pour chaque $x = \sum_{n \in N} x_n \in \sum_{n \in N} X_n$. En effet, les seminormes de $\mathcal{P}$ sont évidemment continues pour la topologie de la somme directe et inversement, si $|\,.\,|$ est une seminorme continue pour la somme directe, alors les restrictions sont continues ; donc pour chaque $n \in N$ il y a $i_n \in I_n$ et $k_n > 0$ tels que :

$$|x| \leq k_n \, |x|_{i_n}$$

pour chaque $x \in X_n$, et

$$|x| = |\sum_{n \in N} x_n| \leq \sum_{n \in N} k_n |x_n|_{i_n} \leq (\sum_{n \in N} 2^{-n})^{1/2} (\sum_{n \in N} 2^n \, k_n^2 \, |x_n|_{i_n}^2)^{1/2} = |x|_{i,c}$$

pour chaque $x = \sum_{n \in N} x_n \in \sum_{n \in N} X_n$, où $i = (i_n) \in \prod_{n \in N} I_n$, et $c = (2^{n/2} k_n) \in s$ c'est-à-dire la seminorme $|\,.\,|$ est continue pour la topologie H-localement convexe engendrée par $\mathcal{P}$. On voit que l'affirmation b) a lieu parceque chaque seminorme bornée sur les ensembles bornés de $X = \sum_{n \in N} X_n$ est continue. Enfin, la dernière assertion du théorème résulte aussitôt de a) et b).

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SPAŢII H-BORNOLOGICE

Rezumat

Prin analogie cu spaţiile bornologice introduse pentru clasa spaţiilor liniare topologice local convexe, se introduc în această Notă spaţiile H-bornologice pentru clasa spaţiilor H-local convexe. Pentru aceste tipuri de spaţii se stabileşte o teoremă de caracterizare cu mai multe condiţii echivalente (teorema 1) şi se pun în evidenţă cîteva proprietăţi de stabilitate (teorema 2).

ALMOST PERIODIC SOLUTIONS BY EXPONENTIAL CONVERGENCE

BY

IOAN MUNTEAN

1. We discuss in this note the system of n ordinary differential equations

$$\dot{x} = X(t, x), \tag{1}$$

where $\dot{x} = dx/dt$, the column vector $x = \text{col}\,(x_1, \ldots, x_n)$ runs over R^n and $X(t, x)$ is a given column vector function $\text{col}\,(X_1(t, x), \ldots, X_n(t, x))$. Assume that $X_i : [0, +\infty[\times R^n \to R$, $i = 1, \ldots, n$, are continuous together with their partial derivatives $X_{ij} = \partial X_i / \partial x_j$, $i, j = 1, \ldots, n$. Solutions of (1) are said to be *exponentially convergent* if there exist positive numbers c and L such that for each pair of solutions $x(t)$, $y(t)$ defined on the interval $[0, +\infty[$ we have

$$|x(t) - y(t)| \leqslant L \, |x(0) - y(0)| \, e^{-ct} \quad \text{for } t \geqslant 0, \tag{2}$$

where $|\cdot|$ is the Euclidean norm in R^n.

Remark. Exponential convergence of solutions implies the uniqueness of periodic or almost periodic solutions of (1).

Denote by M^* the transpose matrix of a matrix M and by J the Jacobian functional matrix $\partial X / \partial x = (X_{ij})_{i,j = 1,2, \ldots, n}$. The following theorem on exponential convergence is known ([4], Theorem 1):

Theorem A. *Suppose that there exist a number $\varepsilon > 0$ and a $n \times n$ positive definite constant symmetric matrix Q such that all eigenvalues of the matrix $J^*Q + QJ$ are $\leqslant -\varepsilon$. Then the solutions of* (1) *are exponentially convergent.*

