ASYMPTOTIC EQUIVALENCE AND FUNCTIONAL DIFFERENTIAL SYSTEMS

BY

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1. Introduction: Recently, Marlin and Struble [3] investigated the problem of asymptotic equivalence of two non-linear differential systems, using a variant of the non-linear variation of constants formula [2, Theorem 2. 6. 3 and 2. 6. 4. pp. 78]. The variant of this formula employed there leads to an improper non-linear integral equation and the important aspect of the result is to prove that the solution of the improper integral equation satisfies the relevant differential equation.

In this paper, we wish to investigate the same problem when one of the equations involved is a functional differential system. The significance of this work is the solutions of the systems as well as the solution of the integral equation need not be bounded. At the end, we give an example in order to show the advantage of our hypotheses.

2. Notations and Definitions: Let R^+ denote the interval $[0, \infty)$, let R^n denote the n-dimensional real Euclidean space and for $\tau \geqslant 0$, let $\mathcal{C}^n = C[[-\tau, 0], R^n]$ denote the space of continuous functions with domain $[-\tau, 0]$ and range in R^n. For any element $\Phi \in \mathcal{C}^n$, define the norm

$$\|\Phi\|_0 = \sup_{-\tau \leqslant s \leqslant 0} \|\Phi(s)\|,$$

where $\|\cdot\|$ is any convenient norm in R^n. Suppose that $x \in C[[-\tau, \infty), R^n]$. For any $t \geqslant 0$, we shall let x_t denote the element of $\mathcal{C}^n$ defined by

$$x_t(s) = x(t+s), \quad -\tau \leqslant s \leqslant 0.$$

Let us consider the real vector space B of all continuous functions from $[t_0 - \tau, \infty)$ into R^n, the topology on B being that induced by the family of pseudo-norms $\{\rho_n(x)\}_{n=1}^{\infty}$, where for $x \in B$,

$$\rho_n(x) = \sup_{t_0 - \tau \leq t \leq n} \| x(t) \|.$$

Note that under this topology, B is a complete, locally convex, linear topological space. With this notation, we write the following differential systems

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{2.1}$$

and

$$y' = F(t, y_t), \quad y_{t_0} = \Phi_0, \tag{2.2}$$

where $f \in C[R^+ \times R^n, R^n]$, $F \in C[R^+ \times \mathcal{C}^n, R^n]$. Furthermore, we assume that $(\partial/\partial x) f(t, x)$ exists and is continuous for $(t, x) \in R^+ \times R^n$.

Notice that the functional differential system (2.2) may be considered as a perturbation of (2.1), since it may be written in the form

$$y' = f(t, y) + R(t, y, y_t), \quad y_{t_0} = \Phi_0, \tag{2.3}$$

where $R(t, y, y_t) = F(t, y_t) - f(t, y)$. We are going to utilize this remark in our considerations subsequently.

Let $x(t, t_0, x_0)$ and $y(t_0, \Phi_0)(t)$ denote any solutions of (2.1) and (2.2) respectively. Let $\Phi(t, t_0, x_0)$ denote the fundamental matrix solution of the variational system

$$z' = f_x(t, x(t, t_0, x_0)) z \tag{2.4}$$

with respect to the solution of (2.1) such that

$$\Phi(t_0, t_0, x_0) = I \text{ (Identity matrix)}.$$

We recall that

$$\Phi(t, t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial x_0},$$

and

$$-\Phi(t, t_0, x_0) \cdot f(t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial t_0}.$$

3. Main Results: Here, we shall show that for a given solution $x(t) = x(t, t_0, x_0)$ of (2.1), there exists a solution $y(t_0, \Phi_0)(t)$ of (2.2) such that

$$\lim_{t \to \infty} [y(t_0, \Phi_0)(t) - x(t, t_0, x_0)] = 0.$$

To achive this, we first prove that the improper functional non-linear integral equation

$$y(t) = x(t) - \int_t^\infty \Phi(t, s, y(s)) R(s, y(s), y_s) ds \tag{3.1}$$

has a solution. Then we prove that any solution of (3.1) is also a solution of (2.2).

