The hypothesis (H_3) of Theorem 4 in [3] fails, since

$$\int_t^\infty \| \Phi(t, s, z(s)) \| \, ds = \int_t^\infty e^{-(t-s)} \, ds = e^{-t} \int_t^\infty e^s \, ds \not\leq N(t), \qquad \text{for } t > t_0.$$

However, our hypotheses (3.6), (3.7) and (3.8) in Theorem 3.2 can be verified easily with

$$K(t, s) = e^{-s}/2, \qquad g(t) = e^{-2t} \qquad \text{and } M(t) = e^t.$$

Hence, the above two equations are asymptotically equivalent.

REFERENCES

1. Corduneanu C. — *Problèmes globaux dans la théorie des équations intégrales de Volterra*; Annali di Matematica pura ed applicata (iv), vol. LXVII (1965) pp. 349–364.
2. Lakshmikantham, V. and Leela, S. — *Differential and Integral Inequalities, theory and applications*; vols. I & II, Academic Press, New York, (1969).
3. Marlin, J. A. and Struble, R. A. — *Asymptotic equivalence of nonlinear systems*. Jour. of Diff. Eqs. 6, (1969) pp. 578–596.

ECHIVALENȚA ASIMPTOTICĂ ȘI SISTEME DIFERENȚIAL-FUNCȚIONALE

Rezumat

În [3] Marlin și Struble au studiat problema echivalenței asimptotice a două sisteme diferențiale neliniare, plecînd de la formula lui Alekseev, o variantă a formulei variației constantelor pentru sisteme neliniare de ecuații diferențiale.

În lucrare se extind rezultatele din [3] la cazul cînd sistemul perturbat este un sistem diferențial reușind să înlăture restricția de mărginire a soluțiilor sistemului (31) care sînt și soluții pentru (2.2). În vederea realizării scopului propus se folosește teorema de punct fix a lui Schauder și Tihonov, alegînd convenabil spațiul de funcții pe care acționează operatorul definit de (31).

A GLOBAL EXISTENCE THEOREM FOR A FUNCTIONAL DIFFERENTIAL EQUATION

BY

DAVID LOWELL LOVELADY

1. Introduction. Let Y be a Banach space with norm $|\ |$, let R be the set of all real numbers, and let R^+ be the set of all nonnegative real numbers. Let $\mathcal{F}[Y]$ be the Fréchet space of all continuous functions from R^+ to Y, with the topology of uniform convergence on compact intervals, and define $\mathcal{F}[R]$ analogously. We propose to obtain a global existence theorem for a functional differential equation of the form

$$u'(t) = A(t, u(t)) + B[t, u]; \quad u(0) = p, \tag{1}$$

where A is a *continuous* function from $R^+ \times Y$ to Y and B is a continuous function from $R^+ \times \mathcal{F}[Y]$ to Y.

We will include such cases as when B is a multiple delay operator, i. e. there are continuous functions $\{r_1, r_2, \ldots, r_n\}$ from R^+ to R^+ such that $r_i(t)$ is in $[0, t]$, whenever t is in R^+ and i is in $\{1, 2, \ldots, n\}$ and continuous functions $\{C_1, C_2, \ldots, C_n\}$ from $R^+ \times Y$ to Y such that B is given by

$$B[t, f] = \sum_{k=1}^n C_k(t, f(r_k(t))), \tag{2}$$

and as when B is a Volterra integral operator, i. e., there is a continuous function K from $R^+ \times R^+ \times Y$ to Y such that B is given by

$$B[t, f] = \int_0^t K(t, s, f(s)) ds. \tag{3}$$

Equations of these forms have been studied by several authors (see, for example, C. Corduneanu [3], S. I. Grossman and R. K. Miller [4], J. J. Levin [6], [7], [8], Levin and J. A. Nohel [9], and P. Talpalaru [12], [13], [14]).

Our hypotheses will involve a dissipativity condition on A and Lipschitz and nonanticipativity conditions on B. In particular, our hypotheses are such that *local* existence follows from [10, Theorem 5], and, in the case that Y is finitedimensional, global existence follows from [10, Theorem 6]. In the present work we shall obtain a global existence theorem without regard to the dimension of Y, and we shall exhibit an iterative scheme which is new even in the finitedimensional case. Under additional hypotheses we shall also obtain information on bounds for solutions. Our techniques and results on bounds will be similar to, but at once more and less general than, results of P. Talpalaru (see [12, Théorème 1] and [13, Théorème 3.1]).

