A, \leqslant)$, it follows

$$\underset{a \in A}{\forall}\ \underset{a_1 \in A}{\exists} \Rightarrow a_0 \leqslant a_1\ \&\ a \leqslant a_1.$$

But from (2) we have

$$a_0 \leqslant a_1 \Rightarrow (f_{a_1}(x), f(x)) \in U_3\ \&\ (f_{a_1}(x_0), f(x_0)) \in U_3. \tag{3}$$

From the continuity of f in x_0, it follows

$$U_3 \in \mathfrak{U}\ \underset{V \in \mathcal{V}(x_0)}{\exists}\ \underset{x \in V}{\forall} \Rightarrow (f(x), f(x_0)) \in U. \tag{4}$$

From (3) and (4) we have

$$(f_{a_1}(x), f_{a_1}(x_0)) \in (U_3 \circ U_3) \circ U_3^{-1}$$

But as $\mathfrak{U}$ is a fundamental system of symmetric neighborhoods, from (1) it results

$$(f_{a_1}(x), f_{a_1}(x_0)) \in U;$$

therefore the condition given in the proposition.

Conversely, let us suppose the net π is uniform convergent to f; at the same time we suppose satisfied the condition given in the proposition. From the properties of the uniform spaces, as we have already shown, we have

$$\underset{U \in \mathfrak{U}}{\forall}\ \underset{U_3 \in \mathfrak{U}}{\exists} \Rightarrow (U_3 \circ U_3) \circ U_3 \subset U. \tag{1}$$

From the uniform convergence to f, of the net π it follows

$$U_3 \in \mathfrak{U}\ \underset{a_0 \in A}{\exists}\ \underset{a \in A}{\forall}\ a_0 \leqslant a\ \underset{x \in X}{\forall} \Rightarrow (f_a(x), f(x)) \in U_3.$$

Therefore

(2) $$U_3 \in \mathfrak{U} \ \forall_{x \in X} \Rightarrow (f_a(x), f(x)) \in U_3 \ \& \ (f_a(x_0), f(x_0)) \in U_3.$$

From the hypothesis it results

(5) $$U_3 \in \mathfrak{U}\ a_0 \in A \quad \exists_{V \in \mathfrak{V}(x_0)} \ \exists_{a_1 \in A} \ a_0 \leqslant a_1 \ \forall_{x \in V} \Rightarrow (f_{a_1}(x), f_{a_1}(x_0)) \in U_3.$$

But from (2) we have

(6) $$a_0 \leqslant a_1 \Rightarrow (f_{a_1}(x),\ f(x)) \in U_3 \ \& \ (f_{a_1}(x_0), f(x_0)) \in U_3.$$

From (5) and (6) it results

$$\forall_{x \in V} \Rightarrow (f(x), f(x_0)) \in (U_3^{-1} \circ U_3) \circ U_3.$$

But as $\mathfrak{U}$ is a fundamental system of symmetric neighborhoods, from (1), it results

$$\forall_{x \in V} \Rightarrow (f(x), f(x_0)) \in U.$$

Therefore

$$\forall_{U \in \mathfrak{U}} \ \exists_{V \in \mathfrak{V}(x_0)} \ \forall_{x \in V} \Rightarrow (f(x), f(x_0)) \in U.$$

And hence (π_8), thus the continuity of f in x_0.

Example 2. Let be

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in Q, \\ 0 & \text{if } x \in R \setminus Q \end{cases}$$

considered in the example 1 and the net π,

$$\pi = \{f_a;\ [0,1] \to R \ \ a \in R\},$$

defined by

$$f_a(x) = x^a - 1 + \frac{a(1-x)}{1 + a(1-x)} + \frac{1}{a}\alpha(x).$$

It is easily seen that although the functions of the net are not continuous, the net converges quasi-uniform to

$$f(x) \equiv 0$$

which is obviously continuous.

Remark. The example given above is a generalization of that one given by C. Jordan [13]. Therefore it is natural to ask for the necessary and sufficient conditions in order that a quasi-uniform convergent net π converge to a continuous function.