We have the following existence theorem.

Theorem 3.1: *Assume that the following conditions with respect to the improper functional non-linear integral equation* (3.1) *hold*:

(i) $x \in B$;

(ii) $\Phi(t, s, y(s))$ *is defined and continuous on* $\Delta' = R^+ \times R^+ \times R^n$, $y \in B$;

(iii) *the mapping* $x \to R(t, x(t), x_t) = F(t, x_t) - f(t, x(t))$ *is a continuous mapping from* B *into* B;

(iv) *there exists a non-negative continuous function* g *defined on* R^+ *into* R^+ *such that* $g(t) \to 0$ *as* $t \to \infty$; *furthermore, for* $0 \leq t \leq T$ *for any* $T \geq 0$ *and* $\| x(t) - z(t) \| \leq g(t)$, *the following inequality is satisfied*

$$\int_T^\infty \| \Phi(t, s, z(s)) R(s, z(s), z_s) \| ds \leq g(T).$$

Then, there exists at least one solution $y \in B$ *of the equation* (3.1) *such that*

$$\| x(t) - y(t) \| \leq g(t), \quad t \in R^+.$$

Proof: Define

$$A = \{ z \in B : \| x(t) - z(t) \| \leq g(t), \ t \in R^+ \}.$$

It is easy to see that A is a closed and convex subset of B. Define

$$(Uz)(t) = x(t) - \int_t^\infty \Phi(t, s, z(s)) R(s, z(s), z_s) ds, \quad t \in R^+ \quad \text{for} \quad z \in A.$$

We will show that

(I) $UA \subset A$, (II) U is continuous, (III) $\overline{UA}$ is compact. From the definition of U and in view of the hypothesis (iv), we have

$$\| (Uz)(t) - x(t) \| \leq \int_t^\infty \| \Phi(t, s, z(s)) R(s, z(s), z_s) \| ds \leq g(t),$$

which implies that $Uz \in A$ and hence (I) is verified.

Now consider $y^n \to y$, whenever $y^n, y \in A$ for $n = 1, 2, \ldots$. We will show that $Uy^n \to Uy$ for every compact interval of R^+. From the definition of U, we have

$$\| (Uy^n)(t) - (Uy)(t) \| \leqslant \int_t^\infty \| \Phi(t, s, y^n(s))\, R(s, y^n(s), y^n_s) -$$

$$(3.2) \qquad - \Phi(t, s, y(s))\, R(s, y(s), y_s) \| \, ds \leqslant \int_t^T \| \Phi(t, s, y^n(s))\, R(s, y^n(s), y^n_s) -$$

$$- \Phi(t, s, y(s))\, R(s, y(s), y_s) \| \, ds + \int_T^\infty \| \Phi(t, s, y^n(s))\, R(s, y^n(s), y^n_s) \| \, ds +$$

$$+ \int_T^\infty \| \Phi(t, s, y(s))\, R(s, y(s), y_s) \| \, ds.$$

Choose T so large that for $t_0 \leqslant t \leqslant T$,

$$\int_T^\infty \| \Phi(t, s, y^n(s)) R(s, y^n(s), y^n_s) \| \, ds < \varepsilon/3 \quad \text{and}$$

$$\int_T^\infty \| \Phi(t, s, y(s))\, R(s, y(s), y_s) \| \, ds < \varepsilon/3 \quad \text{for any} \quad \varepsilon > 0.$$

This is possible in view of the condition (iv). By uniform continuity of Φ and R the remaining term in (3.2) can be made less than $\varepsilon/3$, whenever $y^n \to y$. Hence U is continuous on A into B.