II. The main results. Let α be a continuous function from R^+ to R (note that α may have negative values), and let A and B be continuous functions as in the introduction. Let Δ be the function from $\mathcal{F}[Y]$ to $\mathcal{F}[R]$ having the property that if (t, f) is in $R^+ \times \mathcal{F}[Y]$ then $\Delta[f](t)$ is the least number c such that $|f(s)| \leqslant c$ whenever s is in $[0, t]$. We shall need conditions

C 1: If (t, p, q) is in $R^+ \times Y \times Y$ and c is a positive number then

$$|[p-q] - c[A(t,p) - A(t,q)]| \geqslant [1 - c\alpha(t)]\,|p-q|.$$

C 2: If b is a positive number then there is a positive number c having the property that if (t, f, g) is in $[0, b] \times \mathcal{F}[Y] \times \mathcal{F}[Y]$ then

$$|B[t,f] - B[t,g]| \leqslant c\Delta[f-g](t).$$

Theorem 1: *Suppose that each of* C 1 *and* C 2 *is true, and let p be in Y. Suppose that $\{h_k\}_{k=0}^{\infty}$ is a sequence into $\mathcal{F}[Y]$ such that $h_k(0) = p$ and*

$$h_k'(t) = A(t, h_k(t)) + B[t, h_{k-1}]$$

whenever k is a positive integer and t is a positive number. Then there is a number u of $\mathcal{F}[Y]$ such that $\lim_{k\to\infty} h_k = u$ and such that (1) is true whenever t is a positive number.

Theorem 2: *Suppose that each of* C 1 *and* C 2 *is true. Suppose also that β is a continuous function from $R^+ \times \mathcal{F}[R]$ to R such that each of* (i), (ii), *and* (iii) *is true.*

(i) *β is Lipschitz continuous in the sense of* C 2.

(ii) *If (f, g) is in $\mathcal{F}[Y] \times \mathcal{F}[Y]$ and P in $\mathcal{F}[R]$ is given by $P(t) = |f(t) - g(t)|$ then*

$$|B[t,f] - B[t,g]| \leqslant \beta[t,P]$$

whenever t is in R^+.

(iii) *If (σ, τ) is in $\mathcal{F}[R] \times \mathcal{F}[R]$ and $\sigma(t) \leqslant \tau(t)$ whenever t is in R^+, then $\beta[t, \sigma] \leqslant \beta[t, \tau]$ whenever t is in R^+.*

Let (p, q) be in $Y \times Y$, let u in $\mathcal{F}[Y]$ solve (1) with respect to p, and let v in $\mathcal{F}[Y]$ solve (1) with respect to q. Let σ be that member of $\mathcal{F}[R]$ such that $\sigma(0) = |p-q|$ and such that

$$\sigma'(t) = \alpha(t)\sigma(t) + \beta[t, \sigma]$$

whenever t is a positive number. Then

$$|u(t) - v(t)| \leqslant \sigma(t)$$

whenever t is in R^+.

If each of p and q is in Y then

$$\lim_{\delta\to 0-} (1/\delta)\,(|p+\delta q| - |p|)$$

exists and is $\sup\{(1/\delta)(|p+\delta q| - |p|) : \delta \text{ is a negative number}\}$ (see [1, p. 3]). Let m_- be that function from $Y \times Y$ to R given by

$$m_-[p, q] = \lim_{\delta\to 0-} (1/\delta)(|p+\delta q| - |p|).$$

The above comments make it clear that condition C 1 is equivalent to requiring that

$$m_-[p-q, A(t,p) - A(t,q)] \leqslant \alpha(t)\,|p-q|$$

whenever (t, p, q) is in $R^+ \times Y \times Y$. Note [10, Lemma 6] that $m_-[p, q+r] \leqslant m_-[p, q] + |r|$ whenever (p, q, r) is in $Y \times Y \times Y$. Note also that if u is a function from a subset of R^+ to Y, if c is in the domain of u, if $u^{-1}(c)$ (the left derivative of u at c) exists, and if P is given on the domain of u by $P(t) = |u(t)|$, then $P^{-1}(c)$ exists and $P^{-1}(c) = m_-[u(c), u^{-1}(c)]$ (compare [1, p. 3]).