Theorem 4. *Let π be a net which is quasi-uniform convergent to*

$$f\colon X \to Z.$$

The necessary and sufficient condition for f to be continuous is

$$\forall_{U \in \mathfrak{U}} \ \forall_{x_0 \in X} \ \forall_{a \in A} \ \exists_{a_1, a_2 \in A} \ a \leqslant a_2 \ \exists_{V \in \mathfrak{V}(x_0)} \ \forall_{x \in V} \Rightarrow (f_{a_1}(x), f(x)) \in U \ \& \ (f_{a_2}(x_0), f_{a_1}(x)) \in U.$$

Proof. Let us suppose that π is quasi-uniform convergent to f, and f continuous. From the properties of uniform spaces we know:

(1) $$\forall_{U \in \mathfrak{U}} \ \exists_{U_3 \in \mathfrak{U}} \Rightarrow (U_3 \circ U_3) \circ U_3 \subset U.$$

The net π being quasi-uniform convergent to f, it results (π_4)

$$U_3 \in \mathfrak{U} \ \exists_{(B, \leqslant')} \ \exists_{h\colon B \to A} \ \exists_{\mathcal{G} = \{G_b\}_{b \in B}} \ \bigcup_{b \in B} G_b = X \ \forall_{x \in G_b} \Rightarrow_{h(b)} (f(x), f(x)) \in U_3.$$

Let be $x_0 \in X$ and $b_0 \in B$ so that $x_0 \in G_{b_0}$. Therefore we have

$$\forall_{x \in G_{b_0}} \Rightarrow \left(f_{h(b_0)}(x), f(x) \right) \in U_3.$$

Let be

$$a_1 = h(b_0) \in A;$$

therefore

(7) $$\forall_{x \in G_{b_0}} \Rightarrow (f_{a_1}(x), f(x)) \in U_3.$$

From the continuity of f in x_0, it results

(8) $$U_3 \in \mathfrak{U} \quad \exists_{V_1 \in \mathfrak{V}(x_0)} \ \forall_{x \in V_1} \Rightarrow (f(x), f(x_0)) \in U_3.$$

But as $G_{b_0} \in \mathfrak{V}(x_0)$ are open set, let be

$$V = V_1 \cap G_{b_0} \in \mathfrak{V}(x_0).$$

As the net π is quasi-uniform convergent to f, it converges pointwise to f in the point x_0. Therefore

(9) $$U_3 \in \mathfrak{U} \ \exists_{a \in A} \ \forall_{\bar{a} \in A} \ a_0 \leqslant \bar{a} \Rightarrow (f_{\bar{a}}(x_0), f(x_0)) \in U_3.$$

From the properties of the directed set $(A, \leqslant)$ it results

(10) $$\forall_{a \in A} \ \exists_{a_2 \in A} \Rightarrow a \leqslant a_2 \ \& \ a_0 \leqslant a_2$$

From (9) and (10) it results that:

$$(11)\qquad U_3 \in \mathcal{U}\ \underset{a\in A}{\forall}\ \underset{a_2\in A}{\exists}\ a \leqslant a_2 \Rightarrow (f_{a_2}(x_0), f(x_0)) \in U_3.$$

From (7), (8) and (11) it results

$$(f_{a_2}(x_0), f_{a_1}(x)) \in (U_3 \circ U_3^{-1}) \circ U_3^{-1}.$$

But as $\mathcal{U}$ is a fundamental system of symmetric neighbourhoods, from (1) it results

$$(f_{a_2}(x_0), f_{a_1}(x)) \in U.$$

On the other hand, from (7)

$$(f(x), f_{a_1}(x)) \in U_3^{-1} = U_3 \subset U;$$

it results the condition of the theorem.

Conversely let π be a quasi-uniform convergent net to f, and theorem's condition satisfied. From the properties of the uniform spaces we have.

$$(1)\qquad \underset{U\in\mathcal{U}}{\forall}\ \underset{U_3\in\mathcal{U}}{\exists} \Rightarrow (U_3 \circ U_3) \circ U_3 \subset U.$$

From the quasi-uniform convergence of π to f, it results the pointwise convergence in x_0. Therefore, $x_0 \in X$ being fixed, we have

$$(12)\qquad U_3 \in \mathcal{U}\ \underset{a_0\in A}{\exists}\ \underset{\bar a\in A}{\forall}\ a_0 \leqslant \bar a \Rightarrow (f_{\bar a}(x_0), f(x_0)) \in U_3.$$

From the condition of the theorem we have

$$x_0 \in X, U_3 \in \mathcal{U}\ \underset{a\in A}{\forall}\ \underset{a_1,a_2\in A}{\exists}\ a \leqslant a_2\ \underset{V\in\mathcal{V}(x_0)}{\exists}\ \underset{x\in V}{\forall} \Rightarrow (f(x), f_{a_1}(x)) \in U_3$$

$$\& \ (f_{a_2}(x_0), f_{a_1}(x)) \in U_3.$$

Therefore

$$(13)\qquad a_0 \in A\ \underset{a_1,a_2\in A}{\exists}\ a_0 \leqslant a_2\ \underset{V\in\mathcal{V}(x_0)}{\exists}\ \underset{x\in V}{\forall} \Rightarrow (f(x), f_{a_1}(x)) \in U_3$$

$$\& \ (f_{a_2}(x_0), f_{a_1}(x)) \in U_3.$$

But from (12) $a_0 \leqslant a_2 \Rightarrow (f_{a_2}(x_0), f(x_0)) \in U_3$. Therefore

$$(f(x), f(x_0)) \in (U_3 \circ U_3^{-1}) \circ U_3.$$

As $\mathcal{U}$ is a fundamental system of symmetric neighbourhoods from (1) it results

$$\underset{x\in V}{\forall} \Rightarrow (f(x), f(x_0)) \in U.$$

hence the continuity of f in x_0.