If $t > T$, then continuity of U follows from the condition (iv) of the theorem. Finally we will show that $\bar{U}A$ is compact. Let

$$(Uy)(t) - (Uy)(t_1) = x(t) - x(t_1) + \int_{t_1}^\infty \Phi(t_1, s, y(s))\, R(s, y(s), y_s)\, ds -$$

$$(3.3) \qquad - \int_t^\infty \Phi(t, s, y(s))\, R(s, y(s), y_s) ds = x(t) - x(t_1) + \int_{t_1}^t \Phi(t, s, y(s))\, R(s, y(s),$$

$$y_s) ds + \int_{t_1}^\infty [\Phi(t_1, s, y(s)) - \Phi(t, s, y(s))]\, R(s, y(s), y_s) ds.$$

For any interval $[t_0, T]$, we have for any $\varepsilon > 0$ and sufficiently large T_1 such that $T_1 \geqslant T$,

$$\int_{T_1}^\infty \| \Phi(t_1, s, y(s))\, R(s, y(s), y_s) \| \, ds < \varepsilon/5,$$

$$\int_{T_1}^\infty \| \Phi(t, s, y(s))\, R(s, y(s), y_s) \| \, ds < \varepsilon/5$$

and $\| x(t) - x(t_1) \| < \varepsilon/5$, whenever $|t - t_1| \leqslant \delta_1$.

This together with (3.3), we have

$$\| (Uy)(t) - (Uy)(t_1) \| \leqslant \int_t^{t_1} \| \Phi(t, s, y(s))\, R(s, y(s), y_s) \| \, ds + 3\varepsilon/5 +$$

$$(3.4) \qquad + \int_{t_1}^{T_1} \| [\Phi(t_1, s, y(s)) - \Phi(t, s, y(s))]\, R(s, y(s), y_s) \| \, ds.$$

$$\text{Let } M = \sup_{t_0 \leqslant t \leqslant s \leqslant T} \| \Phi(t, s, y(s)) \|, \quad \widetilde{M} = \sup_{t_0 \leqslant s \leqslant T} \| R(s, y(s), y_s) \|,$$

and choose $\delta_2 = \varepsilon/5 M\widetilde{M}$ such that $|t - t_1| \leqslant \delta_2$. By mean value theorem, the third term in (3.4) becomes

$$|t - t_1| \int_{t_1}^{T_1} \| f_x(\mu(s), x(\mu(s), sy(s)))\, \Phi(\mu(s), s, y(s))\, R(s, y(s), y_s) \| \, ds$$

whenever $t \leqslant \mu(s) \leqslant t_1$ for $t_1 \leqslant s \leqslant T$. Furthermore, by continuity of f_x, Φ and R, there exists a constant M^* which is independent of t_1, t and $y(s)$ as long as $t_0 \leqslant t \leqslant t_1 \leqslant T$, $|t - t_1| \leqslant \varepsilon/5 M^* = \delta_3$ and $\| y(t) - x(t) \| \leqslant g(t)$. Therefore, we have $\| (Uy)(t) - (Uy)(t_1) \| < \varepsilon$, whenever $|t - t_1| < \delta$, where $\delta = \min(\delta_1, \delta_2, \delta_3)$. Hence, $(Uy)(t)$ is uniformly continuous on every compact subinterval of R^+, which implies that UA is equicontinuous on $[t_0, T]$. Therefore, by Ascoli-Arzelà Theorem $\bar{U}A$ is compact in B. Then, by the application of Schauder-Tychonov fixed point Theorem the result follows immediately.

Now, our aim is to show that any solution of the improper non-linear functional integral equation (3.1) is also a solution of the functional differential system (2.2). As before, $x(t) = x(t, t_0, x_0)$ denotes a given solution of (2.1). Define

(3.5) $M(T) = \sup\{\|\Phi(t, T, x_0)\| : t_0 \leqslant t \leqslant T, \|x_0 - y(T)\| \leqslant g(T)\}$.

This together with the Theorem 2. 6. 4. in [2], we have the following lemma, which is used later on.