This author and R. H. Martin, Jr. [11] have shown that if C 1 is true and (q, g) is in $Y \times \mathcal{F}[Y]$ then there is exactly one member v of $\mathcal{F}[Y]$ such that $v(0) = q$ and such that

$$v'(t) = g(t) + A(t, v(t))$$

whenever t is a positive number. In particular, the existence of a sequence $\{h_k\}_{k=0}^{\infty}$ as in Theorem 1 is known, and Theorem 1 truly is an iteratively obtained global existence. It should be noted that C 2 forces B to be *nonanticipative* in the sense that if (t, f, g) is in $R^+ \times \mathcal{F}[Y] \times \mathcal{F}[Y]$ and $f(s) = g(s)$ whenever s is in $[0, t]$ then $B[t, f] = B[t, g]$. It follows from [10, Theorem 5] that C 1 and C 2 together imply *local* existence for (1), and that solutions, as far as they can be continued, are unique. P. Talpalaru [13, Théorème 3.1] has proved a result related to Theorem 2. In particular, Talpalaru obtained a similar inequality assuming that Y has uniformly convex dual space, and B has domain $R^+ \times Y$, but not assuming that A is continuous. Note that, although the continuity of A is used heavily in Theorem 1, it can be dispensed with in Theorem 2 if we assume the existence of u and v.

Proof of theorem 1: Let p be in Y. Let T_p be that function from $\mathcal{F}[Y]$ to $\mathcal{F}[Y]$ such that $g = T_p[f]$ only in case $g(0) = p$, g is differentiable, and

$$g'(t) = A(t, g(t)) + B[t, f]$$

whenever t is a positive number. (It follows from [11] that T_p exists). Our method of proof is to construct a family $\{\mathcal{J}_1, \mathcal{J}_2, \ldots\}$ of seminorms on $\mathcal{F}[Y]$ such that $\{\mathcal{J}_1, \mathcal{J}_2, \ldots\}$ determines the topology on $\mathcal{F}[Y]$ and such that

$$\mathcal{J}_n[T_p[f] - T_p[g]] \leqslant (1/2)\mathcal{J}_n[f - g]$$

whenever (f,g) is in $\mathcal{F}[Y] \times \mathcal{F}[Y]$ and n is a positive integer. The theorem will then follow from a result of G. L. Cain, Jr. and M. Z. Nashed [1, Theorem 2.2].

If n is a positive integer, let c_n be a positive number such that $\alpha(t) \leqslant c_n$ whenever t is in $[0,n]$ and such that $|B[t, f] - B[t, g]| \leqslant c_n \Delta[f - g](t)$ whenever (t, f, g) is in $[0,n] \times \mathcal{F}[Y] \times \mathcal{F}[Y]$. Also, if n is a positive integer, let $b_n = \exp(nc_n)$, let $a_n = b_n c_n$, and let $\mathcal{J}_n$ from $\mathcal{F}[Y]$ to R^+ be given by

$$\mathcal{J}_n[f] = \sup\{\Delta[f](t)\exp(-2a_n t) : t \text{ is in } [0, n]\}.$$

Now clearly each $\mathcal{J}_n$ is a seminorm, and the family $\{\mathcal{J}_1, \mathcal{J}_2, \ldots\}$ determines the topology on $\mathcal{F}[Y]$.

Let n be a positive integer, and let (f,g) be in $\mathcal{F}[Y] \times \mathcal{F}[Y]$. Let $u = T_p[f]$ and let $v = T_p[g]$. Let $Q = \Delta[f - g]$, and let P in $\mathcal{F}[R]$ be given by $P(t) = |u(t) - v(t)|$. Now, if t is in $(0, n]$,

$$P_-'(t) = m_-[u(t) - v(t), u'(t) - v'(t)] = m_-[u(t) - v(t), A(t, u(t)) -$$
$$- A(t, v(t)) + B[t, f] - B[t, g]] \leqslant \alpha(t)P(t) + c_n Q(t) \leqslant c_n P(t) + c_n Q(t).$$

