STOCHASTIC INTEGRAL EQUATIONS OF MIXED TYPE

BY

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1. Introduction. Random integral equations are of great importance in many branches of science, engineering and technology. They occur as models in concrete physical phenomena. Most of the results on random integral equations obtained up till now, can be found in the forth-coming book by Bharucha-Reid [1].

Our aims in this paper is to study a stochastic integral equations of mixed type of the form

$$x(t, w) = f(t, w) + \int_0^t a(t, s, w)\, g(s, x(s, w))ds + \int_0^\infty b(t, s, w)\, h(s, x(s, w))ds, \quad t \geqslant 0. \tag{1.1}$$

We shall prove the existence and uniqueness of wide sense and random solutions (for the definitions see sections 2 and 3) of (1.1) under different set of assumptions. These results are analogous to those obtained by Miller, Nohel and Wong [3]. The main tools are the theory of admissibility and fixed point theorems.

We need the following fixed point theorem proved by Krasnoselskii in [2], which combines both the Banach contraction mapping principle and the Schauder fixed point theorem, in our subsequent discussion.

Theorem 1.1 (Krasnoselskii). *Let B be a Banach space, S be a bounded closed convex subset of B and A, F be operators on S into B such that $Ax + Fy \in S$ for every pair $x, y \in S$. If A is a strict contraction i. e., it satisfies the condition that for all x, $y \in S$*

$$\| Ax - Ay \| \leqslant \alpha \| x - y \|$$

for some α, $0 < \alpha < 1$, and F is completely continuous, then the equation

$$Ax + Fx = x$$

has a solution in S.

2. **Existence of Solutions I.** In this section, we shall obtain an existence theorem analogous to the theorem obtained by Miller, Nohel and Wong [3] for ordinary integral equations of mixed type.

Definition 2.1. *A function $x(t, w)$ on $R^+ \times \Omega$ is said to be a wide sense solution of (1.1) if there exists a set Ω_0 such that $P(\Omega_0) = 0$ and $x(t, w_0)$ satisfies the equation*

$$x(t,w_0) = f(t, w_0) + \int_0^t a(t, s, w_0)\, g(s, x(s,w_0))ds + \int_0^\infty b(t,s,w)h(s,x(s,w_0))ds. \tag{2.0}$$

for every $w_0 \in \Omega - \Omega_0$.

Theorem 2.1. *Let the following conditions hold:*

(H_1) *With probability one,*

$$\sup_{t \geqslant 0} \int_0^t | a(t, s, w) | ds \leqslant A_0(w) < \infty, \tag{2.1}$$

and

$$\sup_{t \geqslant 0} \int_0^\infty | b(t, s, w) | ds \leqslant B_0(w) < \infty, \tag{2.2}$$

(H_2) $g(t, 0) \equiv h(t, 0) \equiv 0$;

(H_3) *for each $\lambda > 0$, there exists a $\delta_1 > 0$ such that*

$$| g(t, x) - g(t, y) | \leqslant \lambda | x - y | \tag{2.3}$$

for all $|x|$, $|y| \leqslant \delta_1$ and uniformly in t;

(H_4) *for each $\eta > 0$, there exists a $\delta_2 > 0$ such that*

$$| h(t, x) - h(t, y) | \leqslant \eta | x - y | \tag{2.4}$$

for all $|x|$, $|y| \leqslant \delta_2$ and uniformly in t; and

(H_5) *for all t,*

$$\int_0^\infty | b(t + \tau, s, w) - b(t, s, w) | ds \to 0 \tag{2.5}$$

as $|\tau| \to 0$ with probability one.

Then, there exists a positive random variable z with probability one such that if $0 < \rho(w) \leqslant z(w)$ then there exists a $\delta(w) > 0$ such that, for $f(t, w)$ satisfying

$$\sup_{t \geqslant 0} | f(t, w) | \leqslant \delta(w), \tag{2.6}$$

there exists a unique wide sense solution $x(t, w)$ of (1.1) on $0 \leqslant t < \infty$ satisfying

$$\sup_{t \geqslant 0} | x(t, w) | \leqslant \rho(w) \tag{2.7}$$

and $x(t, w)$ is continuous in t with probability one.

Proof: In view of assumptions (H_1) and (H_5), there exists a set Ω_0 with probability zero such that for every $w \in \Omega - \Omega_0$, (2.1), (2.2) and (2.5) hold. Let us choose and fix $w_0 \in \Omega - \Omega_0$.