Lemma 3.1 : *Let $y(t)$ be a solution of the improper non-linear functional integral equation (3.1) on $[t_0, \infty)$ and $y\in A$ as in Theorem 3.1. Then,*

$$\|x(t, T, y(T)) - x(t)\| \leqslant M(T)\|y(T) - x(T)\| \text{ for } t_0 \leqslant t \leqslant T.$$

Proof: The proof of this lemma follows, as an application of (3.5) to the integral equation 2. 6. 8. in [2] together with the uniqueness of the solution of (2.1), with appropriate notations.

Theorem 3.2 : *Suppose that the hypotheses of the Theorem 3.1 hold and in addition, assume that*

$$\|[f_x(t, x(t, s, z(s))) - f_x(t, x(t))]\,\Phi(t, s, z(s))\| \leqslant K(t, s), \tag{3.6}$$

where $x(t)$ is a solution of (2.1) and $z\in B$, such that

$$\{z\in B : \|x(t) - x(t, s, z(s))\| \leqslant M(s)\,g(s), \text{ for } t_0 \leqslant t \leqslant s\};$$

and $f_x(t, z) \to f_x(t, x)$ as $z \to x$ uniformly in t; furthermore,

$$\int_t^\infty K(t, s)ds \leqslant \theta < 1, \tag{3.7}$$

for sufficiently large s, and

$$g(s)\,M(s) \to 0, \quad \text{as } s \to \infty. \tag{3.8}$$

Then, there exists a solution $y(t)$ of the functional differential system (2.2) on $[t_0, \infty)$ satisfying the property

$$\|x(t) - y(t)\| \to 0 \text{ as } t \to \infty.$$

Proof: From the hypotheses of the Theorem 3.1, there exists a solution $y(t)$ of the integral equation (3.1) on $[t_0 - \tau, \infty)$, and clearly $\|x(t) - y(t)\| \to 0$ as $t \to \infty$.

Now, we will show that $y(t)$ also satisfies the functional differential system (2.2) on $[t_0, \infty)$. Lemma 3.1 and the hypotheses in the theorem yield

$$\|x(t) - x(t, s, y(s))\| \leqslant M(s)\,g(s), \text{ for } t_0 \leqslant t \leqslant s. \tag{3.9}$$

Hence, $y(t)$ is admissible in (3.6).

(i) First, we shall show that $y'(t)$ exists for $t > t_0$. Differentiating (3.1) with respect to t, we have

$$y'(t) = f(t, x(t)) + R(t, y(t), y_t) - \int_t^\infty f_x(t, x(t, s, y(s)))\,\Phi(t, s, y(s))\,R(s, y(s), y_s)ds. \tag{3.10}$$

From the continuity of $f_x(t, x(t))$, (3.8), (3.9) and together with the hypotheses of the Theorem 3.1, the last two terms of the following equation exist and from the smoothness of f_x, Φ and R, the integral in the second term of the right hand side exists

$$\begin{aligned} y'(t) = f(t, x(t)) + R(t, y(t), y_t) &- \int_t^T [f_x(t, x(t, s, y(s))) - f_x(t, x(t))]\,\Phi(t, s, y(s))\,R(s, y(s), y_s)ds - \\ &- \int_T^\infty [f_x(t, x(t, s, y(s))) - f_x(t, x(t))]\,\Phi(t, s, y(s))\,R(s, y(s), y_s)ds - \\ &- \int_t^\infty f_x(t, x(t))\,\Phi(t, s, y(s))\,R(s, y(s), y_s)ds. \end{aligned}$$

Hence $y'(t)$ exists.

(ii) Now, we shall show that $y'(t)$ is continuous on $[t_0, \infty)$. Let $t_1\in[t_0, \infty)$ be a given point. Consider

$$\begin{aligned} y'(t_1) - y'(t) &= f(t_1, x(t_1)) - f(t, x(t)) + R(t_1, y(t_1), y_{t_1}) \\ &- R(t, y(t), y_t) + \int_t^\infty f_x(t, x(t, s, y(s)))\,\Phi(t, s, y(s))\,R(s, y(s), y_s)ds - \\ &- \int_{t_1}^\infty f_x(t_1, x(t_1, s, y(s)))\,\Phi(t_1, s, y(s))\,R(s, y(s), y_s)ds. \end{aligned} \tag{3.11}$$

For any $\varepsilon > 0$, the first two terms of the right hand side can be made arbitrarily small, whenever $|t - t_1| < \delta_1(t_1, \varepsilon)$.