The hypotheses of this theorem imply that for the quadratic form $V(x) = x^*Qx$ one finds

$$a\,|x|^2 \leqslant V(x) \leqslant b\,|x|^2 \quad \text{for } x \in R^n \tag{3}$$

with positive constants a and b, and

$$\dot V(z(t)) \leqslant -\varepsilon\,|z(t)|^2, \tag{4}$$

where $z(t)$ is the difference of any two solutions of (1). Recently, A. M. Fink [2] showed that the existence of a function $V(x)$ verifying (3) and (4) implies not only the uniqueness, but also the existence of a periodic solution of (1) provided $X(t, x)$ is periodic in t with period T *). The same author poses the problem of the existence of almost periodic solutions of (1) if $X(t, x)$ is almost periodic in t. Using the properties of the asymptotic almost periodic functions and the methods of [4] and [5], under the conditions of Theorem A we give an affirmative answer to this problem.

2. To state our result we recall the definition and some properties of asymptotic almost periodic functions (cf. [3], p. 270—275). A continuous function $x : [0, +\infty[\to R^n$ is said to be *asymptotically almost periodic* if, given $\eta > 0$, there exist $t_1 \geqslant 0$ and $l > 0$ such that every interval in $[0, +\infty[$ of length l contains a point τ with $|x(t+\tau) - x(t)| < \eta$ for $t \geqslant t_1$ and $t + \tau \geqslant t_1$. Every asymptotic almost periodic function $x(t)$ may be uniquely expressed in the form $x(t) = \alpha(t) + \omega(t)$, where $\alpha(t)$ is almost periodic and $\omega(t) \to 0$ as $t \to +\infty$; conversely, each continuous function of this form is asymptotically almost periodic. If $x(t) = \alpha(t) + \omega(t)$ and its derivative are asymptotically almost periodic, then $\dot x(t) = \dot\alpha(t) + \dot\omega(t)$.

Also we make use of the following properties of almost periodic functions (cf. [1], p. 39—40): if $X(t, x)$ is almost periodic in t for every x in a ball $S \subset R^n$ and $X(t, x)$ satisfies the Lipschitz condition in $x \in S$ uniformly in t, then $X(t, x)$ is almost periodic in t uniformly in $x \in S$; moreover, if $x(t)$ is almost periodic and $x(t) \in S$, then $X(t, x(t))$ is almost periodic too.

Theorem B. *In addition to the assumptions of Theorem A, suppose that $X(t, x)$ is almost periodic in t for every $x \in R^n$ and that for every ball S in R^n the derivative $\partial X(t, x)/\partial x$ is bounded on $[0, +\infty[\times S$. Then for the solutions of (1) the following properties hold:*

(i) every solution is defined and bounded on $[0, +\infty[$,

(ii) every solution is asymptotically almost periodic,

(iii) there exists a unique almost periodic solution $\alpha(t)$; its derivative $\dot\alpha(t)$ is almost periodic too,

(iv) every other solution $x(t)$ has the form $x(t) = \alpha(t) + \omega(t)$, with $|\omega(t)| \leqslant L\,|\omega(0)|\,e^{-ct}$ for $t \geqslant 0$ and $\dot\omega(t) \to 0$ as $t \to +\infty$.

Proof. (i) Since $k = \sup\{|X(t, 0)|,\ t \geqslant 0\} < +\infty$, for solutions $x(t)$ with $|x(0)| = 2kb\varepsilon^{-1}$ we get $|x(t)| \leqslant (ba^{-1})^{1/2}\ |x(0)|$ | when $t \geqslant 0$ ([4], Theorem 2), thus these solutions are defined and bounded on the whole interval $[0, +\infty[$. Now, by Theorem A, every solution is defined and bounded on $[0, +\infty[$.

(*) In conditions of Theorem A, the requirement of A. M. Fink that (1) possesses a solution defined on an interval larger than period T can be dropped (cf. [4], Theorem 2).