By the existence theorem for the solutions of ordinary integral equations of mixed type due to Miller, Nohel and Wong [3], there exists a positive number $z(w_0)$ such that if $0 < \rho(w_0) \leqslant z(w_0)$, then there exists a $\delta(w_0) > 0$ such that, for $\sup_{t \geqslant 0} | f(t, w_0) | \leqslant \delta(w_0)$, there exists a unique solution $x(t, w_0)$ on $0 \leqslant t < \infty$ of the integral equation (2.0), continuous in t for each $w_0 \in \Omega - \Omega_0$ and $\sup_{t \geqslant 0} | x(t, w_0) | \leqslant \rho(w_0)$.

Since this is true for every $w_0 \in \Omega - \Omega_0$ and $P(\Omega_0) = 0$, it follows that there exists a unique wide sense solution $x(t, w)$ for the mixed integral equation (1.1) satisfying the conclusions of the theorem. This completes the proof of this theorem.

Corollary 2.1: *If the hypotheses of theorem 2.1 are satisfied and if for each $T > 0$,*

$$\lim_{t \to \infty} \int_0^T | a(t, s, w) | ds = 0 \tag{2.8}$$

with probability one, and

$$\lim_{t \to \infty} \int_0^T | b(t, s, w) | ds = 0 \tag{2.9}$$

with probability one; further, if $f(t, w) \to 0$ as $t \to \infty$ with probability one then $\lim_{t \to \infty} x(t, w) = 0$ with probability one.

Proof: In view of the assumptions (H_2) and (H_5) and the equations (2.8) and (2.9), it follows that there exists a set Ω_0 with $P(\Omega_0) = 0$ such that for $w \in \Omega - \Omega_0$, (2.1), (2.2), (2.5), (2.8) and (2.9) are true. Let us choose and fix $w_0 \in \Omega - \Omega_0$.

Let S be the space of all bounded continuous functions $y(t)$ on $[0, \infty)$ with the norm

$$\| y \| = \sup_{t \geq 0} | y(t) | .$$

Let $S(\rho) = \{ y \in S : \| y \| < \rho \}$, $\rho > 0$.

Define

$$S_0 = \{ z \in S(\rho) : \lim_{t \to \infty} z(t) = 0 \} . \tag{2.10}$$

Clearly S_0 is a closed subspace of $S(\rho)$ under the norm $\| \ \|$. Let $\phi(t, w_0) \in S_0$. Let $\gamma > 0$ be given. By hypothesis, we can choose $T > 0$, such that

$$| f(t_1, w_0) | \leq \frac{\gamma}{5}, \quad | \Phi(t, w_0) | \leq \frac{\gamma}{5\varepsilon_1} \tag{2.11}$$

for all $t \geq T$, where $\varepsilon_1 = \max(\lambda A_0(w_0), \eta B_0(w_0))$. By (2.8) and (2.9), we can find $T_1 \geq T$ so large that

$$\int_0^T | a(t, s, w_0) | \, ds \leq \frac{\gamma}{5 \lambda \rho(w_0)}, \quad (t \geq T_1) \tag{2.12}$$

and

$$\int_0^T | b(t, s, w_0) | \, ds \leq \frac{\gamma}{5 \eta \rho(w_0)}, \quad (t \geq T_1). \tag{2.13}$$

Then, for $t \geq T_1$, we observe that

$$| (A\Phi + F\Phi)(t, w_0) | \leq | f(t, w_0) | + \int_0^t | a(t, s, w_0) | \lambda | \Phi(s, w_0) | \, ds +$$

$$+ \int_0^\infty | b(t, s, w_0) | \eta | \Phi(s, w_0) | \, ds \leq \frac{\gamma}{5} + \lambda \rho(w_0) \int_0^T | a(t, s, w_0) | \, ds +$$

$$+ \lambda \int_T^t | a(t, s, w_0) | \, | \Phi(s, w_0) | \, ds + \eta \rho(w_0) \int_0^T | b(t, s, w_0) | \, ds +$$

$$+ \eta \int_T^\infty | b(t, s, w_0) | \, | \phi(s, w_0) | \, ds \leq \frac{\gamma}{5} + \lambda \rho(w_0) \frac{\gamma}{5 \lambda \rho(w_0)} +$$