4. The net $\tilde{\pi}$. *Example 3.* Let us take the directed set $(R, \leqslant)$, and consider

$$\alpha(u) = \begin{cases} \dfrac{1}{u} \text{ if } u \neq 0 \\ 0 \text{ if } u = 0 \end{cases} \qquad u \in R.$$

Let by the net

$$\tilde{\pi} = \{f_a : [-1,1] \times [-1,1] \to R;\ a \in R\}$$

be defined by

$$f_a(x, y) = (|xy| - 1)^{2a} + \alpha(x) + \alpha(y).$$

It is easly seen that the functions of the net $\tilde{\pi}$ are not continuous in (0,0) but are bidimensionally continuous. It is also easily checked that the net $\tilde{\pi}$ converges pointwise to

$$f(x,y) = \begin{cases} \dfrac{1}{x} + \dfrac{1}{y} \text{ if } x \neq 0 \text{ and } y \neq 0 \\ 1 + \dfrac{1}{x} \text{ if } x \neq 0 \text{ and } y = 0 \\ 1 + \dfrac{1}{y} \text{ if } x = 0 \text{ and } y \neq 0 \\ 1 \quad\ \text{ if } x = 0 \text{ and } y = 0. \end{cases}$$

From

$$f(x, 0) + f(y, 0) - f(x, y) - f(0,0) = 1,$$

it results that f is not bidimensionally continuous in (0,0). Therefore the pointwise convergence of a net $\tilde{\pi}$ of continuous functions, is not sufficient to insure the bidimensional continuity of the limit.

Theorem 5. *A net $\tilde{\pi}$, of bidimensionally continuous functions, uniform convergent, have, as limit a bidimensional continuous function.*

Proof. From the properties of the linear topological spaces [14] it results

$$(14)\qquad \underset{V\in\mathcal{V}(0)}{\forall}\ \underset{W\in\mathcal{V}(0)}{\exists} \Rightarrow -W - W + W + W + W \subset V.$$

From the fact that the net $\tilde{\pi}$ is uniform convergent, it results

$$W \in \mathcal{V}(\theta)\ \underset{a_0\in A}{\exists}\ \underset{a\in A}{\forall}\ a_0 \leqslant a \quad \underset{(x,y)\in X\times Y}{\forall} \Rightarrow f_a(x,y) - f(x,y) \in W.$$

Therefore

$$(15)\qquad \begin{matrix} f_a(x,y_0)-f(x,y_0)\in W & f_a(x,y)-f(x,y)\in W \\ f_a(x_0,y)-f(x_0,y)\in W & f_a(x_0,y_0)-f(x_0,y_0)\in W. \end{matrix}$$

From the bidimensional continuity of f_{a_0} in (x_0,y_0) it results

$$(16)\qquad \underset{W\in\mathcal{V}(0)}{\forall}\ \underset{U\in\mathcal{V}(x_0,y_0)}{\exists}\ \underset{(x,y)\in U}{\forall} \Rightarrow f_{a_0}(x_0,y)+f_{a_0}(x,y_0)-f_{a_0}(x,y)-f_{a_0}(x_0,y_0)\in W.$$

But

$$f(x_0,y)+f(x,y_0)-f(x,y)-f(x_0,y_0)=[f(x,y_0)-f_{a_0}(x,y_0)]+[f(x_0,y)-$$
$$-f_{a_0}(x_0,y)]+[f_{a_0}(x,y)-f(x,y)]+[f_{a_0}(x_0,y_0)-f(x_0,y_0)]+[f_{a_0}(x_0,y)+$$
$$+f_{a_0}(x,y_0)-f_{a_0}(x,y)-f_{a_0}(x_0,y_0)].$$

As $a_0\leqslant a_0$, from (15) and (16) it results:

$$f(x_0,y)+f(x,y_0)-f(x,y)-f(x_0,y_0)\in -W-W+W+W+W.$$

Adding (1) we get

$$\underset{V\in\mathcal{V}(0)}{\forall}\ \underset{U\in\mathcal{V}(x_0,y_0)}{\exists}\ \underset{(x,y)\in U}{\forall} \Rightarrow f(x_0,y)+f(x,y_0)-f(x,y)-f(x_0,y_0)\in V;$$

therefore f is bidimensionally continuous in (x_0,y_0).