(ii) Let $x(t)$ be a solution and $m \geqslant 0$ a number with $|x(t)| \leqslant m$ for $t \geqslant 0$. Let $\eta > 0$, $t_1 = 16(2 + \sqrt{2})m^2b^2\,(a\varepsilon\eta^2)^{-1} \geqslant 0$ and $S = \{x \in R^n,\ |x| \leqslant m\}$. According to the boundedness of $\partial X(t, x)/\partial x$ on $[0, +\infty[\times S$, $X(t, x)$ satisfies the Lipschitz condition in $x \in S$ uniformly in t, hence $X(t, x)$ is almost periodic in t uniformly in $x \in S$. Thus, for $\delta = 2^{-5/2}\,b^{-2}\,a\varepsilon\eta > 0$, there is an $l > 0$ such that every interval in $[0, +\infty[$ of length l contains a point τ with $|X(t+\tau, x) - X(t, x)| < \delta$ for $t \geqslant 0$ and $x \in S$, so $|X(t+\tau, x(t)) - X(t, x(t))| < \delta$ for $t \geqslant 0$.

For the difference $z(t) = x(t+\tau) - x(t)$ we obtain

$$\dot z(t) = (X(t+\tau, x(t+\tau)) - X(t+\tau, x(t))) + (X(t+\tau, x(t)) - X(t, x(t))) = \int_0^1 \frac{\partial}{\partial x} X(t+\tau, sz(t) + x(t))\, z(t)\, ds + (X(t+\tau, x(t)) - X(t, x, (t))).$$

whence for the derivative of $V(z) = z^*Qz$ along the function $z(t)$ we get

$$\dot V(z(t)) = \int_0^1 z^*(t)\,(J^*Q + QJ)\,z(t)ds + (X(t+\tau, x(t)) - X(t, x(t)))^*\,Qz(t) + z^*(t)Q(X(t+\tau, x(t)) - X(t, x(t))) \leqslant \int_0^1 \sup\,\{\lambda_1(s, t), \ldots, \lambda_n(s, t)\}\,|z(t)|^2\,ds + 2b\delta\,|z(t)| \leqslant -\varepsilon\,|z(t)|^2 + 2b\delta\,|z(t)|,$$

where $\lambda_1(s, t), \ldots, \lambda_n(s, t)$ are the eigenvalues of the matrix $J^*Q + QJ$. Now, setting $p = \varepsilon b^{-1}$ and $q = 2ba_{-}^{1/2}$, from (3) we arrive at the inequality

$$\dot V(z(t)) \leqslant -pV(z(t)) + \delta q(V(z(t)))^{1/2}. \tag{5}$$

At the points $t \geqslant 0$ with $V(z(t)) \geqslant 2q^2p^{-2}\delta^2$ we have $-p(V(z(t)))^{1/2} + \delta q \leqslant (1 - \sqrt{2})\delta q$ and, by (5),

$$\dot V(z(t)) \leqslant (V(z(t)))^{1/2}[-p(V(z(t)))^{1/2} + \delta q] \leqslant \sqrt{2}qp^{-1}\delta(1 - \sqrt{2})\,\delta q = -(2 - \sqrt{2})q^2p^{-1}\delta^2.$$

Therefore, setting $G = 2q^2p^{-2}\delta^2$ and $H = (2 - \sqrt{2})q^2p^{-1}\delta^2$, we have $\dot V(z(t)) \leqslant -H$ at all points $t \geqslant 0$ with $V(z(t)) \geqslant G$. From this it is not hard to see that there is a $t_2 \geqslant 0$ with $V(z(t_2)) < G$ and $V(z(t)) \leqslant G$ for all $t \geqslant t_2$, consequently either $V(z(t)) \leqslant G$ for all $t \geqslant 0$ or there is a unique $t_3 > 0$

with $V(z(t)) > G$ when $t < t_3$ and with $V(z(t)) < G$ when $t > t_3$. If the second case occurs then $t_1 \geqslant t_3$, for $t_1 < t_3$ yields $V(z(t)) > G$ when $t \in [0, t_1]$, hence