$$+ \lambda \frac{\gamma}{5\varepsilon_1} \int_0^t | a(t, s, w_0) | \, ds + \eta \rho(w_0) \frac{\gamma}{5 \eta \rho(w_0)} + \eta \frac{\gamma}{5\varepsilon_1} \int_0^\infty | b(t, s, w_0) | \, ds \leq$$

$$\leq \frac{\gamma}{5} + \frac{\gamma}{5} + \lambda \frac{\gamma}{5\varepsilon_1} A_0(w_0) + \frac{\gamma}{5} + \eta \frac{\gamma}{5\varepsilon_1} B_0(w_0) \leq \gamma,$$

$$\text{since } \varepsilon_1 = \max(\lambda A_0(w_0), \eta B_0(w_0))$$

by using (2.1), (2.2), (2.5), (2.8), (2.9), (2.10) and (2.13). Since γ is arbitrary, it follows that $(A\Phi + F\Phi)(t, w_0) \to 0$ as $t \to \infty$. Hence $\phi \in S_0$ implies that $A\Phi + F\Phi \in S_0$. Therefore, by Krasnoselskii theorem, there exists a solution of $A\Phi + F\Phi = \Phi \in S_0$. Therefore for any given $w_0 \in \Omega - \Omega_0$, (2.0) has a solution in S_0 i. e. $x(t, w_0) \to 0$ as $t \to \infty$. Since this is true for every $w_0 \in \Omega - \Omega_0$ and $P(\Omega_0) = 0$, it follows that the solution of (1.1) is almost surely stable.

Remark: We have shown the existence of a unique wide sense solution. We were unable to show that the solution is random in the sense that the solution $x(t, w)$ is a measurable process. But we note that the solution has the property that $x(t, w)$ is continuous in t with probability one.

3. **Existence of Solution II.** In this section, we shall obtain a theorem on the existence of solution for (1.1) under a different set of conditions. The analysis, we used in this section, is the theory of admissibility. This study includes some results of T s o k o s [4] as special cases.

We shall consider the random solution $x(t, w)$ and the stochastic free term $f(t, w)$ to be the functions of the real argument t with values in the space $L_2(\Omega, A, \mu)$ where Ω is an abstract space, A is a σ-field of subsets of Ω and μ is a probability measure on Ω. The functions $g(t, x(t, w))$ and $h(t, x(t, w))$, under convenient conditions, will also be functions of t with values in $L_2(\Omega, A, \mu)$. The stochastic kernels $a(t, s, w)$ and $b(t, s, w)$ are essentially bounded functions for every t and s such that $0 \leq s \leq t < \infty$. The values of the stochastic kernels for fixed t and s will be $L_\infty(\Omega, A, \mu)$, so that the products of $a(t, s, w)\, g(s, x(s, w))$ and $b(t, s, w)\, h(s, x(s, w))$ will always be in $L_2(\Omega, A, \mu)$. We shall assume that the mappings

$$(t, s) \to a(t, s, w) \ ; \quad (t, s) \to b(t, s, w)$$

from the set $\Delta = \{(t, s) : \ 0 \leq s \leq t < \infty\}$ into $L_\infty(\Omega, A, \mu)$ are continuous i. e.,

$$\mu - \operatorname*{ess\,sup}_{w} | a(t_n, s_n, w) - a(t, s, w) | \to 0$$

and

$$\mu - \operatorname{ess\,sup}_{w} | b(t_n, s_n, w) - b(t, s, w) | \to 0$$

as $n \to \infty$ whenever $(t_n, s_n) \to (t, s)$ as $n \to \infty$.

Let $C_c = C_c(R^+, L_2(\Omega, A, \mu))$ be the space of all continuous functions from R^+ into $L_2(\Omega, A, \mu)$ with the topology of uniform convergence on every interval $[0, \beta]$ for $\beta > 0$. The space C_c is a locally convex space whose topology is defined by the following family of semi-norms:

$$\| x(t, w) \|_n = \sup_{0 < t \leq n} \left\{ \int | x(t, w) |^2 \, d\mu(w) \right\}^{1/2}, \quad n = 1, 2, 3, \ldots$$

Let B and D be a pair o Banach spaces such that $B \subset C_c$ and $D \subset \subset C_c$ and let T be a linear operator from C_c into itself. Then with respect to these objects, we introduce the following definitions.