Consider the last two terms of (3.11). For $t_0 \leqslant t_1 \leqslant t \leqslant s$

$$-\int_{t_1}^{t} f_x(t, x(t, s, y(s)))\, \Phi(t, s, y(s))\, R(s, y(s), y_s)ds +$$

(3.12)

$$+\int_{t_1}^{\infty} [f_x(t, x(t,s, y(s)))\, \Phi(t,s, y(s)) - f_x(t_1, s, y(s)))\, \Phi(t_1, s, y(s))]\, R(s, y(s), y_s)ds.$$

In (3.12), the first term can be made arbitrarily small, whenever $|t - t_1| < < \delta_2(t_1, \varepsilon)$. The last term in (3.12) can be written as

$$\int_{t_1}^{T} [f_x(t, x(t, s, y(s))) - f_x(t_1, x(t_1, s, y(s)))]\, \Phi(t, s, y(s))\, R(s, y(s), y_s)ds +$$

$$+\int_{t_1}^{T} f_x(t_1, x(t_1, s, y(s)))[\Phi(t, s, y(s)) - \Phi(t_1, s, y(s))]\, R(s, y(s), y_s)ds +$$

$$+\int_{T}^{\infty} [f_x(t, x(t, s, y(s))) - f_x(t, x(t))]\, \Phi(t, s, y(s))\, R(s, y(s), y_s)ds +$$

(3.13)

$$+\int_{T}^{\infty} [f_x(t, x(t)) - f_x(t_1, x(t_1))]\, \Phi(t, s, y(s))\, R(s, y(s), y_s)\, ds +$$

$$+\int_{T}^{\infty} [f_x(t_1, x(t_1)) - f_x(t_1, x, (t_1, s, y(s)))]\, \Phi(t_1, s, y(s)) R(s, y(s), y_s)ds +$$

$$+\int_{T}^{\infty} f_x(t_1, x(t_1))\, [\Phi(t, s, y(s)) - \Phi(t_1, s, y(s))]\, R(s, y(s), y_s)ds.$$

From the continuity of $f_x(t, x)$, $\Phi(t, s, y(s))$, $R(s, y(s,, y_s)$, the first term can be made arbitrarily small for $|t - t_1| \leq \delta_3$. From the hypotheses (3.8) and (3.9), together with the continuity of $f_x(t, x)$, the third, fifth and sixth terms can be made arbitrarily small. Finally, the fourth and second terms can be made arbitrarily small, in view of the continuity of f_x, the hypotheses of the Theorem 3.2 and the mean value Theorem. Thus, the continuity of $y'(t)$ for $t \geq t_0$ follows by choosing appropriate δ and combining the relations (3.11), (3.12) and (3.13).

Now, using the fact that

$$\frac{dx}{ds}(t, s, y(s)) = \Phi(t, s, y(s))\, [y'(s) - f(s, y(s))],$$

we obtain the following equality in view of the Lemma 3.1 and the equation (3.8);

$$y(t) - x(t) = \lim_{T\to\infty} [y(t) - x(t, T, y(T))] = \lim_{T\to\infty} \int_{T}^{t} dx/ds(t, s, y(s))ds =$$

$$= \int_{t}^{\infty} \Phi(t, s, y(s))[f(s, y(s)) - y'(s)]ds, \qquad t \geq t_0.$$

This together with (3.1), gives

$$0 = \int_{t}^{\infty} \Phi(t, s, y(s))\, [R(s, y(s), y_s) + f(s, y(s)) - y'(s)]ds.$$

which implies that

$$0 = \int_{t}^{\infty} \Phi(t, s, y(s))\, [F(s, y_s) - y'(s)]ds.$$

Set $w(s) = F(s, y_s) - y'(s)$. Then,

$$0 = \int_{t}^{\infty} \Phi(t, s, y(s))\, w(s)\, ds, \qquad t \geq t_0. \tag{3.14}$$