$$G < V(z(t_1)) \leqslant V(z(0)) - Ht_1 \leqslant 4m^2b - Ht_1 = 0,$$

which is a contradiction. Therefore, for all $t \geqslant t_1$ we always have $a|z(t)|^2 \leqslant V(z(t)) \leqslant G$, thus

$$|x(t+\tau) - x(t)| = |z(t)| \leqslant a^{-1/2}(V(z(t)))^{1/2} \leqslant a^{-1/2} G^{-1/2} =$$
$$= \sqrt{2}a^{-1/2}\, qp^{-1}\delta = \eta/2 < \eta.$$

(iii) The asymptotic almost periodic solution $x(t)$ may be uniquely expressed in the form $x(t) = \alpha(t) + \omega(t)$, where $\alpha(t)$ is an almost periodic function and $\omega(t) \to 0$ as $t \to +\infty$, thus

$$\dot{x}(t) = X(t, x(t)) = X(t, \alpha(t) + \omega(t)) = X(t, \alpha(t)) + \omega_1(t),$$

where

$$\omega_1(t) = \frac{\partial}{\partial x} X(t, \alpha(t) + \theta(t)\,\omega(t))\,\omega(t), \quad 0 < \theta(t) < 1.$$

But $|\alpha(t) + \theta(t)\,\omega(t)| \leqslant m_1$ and $\partial X(t,x)/\partial x$ is bounded on $[0, +\infty[\times S_1$, where $S_1 = \{x \in R^n, |x| \leqslant m_1\}$, so $\omega_1(t) \to 0$ as $t \to +\infty$. Since $X(t, \alpha(t))$ is almost periodic, the derivative $\dot{x}(t)$ is asymptotically almost periodic and $\dot{x}(t) = \dot{\alpha}(t) + \dot{\omega}(t)$. According to the uniqueness of the representation of aymptotic almost periodic functions, it is necessary to have $\dot{\alpha}(t) = X(t, \alpha(t))$, i. e. $\alpha(t)$ is a solution of (1). By Remark of Section 1, this is the unique almost periodic solution.

(iv) Finally, by (ii) and (iii), for any other solution $x(t)$ we derive $x(t) = \alpha(t) + \omega(t)$, where, in view of (2), $|\omega(t)| = |x(t) - \alpha(t)| \leqslant \leqslant L|\omega(0)|e^{-ct}$, $t \geqslant 0$. Since $\dot{x}(t) = \dot{\alpha}(t) + \dot{\omega}(t)$ is asymptotically almost periodic, we have $\dot{\omega}(t) \to 0$ as $t \to +\infty$. This completes the proof of Theorem B.

Remark 1. In more general conditions a related result of existence of almost periodic solutions has been given by W. A. Coppel ([1], Theorem 5). Under our conditions, which are expressed directly on the right hand of (1) and do not involve any data on the solutions of (1), we establish at the same time the existence and uniqueness of the almost periodic solution and the form and behaviour at $+\infty$ of all solutions of (1).

Remark 2. If the system (1) has the particular form considered by A. M. Fink

$$\dot{x} = e(t) + Y(x), \tag{1'}$$

where $e(t)$ is almost periodic and $Y(x)$ is continuous together with their partial derivatives of the first order, the following corollary can be stated :

Corollary. *If the hypotheses of Theorem A hold, then the solutions of* (1′) *possess all properties in Theorem B.*

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SOLUȚII APROAPE PERIODICE CU CONVERGENȚĂ EXPONENȚIALĂ

Rezumat

Pentru sistemul de ecuații diferențiale $\dot{x} = X(t, x)$, cu $X(t, x)$ aproape periodică în t pentru fiecare x, se dau condiții care asigură existența și unicitatea unei soluții aproape periodice și se stabilește forma și comportarea la $+\infty$ a tuturor soluțiilor acestui sistem.