Definition 3.1. *The pair of spaces (B, D) will be called admissible with respect to the operator*

$$T : C_c(R^+, L_2(\Omega, A, \mu)) \to C_c(R^+, L_2(\Omega, A, \mu))$$

if and only if $TB \subset D$.

Definition 3.2. *We call $x(t, w)$ a random solution of the integral equation (1.1) if for every fixed t belonging to R^+, $x(t, w) \in L_2(\Omega, A, \mu)$ and $x(t, w)$ satisfies the equation (1.1) μ a. e.*

Lemma 3.1. *Let T be a continuous operator from $C_c(R^+, L_2(\Omega, A, \mu))$ into itself. If B and D are Banach spaces stronger than C_c and if the pair (B, D) is admissible with respect to T, then T is a continuous operator from B to D.*

The proof of this lemma can be obtained by using closed graph theorem.

Remark 3.1. Since T is a continuous operator, it is also bounded. Hence there exists a constant $k > 0$ such that

$$\| (Tx)(t, w) \|_D \leq k \| x(t, w) \|_B .$$

We shall now state the main existence theorem.

Theorem 3.1. *Assume that the following conditions are satisfied:*

(i) *Let B and D be Banach spaces stronger than C_c such that (B, D) is admissible with respect to the operators T and U defined by*

$$(Tx)(t, w) = \int_0^t a(t, s, w) \, x(s, w) \, ds \tag{3.1}$$

and

$$(Ux)(t, w) = \int_0^\infty b(t, s, w) \, x(s, w) \, ds \tag{3.2}$$

(ii) *$x(t, w) \to g(t, x(t, w))$ and $x(t, w) \to h(t, x(t, w))$ are operators on D with values in B satisfying*

$$\| g(t, x(t, w)) - g(t, y(t, w)) \|_B \leq \lambda \| x(t, w) - y(t, w) \|_D, \tag{3.3}$$

and

$$\| h(t, x(t, w)) - h(t, y(t, w)) \|_B \leq \eta \| x(t, w) - y(t, w) \|_D, \tag{3.4}$$

for $x(t, w)$, $y(t, w) \in D$, where λ and η are positive constants.

$$\text{(iii)} \qquad g(t, 0) = h(t, 0) = 0. \tag{3.5}$$

$$\text{(iv)} \qquad f(t, w) \in D. \tag{3.6}$$

(v) *The pair (B, D) is admissible with respect to the operators $\{U_\tau, \tau \geq 0\}$ defined by*

$$(U_\tau x)(t, w) = \int_0^\infty b(t + \tau, s, w) \, x(s, w) \, ds \tag{3.7}$$

and $\{U_\tau, \tau \geq 0\}$ satisfy

$$\| (U_\tau x)(t, w) - (Ux)(t, w) \|_D \leq \Phi(\tau) \| x(t, w) \|_D \tag{3.8}$$

where $\Phi(\tau) \to 0$ as $\tau \to 0$.

Under these assumptions, there exists a number $\varepsilon_0 > 0$ such that for $0 < < \varepsilon \leq \varepsilon_0$, if $\| f(t, w) \|_D \leq \delta\varepsilon$ for some fixed $\delta \in (0, 1)$, then there exists a unique random solution $x(t, w)$ of (1.1), $x(t, w) \in D$ and in addition $\| x(t,w) \|_D \leq \varepsilon$.

Proof: Let k and k' be the norms of the operators T and U respectively. Let $\eta > 0$ such that $\eta k' < 1$. By (3.4) and (3.5), we can find $\delta_1 > 0$ such that

$$\| h(t, x(t, w)) \|_B \leq \eta \| x(t, w) \|_D \tag{3.9}$$

whenever $\| x(t, w) \|_D \leq \delta_1$. Let $\lambda > 0$ such that $\lambda k \leq \delta$ where $\delta = (1 - - \eta k')/2$. Choose $\delta_2 > 0$ such that

$$\| g(t, x(t, w)) - g(t, y(t, w)) \|_B \leq \lambda \| x(t, w) - y(t, w) \|_D \tag{3.10}$$

whenever $\| x(t, w) \|_D \leq \delta_2$ and $\| y(t, w) \| \leq \delta_2$. Let $\varepsilon_0 = \min(\delta_1, \delta_2)$. For any ε in $(0, \varepsilon_0]$, define