Example 4. Let us consider the directed set $(R,\leqslant)$ and the function

$$\alpha(u)=\begin{cases}\dfrac{1}{u} & \text{if } u\neq 0\\ 0 & \text{if } u=0\end{cases}\qquad u\in R.$$

Let the net $\tilde{\pi}$

$$\tilde{\pi}=\left\{f_a:\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]\to R;\ a\in R\right\}$$

be defined by

$$f_a(x,y)=(x+y)^a-1+\frac{a(1-x-y)}{1+a(1-x-y)}+\alpha(x)+\alpha(y).$$

It is easily seen that the functions of the net $\tilde{\pi}$, are bidimensionally continuous in $[0,1/2]\times[0,1/2]$.

One can easily show that the net $\tilde{\pi}$ converges pointwise to

$$f(x,y)=\alpha(x)+\alpha(y)$$

which is bidimensionally continuous.

If

$$x=\frac{1}{2}-\frac{1}{2a}\in\left[0,\frac{1}{2}\right];\quad y=\frac{1}{2}-\frac{1}{2a}\in\left[0,\frac{1}{2}\right],$$

it results

$$|f_a(x,y)-f(x,y)|>\frac{1}{2}-\frac{1}{e}.$$

Therefore the net $\tilde{\pi}$, does not converge uniformly.

Thus the uniform convergence is not necessary for a net $\tilde{\pi}$ of bidimensionally continuous functions, to have as limit a bidimensionally continuous function.

Theorem 6. *A quasi-uniform convergent net $\tilde{\pi}$ of bidimensionally continuous functions, has as limit a function.*

$$f\colon X\times Y\to E,$$

which is bidimensionally continuous.

Proof. From the properties of the linear topological spaces, we have

$$(14)\qquad \underset{V\in\mathcal{V}(0)}{\forall}\ \underset{W\in\mathcal{V}(0)}{\exists}\Rightarrow -W-W+W+W+W\subset V.$$

From the quasi-uniform convergence of $\tilde{\pi}$ to f, it results (π_7)

$$(17)\qquad \underset{W\in\mathcal{V}(0)}{\forall}\ \underset{(B,\leqslant')}{\exists}\ \underset{h:\ B\to A}{\exists}\ \underset{\tilde{\mathcal{G}}=\{G_b\}_{b\in B}}{\exists}\ \bigcup_{b\in B}G_b=X\times Y\ \underset{(x,y)\in G_b}{\forall}\Rightarrow f_{h(b)}(x,y)-$$
$$-f(x,y)\in W,$$

where $(B,\leqslant')$, h, $\tilde{\mathcal{G}}$, have the corresponding meanings. Let be

$$(x_0,y_0)\in X\times Y.$$

Therefore

$$\underset{b_0\in B}{\exists}\Rightarrow (x_0,y_0)\in G_{b_0}.$$

But as G_{b^0} is an open set, it results

$$G_{b_0}\in\mathcal{V}(x_0,y_0).$$

From the properties of the product space of topological spaces [11], it results

$$\underset{U_1\in\mathcal{V}(x_0)}{\exists}\ \underset{U_2\in\mathcal{V}(y_0)}{\exists}\Rightarrow U'=U_1\times U_2\in\mathcal{V}(x_0,y_0)\ \&\ U'\subset G_{b_0}.$$

From (17) it follows

$$\exists_{(x,y)\in U'} \Rightarrow f_{h(b_0)}(x,y) - f(x,y) \in W,$$

or

$$(18)\qquad \begin{array}{ll} f_{h(b_0)}(x_0,y) - f(x_0,y) \in W, & f_{h(b_0)}(x,y) - f(x,y) \in W, \\ f_{h(b_0)}(x,y_0) - f(x,y_0) \in W, & f_{h(b_0)}(x_0,y_0) - f(x_0,y_0) \in W. \end{array}$$