From (3.10), we have

$$F(t, y_t) - y'(t) = f(t, y(t)) - f(t, x(t)) +$$

$$+\int_{t}^{\infty} f_x(t, x(t, s, y(s)))\, \Phi(t, s, y(s))\, R(s, y(s), y_s)ds,$$

which can be written as

$$w(t) = f(t, y(t)) - f(t, x(t)) + \int_{t}^{\infty} f_x(t, x(t, s, y(s)))\, \Phi(t, s, y(s))\, R(s, y(s), y_s)ds. \tag{3.15}$$

Our aim is to show that $w(t)$ is bounded. For this purpose (3.15) can be written as

$$w(t) = \{f(t, y(t)) - f(t, x(t)) - f_x(t, x(t))\,[y(t) - x(t)]\} +$$

$$+ \int_t^\infty [f_x(t, x(t, s, y(s))) - f_x(t, x(t))]\,\Phi(t, s, y(s))\,R(s, y(s), y_s)\,ds. \tag{3.16}$$

From the hypotheses on f_x and the relations (3.8) and (3.9), there exists a number $T > 0$ such that, for $s \geqslant t \geqslant T$, the last term in (3.16) can be made small enough. In view of the mean value Theorem, the first term can also be made arbitrarily small. Hence, $w(t)$ is arbitrarily small for $t \geqslant T$. This together with the continuity of w implies that $w(t)$ is bounded. On the other hand

$$\frac{df}{ds}(t, x(t, s, y(s))) = f_x(t, x(t, s, y(s)))\,\Phi(t, s, y(s))\,[y'(s) - f(s, y(s))].$$

Integrating, we have

$$f(t, x(t)) - f(t, y(t)) = \int_t^\infty f_x(t, x(t, s, y(s)))\,\Phi(t, s, y(s))\,[y'(s) - f(s, y(s))]\,ds.$$

This together with (3.15), gives

$$w(t) = \int_t^\infty f_x(t, x(t, s, y(s)))\,\Phi(t, s, y(s))\,[F(s, y_s) - y'(s)]\,ds,$$

which implies that

$$w(t) = \int_t^\infty f_x(t, x(t, s, y(s)))\,\Phi(t, s, y(s))\,w(s)\,ds. \tag{3.17}$$

Upon multiplying (3.14) by $f_x(t, x(t))$ and subtracting the result from (3.17), we have

$$w(t) = \int_t^\infty [f_x(t, x(t, s, y(s))) - f_x(t, x(t))]\,\Phi(t, s, y(s))\,w(s)\,ds. \tag{3.18}$$

This together with the Lemma 3.1, (3.6) and (3.9) yields

$$\| w(t) \| \leqslant \sup_{s \geqslant t} \| w(s) \| \int_t^\infty K(t, s)\,ds. \tag{3.19}$$

for sufficiently large $t, s \geqslant t \geqslant T$ and in view of (3.7), we get

$$\| w(t) \| \leqslant \sup_{s \geqslant t} \| w(s) \|.$$

This implies that $w(t) = 0$ for $t \geqslant T$. Thus (3.17) reduces to

$$w(t) = \int_t^T f_x(t, x(t, s, y(s)))\,\Phi(t, s, y(s))w(s)ds. \tag{3.20}$$

Now, it is easy to show by using a standard argument that this linear integral equation has only the trivial solution $w(t) \equiv 0$ on $[t_0, T]$. Hence, $w(t) \equiv 0$ for all $t \geqslant t_0$. This proves that $y(t)$ satisfies the functional differential system (2.2) and the proof is complete.

Remark 3.1: In proving that $y'(t)$ exists for $t \geqslant t_0$, we only used the continuity of $f_x(t, x)$ and the hypothesis (3.8). Furthermore, contrary to the case in [3], boundedness of solutions is not needed in the proof of the theorem.