Hence [5, Theorem 1.4.1, p. 15],

$$P(t) \leqslant (c_n)\int_0^t Q(s)\exp(c_n t - c_n s)ds \leqslant (c_n b_n)\int_0^t Q(s)ds = (a_n)\int_0^t Q(s)ds$$

whenever t is in $[0, n]$. But since Q has only nonnegative values this says that if t is in $[0, n]$ then

$$\Delta[u - v](t) \leqslant (a_n)\int_0^t Q(s)ds = (a_n)\int_0^t Q(s)\exp(-2a_n s)\exp(2a_n s)ds$$

$$\leqslant a_n \mathcal{J}_n[f - g]\int_0^t \exp(2a_n s)ds \leqslant (1/2)\exp(2a_n t)\mathcal{J}_n[f - g]$$

and

$$\Delta[u - v](t)\exp(-2a_n t) \leqslant (1/2)\mathcal{J}_n[f - g].$$

Thus

$$\mathcal{J}_n[T_p[f] - T_p[g]] = \mathcal{J}_n[u - v] \leqslant (1/2)\mathcal{J}_n[f - g].$$

With an application of the aforementioned result of Cain and Nashed [1, Theorem 2.2], the proof is now complete.

Before proving Theorem 2 we need a lemma. Note that the lemma is related to, but neither contains nor is contained by [12, Théorème 1].

Lemma 1: *Let β be as in Theorem 2, and let (φ, ψ) be in $\mathcal{F}[R] \times \mathcal{F}[R]$. Suppose that φ is left differentiable and that ψ is differentiable. Suppose also that*

$$\varphi(0) \leqslant \psi(0),$$
$$\varphi_-'(t) \leqslant \alpha(t)\varphi(t) + \beta[t, \varphi],$$

and

$$\psi'(t) = \alpha(t)\psi(t) + \beta[t, \psi]$$

whenever t is a positive number. Then $\varphi(t) \leqslant \psi(t)$ whenever t is in R^+.

Proof: Let λ in $\mathcal{F}[R]$ be given by

$$\lambda(t) = \psi(0)\exp[\int_0^t \alpha(s)ds] + \int_0^t \beta[s, \varphi]\exp[\int_s^t \alpha(r)dr]ds.$$

Now $\varphi(0) \leqslant \psi(0) = \lambda(0)$ and

$$\lambda'(t) = \alpha(t)\lambda(t) + \beta[t, \varphi]$$

whenever t is a positive number. Thus [5, Theorem 1. 4. 1, p. 15], $\varphi(t) \leqslant \lambda(t)$ whenever t is in R^+. Let $\{\gamma_k\}_{k=1}^{\infty}$ be an $\mathcal{F}[R]$-valued sequence such that $\gamma_1 = \lambda$ and $\gamma_{k+1}(0) = \psi(0)$ and

$$\gamma'_{k+1}(t) = \alpha(t)\gamma_{k+1}(t) + \beta(t, \gamma_k]$$

whenever t is a positive number and k is a positive integer. Now, since (iii) of Theorem 2 holds, and since

$$\gamma_{k+1}(t) = \varphi(0) \exp [\int_0^t \alpha(s)ds] + \int_0^t \beta[s, \gamma_k] \exp [\int_s^t \alpha(r)dr]ds$$

whenever t is in R^+ and k is a positive integer, an easy induction argument shows that $\varphi(t) \leqslant \gamma_k(t) \leqslant \gamma_{k+1}(t)$ whenever t is in R^+ and k is a positive integer. But Theorem 1 tells us that $\psi(t) = \lim\limits_{k \to \infty} \gamma_k(t)$ whenever t is in R^+ so the proof is complete.

Proof of theorem 2: Let β, p, q, u, v, and σ be as in Theorem 2. Let P in $\mathcal{F}|R|$ be given by $P(t) = |u(t) - v(t)|$. Now, if t is a positive number,

$$P_-'(t) = m_-[u(t) - v(t), u'(t) - v'(t)] = m_-[u(t) - v(t),$$
$$A(t, u(t)) - A(t, v(t)) + B[t, u] - B(t, v]] \leqslant \alpha(t)P(t) + | B[t,$$
$$u] - B[t, v] | \leqslant \alpha(t) P(t) + \beta[t, P].$$

Theorem 2 now follows directly from Lemma 1.

III. Examples. We shall show how our results apply to the two examples mentioned in the introduction.

Example 1: Let n be a positive integer, let $\{C_1, \ldots, C_n\}$ be a set of continuous functions from $R^+ \times Y$ to Y, and let each of $\{r_1, \ldots, r_n\}$ and $\{\gamma_1, \ldots, \gamma_n\}$ be a set of continuous functions from R^+ to R^+. Suppose that $r_i(t)$ is in $[0, t]$ whenever i is in $\{1, \ldots, n\}$ and t is in R^+, and suppose that

$$| C_i(t, p) - C_i(t, q) | \leqslant \gamma_i(t) | p - q |$$

whenever (i, t, p, q) is in $\{1, \ldots, n\} \times R^+ \times Y \times Y$. Now, if we let B be given according to (2) and let β be given by

$$\beta[t, \varphi] = \sum_{k=1}^{n} \gamma_k(t) \varphi(r_k(t)),$$

then both of Theorem 1 and Theorem 2 apply.