From the bidimensional continuity of $f_{h(b_0)}$ in (x_0, y_0) it results

$$(19)\qquad W \in \mathcal{V}(\theta)\ \exists_{U''\in\mathcal{V}(x_0,y_0)}\ \forall_{(x,y)\in U''} \Rightarrow f_{h(b_0)}(x_0,y) + f_{h(b_0)}(x,y_0) - f_{h(b_0)}(x,y) -$$

$$- f_{h(b_0)}(x_0,y_0) \in W.$$

But

$$f(x_0,y) + f(x,y_0) - f(x,y) - f(x_0,y_0) = [f(x_0,y) - f_{h(b_0)}(x_0,y)] + [f(x,y_0) -$$

$$- f_{h(b_0)}(x,y_0)] + [f_{h(b_0)}(x,y) - f(x,y)] + [f_{h(b_0)}(x_0,y_0) - f(x_0,y_0)] + [f_{h(b_0)}(x_0,y)$$

$$+ f_{h(b_0)}(x,y_0) - f_{h(b_0)}(x,y) - f_{h(b_0)}(x_0,y_0)].$$

Denoting

$$U = U' \cap U'' \in \mathcal{V}(x_0,y_0),$$

from (18), (19) and due to (14) we have

$$\forall_{(x,y)\in U} \Rightarrow f(x_0,y) + f(x,y_0) - f(x,y) - f(x_0,y_0) \in V;$$

therefore f is bidimensionally continuous in (x_0, y_0).

Example 5. Let be

$$\varphi(x,y) = \begin{cases} 1 & \text{if } x, y \in Q, \\ \dfrac{1}{2} & \text{if } x \in Q;\ y \in R\setminus Q, \\ \dfrac{1}{2} & \text{if } x \in R\setminus Q;\ y \in Q, \\ 1 & \text{if } x, y \in R\setminus Q. \end{cases}$$

Let the net $\tilde{\pi}$

$$\tilde{\pi} = \{f_a\colon R\times R \to R;\ a \in R\}$$

be defined by

$$f(x,y) = \frac{1}{a}\,\varphi(x,y).$$

It is easily checked that although the functions of the net are not bidimensionally continuous, the net uniformly converges to the bidimensionally continuous function

$$f(x) = 0.$$

Therefore it is natural to look for necessary and sufficient conditions that a net of functions, not necessarly bidimensionally continuous converge to a bidimensionally continuous function.

Theorem 7. *Let* $\tilde{\pi}$ *be a net which is uniformly convergent to*

$$f\colon X\times Y \to E.$$

The necessary and sufficient condition for f *to be bidimensionally continuous is*

$$\forall_{(x_0,y_0)\in X\times Y}\ \forall_{V\in\mathcal{V}(\theta)}\ \forall_{a\in A}\ \exists_{a_1\in A}\ a \leq a_1\ \exists_{U\in\mathcal{V}(x_0,y_0)}\ \forall_{(x,y)\in U} \Rightarrow f_{a_1}(x_0,y) + f_{a_1}(x,y_0) -$$

$$- f_{a_1}(x,y) - f_{a_1}(x_0,y_0) \in V.$$

Proof. Let us suppose that the net $\tilde{\pi}$ is uniformly convergent to f and f bidimensionally continuous. From the properties of the linear topological spaces, we have

$$(20)\qquad \forall_{V\in\mathcal{V}(\theta)}\ \exists_{W\in\mathcal{V}(\theta)} \Rightarrow W + W - W - W + W \subset V.$$

From the uniform convergence of $\tilde{\pi}$ to f, it results

$$(21)\qquad W \in \mathcal{V}(\theta)\ \exists_{a_0\in A}\ \forall_{a\in A}\ a_0 \leq a\ \forall_{(x,y)\in X\times Y} \Rightarrow f_a(x,y) - f(x,y) \in W.$$

From the fact that $(A, \leq)$ is a directed set, it results:

$$\forall_{a\in A}\ \exists_{a_1\in A} \Rightarrow a \leq a_1,\ \&\ a_0 \leq a_1$$

From $a_0 \leq a_1$ it results in agreement with (21)

$$(22)\qquad \begin{array}{ll} f_{a_1}(x_0,y) - f(x_0,y) \in W, & f_{a_1}(x,y) - f(x,y) \in W, \\ f_{a_1}(x,y_0) - f(x,y_0) \in W, & f_{a_1}(x_0,y_0) - f(x_0,y_0) \in W. \end{array}$$

From the bidimensional continuity of f in (x_0, y_0) we obtain:

$$(23)\qquad W \in \mathcal{V}(\theta)\ \exists_{U\in\mathcal{V}(x_0,y_0)}\ \forall_{(x,y)\in U} \Rightarrow f(x_0,y) + f(x,y_0) - f(x,y) - f(x_0,y_0) \in W.$$

But as

$$f_{a_1}(x_0,y) + f_{a_1}(x,y_0) - f_{a_1}(x,y) - f_{a_1}(x_0,y_0) = [f_{a_1}(x_0,y) - f(x_0,y)] + [f_{a_1}(x,y_0) -$$

$$- f(x,y_0)] + [f(x,y) - f_{a_1}(x,y)] + [f(x_0,y_0) - f_{a_1}(x_0,y_0)] + [f(x_0,y) +$$

$$+ f(x,y_0) - f(x,y) - f(x_0,y_0)]$$

from (22), (23) and (20), it results

$$\underset{(x,y)\in U}{\forall} \Rightarrow f_{a_1}(x, y_0) + f_{a_1}(x_0, y) - f_{a_1}(x, y) - f_{a_1}(x_0, y_0) \in V$$

the theorem.