$$S(\varepsilon) = \{ x \in D : \| x(t, w) \|_D \leq \varepsilon \} \tag{3.11}$$

and define the operators A and F from $S(\varepsilon)$ into D by the relations

$$(Ax)(t, w) = f(t, w) + \int_0^t a(t, s, w) \, g(s, x(s, w)) \, ds, \tag{3.12}$$

and

$$(3.13)\qquad (Fy)\,(t, w) = \int_0^\infty b(t, s, w)\, h(s, y(s, w))\; ds,$$

for $x(t, w)$, $y(t, w) \in S(\varepsilon)$.
Using (3.1), (3.2), (3.9)—(3.13) and the assumption (iii) of the hypothesis, one obtains that

$$\| (Ax)\,(t, w) + (Fy)\,(t, w) \|_D \leqslant \| f(t, w) \|_D +$$

$$(3.14)\qquad + \left\| \int_0^t a(t, s, w)\; g(s, x(s, w))\, ds \right\|_D + \left\| \int_0^\infty b(t, s, w)\; h(t, y(s, w)\; ds \right\|_D$$

$$\leqslant \| f(t, w) \|_D + \lambda k \| x(t, w) \|_D + \eta k' \| y(t, w) \|_D \leqslant \delta\varepsilon + \lambda k \varepsilon + \eta k' \varepsilon = \varepsilon.$$

This shows that $x \in S(\varepsilon)$, $y \in S(\varepsilon)$ imply that $Ax + Fy \in S(\varepsilon)$. On the other hand, using (3.1), (3.2), (3.10) and (3.12), we obtain

$$(3.15)\qquad \| (Ax)\,(t, w) - (Ay)\,(t, w) \|_D \leqslant \lambda k \| x(t, w) - y(t, w) \|_D$$

for any $x \in S(\varepsilon)$ and $y \in S(\varepsilon)$. Hence A is a contraction on $S(\varepsilon)$ since $\lambda k < 1$. Further, the operator F is equi-continuous on $S(\varepsilon)$ since

$$\| (Fx)\,(t + \tau, w) - (Fx)\,(t, w) \|_D$$

$$(3.16)\qquad = \left\| \int_0^\infty \{b(t + \tau, s, w) - b(t, s, w)\}\, h(s,\; x(s, w))\; ds \right\|_D$$

$$\leqslant \Phi(\tau) \eta k' \| x(t, w) \|_D \leqslant \Phi(\tau)\; \eta k' \varepsilon,$$

by assumption (v) in the hypothesis and the terms on the right hand side of (3.16) tends to zero as $\tau \to 0$. Clearly the operator F is uniformly bounded on $S(\varepsilon)$, because

$$\| (Fx)\,(t, w) \|_D \leqslant \eta k' \| x(t, w) \|_D \leqslant \eta k' \varepsilon.$$

Since the operator F is equi-continuous and uniformly bounded, it is completely continuous on $S(\varepsilon)$. Hence, by theorem 1.1 due to Krasnoselskii, there exists a random solution to the integral equation (1.1) with $x(t, w) \in D$ and $\| x(t, w) \| \leqslant \varepsilon$.

We shall now prove the uniqueness of the random solution. Let $x_1(t, w)$ and $x_2(t, w)$ be two random solutions of (1.1) and let $v(t, w) = x_1(t, w) - x_2(t, w)$. Then, it is clear that

$$(3.17)\qquad \| v(t, w) \|_D \leqslant \lambda k \| v(t, w) \|_D + \eta k' \| v(t, w) \|_D$$

which in turn implies that

$$(3.18)\qquad \| v(t, w) \|_D\, (1 - \lambda k - \eta k') \leqslant 0.$$

Since $1 - \lambda k - \eta k' > 0$, from (3.18), we have $\| v(t, w) \|_D = 0$. This completes the proof of the theorem.

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ECUAŢII INTEGRALE STOCASTICE DE TIP MIXT

Rezumat

În prima parte a lucrării sînt extinse rezultatele obţinute de Miller, Nohel şi Wong [3] la ecuaţii integrale stocastice de tip mixt, obţinîndu-se o teoremă de existenţă (teorema 2.1) şi unicitate a unei soluţii în sens larg (definiţia 1.1).

În partea a doua, folosind teoria admisibilităţii pentru astfel de ecuaţii (C. P. Tsokos [4]), se demonstrează teorema 3.1 de existenţă şi unicitate a unei soluţii întîmplătoare (random solution — definiţia 3.2).

Peste tot este utilizată teorema de punct fix a lui Krasnoselski.