To show that $w(t)$ in (3.15) is bounded, we have used the hypotheses on $f_x(t, x)$ and the hypothesis (3.8). The hypotheses (L) and (iii) in Theorem 4 of [3] are unnecessary in our paper.

Remark 3.2: In general, we have very few systems which have bounded solutions and also satisfy the hypothesis (H_2) in [3]. Moreover, (H_3) does not allow the systems whose trivial solution is stable or asymptotically stable. But our hypothesis (3.6) does allow such a system.

The other half of the asymptotic equivalence theorem has been treated in [2].

4. *Example*: In the following, we give a simple example, in order to show the fruitfullness of our hypothesis (3.6) and justification of the remark 3.2.

Consider the system,

$$x' = -x\,; \qquad x(t_0) = x_0$$
$$y' = -y + e^{-2t}\,; \quad y(t_0) = y_0.$$

Thus,

$$x(t, t_0, x_0) = x_0 e^{-(t-t_0)} \text{ and } \Phi(t, t_0, x_0) = e^{-(t-t_0)}.$$

The hypothesis (H_3) of Theorem 4 in [3] fails, since

$$\int_t^\infty \| \Phi(t, s, z(s)) \| \, ds = \int_t^\infty e^{-(t-s)} \, ds = e^{-t} \int_t^\infty e^s \, ds \not< N(t), \qquad \text{for } t \geq t_0.$$

However, our hypotheses (3.6), (3.7) and (3.8) in Theorem 3.2 can be verified easily with

$$K(t, s) = e^{-s}/2, \qquad g(t) = e^{-2t} \qquad \text{and } M(t) = e^t.$$

Hence, the above two equations are asymptotically equivalent.

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ECHIVALENȚA ASIMPTOTICĂ ȘI SISTEME DIFERENȚIAL-FUNCȚIONALE

Rezumat

În [3] Marlin și Struble au studiat problema echivalenței asimptotice a două sisteme diferențiale neliniare, plecînd de la formula lui Alekseev, o variantă a formulei variației constantelor pentru sisteme neliniare de ecuații diferențiale.

În lucrare se extind rezultatele din [3] la cazul cînd sistemul perturbat este un sistem diferențial reușind să înlăture restricția de mărginire a soluțiilor sistemului (31) care sînt și soluții pentru (2.2). În vederea realizării scopului propus se folosește teorema de punct fix a lui Schauder și Tihonov, alegînd convenabil spațiul de funcții pe care acționează operatorul definit de (31).

A GLOBAL EXISTENCE THEOREM FOR A FUNCTIONAL DIFFERENTIAL EQUATION

BY

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1. Introduction. Let Y be a Banach space with norm $|\ |$, let R be the set of all real numbers, and let R^+ be the set of all nonnegative real numbers. Let $\mathcal{C}[Y]$ be the Fréchet space of all continuous functions from R^+ to Y, with the topology of uniform convergence on compact intervals, and define $\mathcal{C}[R]$ analogously. We propose to obtain a global existence theorem for a functional differential equation of the form

$$u'(t) = A(t, u(t)) + B[t, u]; \quad u(0) = p, \tag{1}$$

where A is a *continuous* function from $R^+ \times Y$ to Y and B is a continuous function from $R^+ \times \mathcal{C}[Y]$ to Y.

We will include such cases as when B is a multiple delay operator, i. e. there are continuous functions $\{r_1, r_2, \ldots, r_n\}$ from R^+ to R^+ such that $r_i(t)$ is in $[0, t]$, whenever t is in R^+ and i is in $\{1, 2, \ldots, n\}$ and continuous functions $\{C_1, C_2, \ldots, C_n\}$ from $R^+ \times Y$ to Y such that B is given by

$$B[t, f] = \sum_{k=1}^{n} C_k(t, f(r_k(t))), \tag{2}$$

and as when B is a Volterra integral operator, i. e., there is a continuous function K from $R^+ \times R^+ \times Y$ to Y such that B is given by

$$B[t, f] = \int_0^t K(t, s, f(s)) \, ds. \tag{3}$$