Conversely, let $\tilde{\pi}$ be uniform convergent to f and the hypothesis of the theorem satisfied.

From the properties of the linear topological spaces it results:

$$(24) \qquad \underset{V\in\mathcal{V}(0)}{\forall} \ \underset{W\in\mathcal{V}(0)}{\exists} \Rightarrow -W - W + W + W + W \subset V$$

From the uniforn convergence of $\tilde{\pi}$, it results:

$$(25) \qquad W \in \mathcal{V}(0) \ \underset{a_0\in A}{\exists} \ \underset{\bar{a}\in A}{\forall} \ a_0 \leqslant \bar{a} \ \underset{(x,y)\in X\times Y}{\forall} \rightarrow f_{\bar{a}}(x,y) - f(x,y) \in W.$$

From the hypothesis of the theorem, it results:

$$(26) \qquad W \in \mathcal{V}(\theta) \underset{(x_0,y_0)\in X\times Y}{\forall} \ a_0 \in A \ \underset{a_1\in A}{\exists} \ a_0 \leqslant a_1 \underset{U\in\mathcal{V}(x_0,y_0)}{\exists} \ \underset{(x,y)\in U.}{\forall} \Rightarrow f_{a_1}(x_0, y) +$$

$$+ f_{a_1}(x, y_0) - f_{a_1}(x, y) - f_{a_1}(x_0, y_0) \in W.$$

But from (25) we have

$$a_0 \leqslant a_1 \Rightarrow \underset{(x,y)\in X\times Y}{\forall} f_{a_1}(x, y) - f(x, y) \in W$$

or

$$(27) \qquad \begin{array}{ll} f_{a_1}(x_0, y) - f(x_0, y) \in W, & f_{a_1}(x, y) - f(x, y) \in W, \\ f_{a_1}(x, y_0) - f(x, y_0) \in W, & f_{a_1}(x_0, y_0) - f(x_0, y_0) \in W. \end{array}$$

As

$$f(x_0, y) + f(x, y_0) - f(x, y) - f(x_0, y_0) = [f(x_0, y) - f_{a_1}(x_0, y)] + [f_{a_1}(x, y_0) - f(x, y_0)] + [f_{a_1}(x, y) - f(x, y)] + [f_{a_1}(x_0, y_0) - f(x_0, y_0)] + [f_{a_1}(x_0, y) - f_{a_1}(x, y_0) - f_{a_1}(x, y) - f_{a_1}(x_0, y_0)].$$

From (26), (27) and (24) it results

$$\underset{(x,y)\in U}{\forall} \Rightarrow f(x_0, y) + f(x, y_0) - f(x, y) - f(x_0, y_0) \in V;$$

therefore f is bidimensionally continuous in (x_0, y_0).

Theorem 8. *Let the net* $\tilde{\pi}$ *be quasi-uniform convergent to*

$$f\colon X \times Y \rightarrow E$$

and sufficient condition for f *to be bidimensionally continuous is*

$$\underset{(x_0,y_0)\in X\times Y}{\forall} \ \underset{V\in\mathcal{V}(0)}{\forall} \ \underset{a\in A}{\forall} \ \underset{a_1,a_2\in A}{\exists} \ a \leqslant a_2 \ \underset{U\in\mathcal{V}(x_0,y_0)}{\exists} \ \underset{(x,y)\in U}{\forall} \Rightarrow f_{a_1}(x, y) - f(x, y) \in V$$

$$\& \ f_{a_1}(x_0, y) + f_{a_1}(x, y_0) - f_{a_1}(x, y) - f_{a_1}(x_0, y_0) \in V.$$

Proof. From the properties of the linear topological spaces, we have

$$(28) \qquad \underset{V\in\mathcal{V}(0)}{\forall} \ \underset{W\in\mathcal{V}(0)}{\exists} \Rightarrow W + W - W + W - W \subset V.$$