Example 2: Let K be a continuous function from $R^+ \times R^+ \times Y$ to Y, and let $\varkappa$ be a continuous function from $R^+ \times R^+$ to R^+. Suppose that

$$| K(t, s, p) - K(t, s, q) \leqslant \varkappa(t, s) | p - q |$$

whenever (t, s, p, q) is in $R^+ \times R^+ \times Y \times Y$ and s is in $[0, t]$. Now, if we let B be given according to (3) and let β be given by

$$\beta[t, \varphi] = \int_0^t \varkappa(t, s)\varphi(s)ds,$$

then both of Theorem 1 and Theorem 2 apply.

Departmen of Mathematics
Florida State University
Pallahassee, Florida 32306
U. S. A.

REFERENCES

1. G. L. Cain, Jr. and M. Z. Nashed — *Fixed points and stability for a sum of two operators in locally convex spaces*, Pacific J. Math. 39 (1971) 581—592.
2. W. A. Coppel — *Stability and asymptotic behavior of differential equations*. D. C. Heath, Boston, 1965.
3. C. Corduneanu — *Sur certaines équations fonctionnelles de Volterra*, Funkcialaj Ekvacioj. 9 (1966), 119—127.
4. S. I. Grossman and R. K. Miller — *Perturbation theory for Volterra integrodifferential systems*, J. Diff. Eqns., 8 (1970), 457—474.
5. V. Lakshmikantham and S. Leela — *Differential and integral inequalities*, vol. 1, Academic Press, New York, 1969.
6. J. J. Levin — *The asymptotic behavior of the solution of a Volterra equation*, Proc. Amer. Math. Soc., 14 (1963), 534—541.
7. — *The qualitative behavior of a nonlinear Volterra equation*, Proc. Amer. Math. Soc., 16 (1965), 711—718.
8. — *A nonlinear Volterra equation not of convolution type*, J. Diff. Eqns., 4 (1968), 176—186.
9. J. Levin and J. A. Nohel — *Perturbations of a nonlinear Volterra equation*, Michigan Math. J., 12 (1965), 431—447.
10. D. L. Lovelady — *A functional differential equation in a Banach space*, Funkcialaj Ekvacioj, 14 (1971), 111—122.
11. D. L. Lovelady and R. H. Martin, Jr. — *A global existence theorem for a nonautonomous differential equation in a Banach space*, Proc. Amer. Math. Soc., (to appear).
12. P. Talpalaru — *Inégalités différentielles pour une certaine équation fonctionelle de Volterra*, An. şt. Univ. Iaşi, Matematică, XIV (1968), 313—319.
13. P. Talpalaru, — *Quelques problèmes qualitatifs de la théorie des équations différentielles dans l'espace de Banach*, An. şt. Univ. Iaşi, Matematică, XVI (1970), 327—337.
14. P. Talpalaru — *Quelques problèmes concernant l'équivalence asymptotique des systèmes différentiels*, Boll. Un. Mat. Ital. (4), 4 (1971), 164—186.

O TEOREMĂ DE EXISTENŢĂ GLOBALĂ PENTRU O ECUAŢIE DIFERENŢIALĂ FUNCŢIONALĂ

Rezumat

În lucrare se demonstrează o teoremă de existenţă globală pentru ecuaţia diferenţială funcţională (1), unde A este o funcţie continuă de la $R^+ \times Y$ la $Y(R^+ = [0, \infty)$, Y un spaţiu Banach) iar B este o funcţie continuă de la $R^+ \times \mathcal{F}[Y]$ la Y. ($\mathcal{F}[Y]$ fiind spaţiul Fréchet al funcţiilor continue pe R^+ cu valori în Y). În vederea acestui scop se utilizează o condiţie de disipativitate relativ la A şi o condiţie de nonanticipativitate relativ la B.

Added in proof: The author has learned that the results of [11], on which the present work is based, were obtained independently by N. Pavel, „Sur certaines équations différentielles abstraites", Boll. Un. Mat. Ital., to appear.