From the quasi-uniform convergence of the net $\tilde{\pi}$ to f, results

$$(29) \qquad W \in \mathcal{V}(0) \underset{(B,\leqslant')}{\exists} \ \underset{h\colon B\rightarrow A}{\exists} \ \underset{\tilde{\mathcal{G}}=\{G_b\}_{b\in B}}{\exists} \ \bigcup_{b\in B} G_b = X \times Y \ \underset{(x,y)\in G_b}{\forall} \Rightarrow f_{h(b)}(x, y) -$$

$$f(x, y) \in W,$$

where $(B, \leqslant')$, h, $\tilde{\mathcal{G}}$ have the corresponding meanings. But

$$\bigcup_{b\in B} G_b = X \times Y \Rightarrow \underset{b_0\in B}{\exists} (x_0, y_0) \in G_{b_0}.$$

Let be

$$a_1 = h(b_0) \in A.$$

As G_{b_0} is an open set in the product topology, we have

$$G_{b_0} \in \mathcal{V}(x_0, y_0),$$

and

$$\underset{U_1\in\mathcal{V}(x_0)}{\exists} \ \underset{U_2\in\mathcal{V}(y_0)}{\exists} \Rightarrow U' = U_1 \times U_2 \in \mathcal{V}(x_0, y_0) \ \& \ U' \subset G.$$

From (29) it results

$$\underset{(x,y)\in U'}{\forall} \Rightarrow f_{a_1}(x, y) - f(x, y) \in W;$$

therefore

$$(30) \qquad \begin{array}{ll} f_{a_1}(x_0, y) - f(x_0, y) \in W, & f_{a_1}(x, y) - f(x, y) \in W, \\ f_{a_1}(x, y_0) - f(x, y_0) \in W, & f_{a_1}(x_0, y_0) - f(x_0, y_0) \in W. \end{array}$$

From the bidimensional continuity of f in (x_0, y_0), it results

$$(31) \qquad W \in \mathcal{V}(0) \underset{U''\in\mathcal{V}(x_0,y_0)}{\exists} \ \underset{(x,y)\in U''}{\forall} \Rightarrow f(x_0, y) + f(x, y_0) - f(x, y) - f(x_0, y_0) \in W.$$

From the quasi-uniform convergence of the net $\tilde{\pi}$, it results its pointwise convergence. Therefore

(32) $(x_0, y_0) \in X \times Y; W \in \mathcal{V}(0) \ \exists_{a_0 \in A} \forall_{\bar{a} \in A} \ a_0 \leq \bar{a} \Rightarrow f_{\bar{a}}(x_0, y_0) - f(x_0, y_0) \in W.$

From the fact that $(A, \leq)$ is a directed set, it results

(33) $$\forall_{a \in A} \exists_{a_2 \in A} \Rightarrow a_0 \leq a_2 \ \& \ a \leq a_2.$$

But from (32)

$$a_0 \leq a_2 \Rightarrow f_{a_2}(x_0, y_0) - f(x_0, y_0) \in W.$$

because

$$f_{a_1}(x_0, y) + f_{a_1}(x, y_0) - f_{a_1}(x, y) - f_{a_2}(x_0, y_0) = [f_{a_1}(x_0, y) - f(x_0, y)] + [f_{a_1}(x, y_0) - f(x, y_0)] + [f(x, y) - f_{a_1}(x, y)] + [f(x_0, y_0) - f_{a_2}(x_0, y_0)] + [f(x_0, y) + f(x, y_0) - f(x, y) - f(x_0, y_0)].$$

But in this case from (30), (31), (33) and (28) it results, if

$$U = U' \cap U'' \in \mathcal{V}(x_0, y_0)$$

that

$$\forall_{(x, y) \in U} \Rightarrow f_{a_1}(x_0, y) + f_{a_1}(x, y_0) - f_{a_1}(x, y) - f_{a_2}(x_0, y_0) \in V.$$

And from (30)

$$\forall_{(x,y) \in U} \Rightarrow f_{a_1}(x, y) - f(x, y) \in W \subset V.$$

therefore the hypothesis of the theorem. Conversely, let us suppose that the net $\tilde{\pi}$ is quasi-uniform convergent to f and the theorem are satisfied. From the properties of the linear topological spaces, it results

(34) $$\forall_{V \in \mathcal{V}(0)} \exists_{W \in \mathcal{V}(0)} \Rightarrow - W - W + W + W + W \subset V.$$

The net $\tilde{\pi}$ being quasi-uniform convergent, is pointwise convergent to f. Therefore

(35) $(x_0, y_0) \in X \times Y, W \in \mathcal{V}(0) \ \exists_{a_0 \in A} \forall_{\bar{a} \in A} \ a_0 \leq \bar{a} \Rightarrow f_{\bar{a}}(x_0, y_0) - f(x_0, y_0) \in W.$

From the hipothesis of the theorem, it results

(36) $W \in \mathcal{V}(0), (x_0, y_0) \in X \times Y \ \ a_0 \in A \ \exists_{a_1, a_2 \in A} \ a_0 \leq a_2 \ \exists_{U' \in \mathcal{V}(x_0, y_0)} \forall_{(x,y) \in U'} \Rightarrow$

$$f_{a_1}(x, y) - f(x, y) \in W \ \& \ f_{a_1}(x_0, y) + f_{a_1}(x, y_0) - f_{a_1}(x, y) - f_{a_2}(x_0, y_0) \in W.$$

As

$$U' \in \mathcal{V}(x_0, y_0),$$

in the product topology of $X \times Y$, we have

$$\exists_{U_1 \in \mathcal{V}(x_0)} \exists_{U_2 \in \mathcal{V}(y_0)} \Rightarrow U = U_1 \times U_2 \in \mathcal{V}(x_0, y_0) \ \& \ U \subset U'.$$

From (36), it results:

$$\forall_{(x,y) \in U} f_{a_1}(x, y) - f(x, y) \in W;$$

or

(37) $$f_{a_1}(x_0, y) - f(x_0, y) \in W, \quad f_{a_1}(x, y) - f(x, y) \in W,$$
$$f_{a_1}(x, y_0) - f(x, y_0) \in W, \quad f_{a_1}(x_0, y_0) - f(x_0, y_0) \in W,$$
$$f_{a_1}(x, y_0) + f_{a_1}(x_0, y) - f_{a_1}(x, y) - f_{a_2}(x_0, y_0) \in W,$$

as

$$a_0 \leq a_2.$$

From (35), it results:

(38) $$f_{a_2}(x_0, y_0) - f(x_0, y_0) \in W,$$

because

$$f(x_0, y) + f(x, y_0) - f(x, y) - f(x_0, y_0) = [f(x_0, y) - f_{a_1}(x_0, y)] + [f(x, y_0) - f_{a_1}(x, y_0)] + [f_{a_1}(x, y) - f(x, y)] + [f_{a_1}(x_0, y) + f_{a_1}(x, y_0) - f_{a_1}(x, y) - f_{a_2}(x_0, y_0)] + [f_{a_2}(x_0, y_0) - f(x_0, y_0)].$$

In these conditions, from (37), (38), and (34) it results

$$\forall_{(x,y) \in U} \Rightarrow f(x_0, y_0) + f(x, y_0) - f(x, y) - f(x_0, y_0) \in V,$$

therefore the bidimensional continuity of f in (x_0, y_0).

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ŞIRURI GENERALIZATE DE FUNCŢII

Rezumat

Se consideră şirurile generalizate de funcţii:

$$\pi = \{f_a : X \to Z;\ a \in A\},$$

$$\pi = \{f_a : X \times Y \to E,\ a \in A\}$$

unde X şi Y sînt spaţii topologice, Z un spaţiu uniform, E un spaţiu vectorial topologic, $(A, \leqslant)$ o mulţime dirijată. Se găsesc condiţii necesare şi suficiente pentru ca limita să fie continuă sau bidimensional continuă.

A RESULT ON GENERALIZED HYPERGEOMETRIC FUNCTION AND GENERALIZED MEIJER FUNCTION OF TWO VARIABLES

BY

MANILAL SHAH

The object of the present paper is to evaluate an infinite integral involving product of hypergeometric function and generalized Meijer function of two variables with the help of known expansion and integral formulae. On specializing the parameters, the function $S\begin{bmatrix} x \\ y \end{bmatrix}$ incorporates not only Meijer's G-function, or the product of two G-functions, as its particular cases, but it also includes most of the commonly used functions in two arguments e.g. Kampé de Fériet's [5] double hypergeometric function F which in turn yields the Appell functions [2] F_1, F_2, F_3 and F_4, the Whittaker function of two variables etc. However, some interesting results are obtained as particular cases of the integral.

1. The integral. The integral to be evaluated is

$$(1.1)\quad \int_0^\infty x^{u+v}\,{}_mF_{n+1}\left[\begin{matrix} \alpha_m \\ \Delta(2,u+1),\ \beta_{n-1}; \end{matrix}\ -4x^2yz^2\right] S\begin{bmatrix} \alpha x^{2\rho} \\ \beta x^{2\rho} \end{bmatrix} dx = \frac{\rho^{v}}{(2z)^{u+v+1}}$$

$$\sum_{\sigma=0}^{\infty} \frac{(u/2+\sigma)\,\Gamma(u+\sigma)}{\sigma!}\ {}_{m+2}F_{n+1}\left[\begin{matrix} -\sigma,\ u+\sigma,\ \alpha_m;\ y \\ \Delta(2,u+1),\ \beta_{n-1} \end{matrix}\right]$$