ON A GENERALIZED TRANSFORM OF TWO VARIABLES

BY

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1. Introduction. R. D. Agarwal [1] has given a generalised form of the Laplace Transform

$$\Phi(p) = p \int_0^\infty e^{-px} f(x)\, dx \tag{1.1}$$

in the form

$$\Phi(p) = p \int_0^\infty e^{-(1/2)px} (px)^{-\lambda-1/2} M_{k+1/2,m}(px) f(x)\, dx \tag{1.2}$$

and he has denoted this transform symbolically as

$$\Phi(p) \xrightarrow[\lambda,\, k+\frac{1}{2},\, m]{M} f(x).$$

Now, in this paper, we define the Laplace Transform of two variables as

$$\Phi(p, q) = pq \int_0^\infty \int_0^\infty e^{-1/2(px+qy)} (px)^{-\lambda_1-1/2} (qy)^{-\lambda_2-1/2} M_{k_1+1/2,\, m_1}(px) \times \tag{1.3}$$

$$\times M_{k_2+1/2,\, m_2}(qy) f(x, y)\, dx\, dy$$

which we shall denote by

$$\Phi(p,q) \xrightarrow[\lambda_1,\,\lambda_2,\,m_1,\,m_2]{k_1+\frac{1}{2},\,k_2+\frac{1}{2}} f(x,y).$$

The Laplace Transform of two variables is defined as

$$\Phi(p,q) = pq\int_0^\infty\!\!\int_0^\infty e^{-(px+qy)} f(x,y)\,dx\,dy \tag{1.4}$$

and is denoted symbolically as

$$\Phi(p,q) \;\underset{\cdot\;\cdot}{\overset{\cdot\;\cdot}{=\!=\!=}}\; f(x,y).$$

The Transform (1.3) reduces to the Transform (1.4) when $k_1=\lambda_1=m_1$ and $k_2=\lambda_2=m_2$.

Our Transform (1.3) exists when

(i) $\mathrm{Re}\,(\delta_1+m_1-\lambda_1+1)>0$ where $f(x,y)=O\,(x^{\delta_1})$ for small x and $\mathrm{Re}\,(\delta_2+m_2-\lambda_2+1)>0$ where $f(x,y)=O(y^{\delta_2})$ for small y.

2. Theorem 1. *If*

$$\Phi_1(p,q) \xrightarrow[\lambda_1,\,\lambda_2,\,m_1,\,m_2]{k_1+\frac{1}{2},\,k_2+\frac{1}{2}} f_1(x,y)$$

and

$$\Phi_2(p,q) \xrightarrow[\lambda_1,\,\lambda_2,\,m_1,\,m_2]{k_1+\frac{1}{2},\,k_2+\frac{1}{2}} f_2(x,y)$$

then

$$\int_0^\infty\!\!\int_0^\infty \Phi_1(u,v)f_2(u,v)\,\frac{du}{u}\,\frac{dv}{v} = \int_0^\infty\!\!\int_0^\infty \Phi_2(t,s)f_1(t,s)\,\frac{dt}{t}\,\frac{ds}{s}$$

provided $f_1(x,y)$, $f_2(x,y)$ are bounded and absolutely integrable in $(0,\infty)$ in each of the variables x and y.

Proof. Here we have

$$\Phi_1(u,v) = uv\int_0^\infty\!\!\int_0^\infty e^{-(1/2)(ut+vs)}(ut)^{-\lambda_1-1/2}(vs)^{-\lambda_2-1/2}M^{(ut)}_{k_1+1/2,\,m_1}\times M^{(vs)}_{k_2+1/2,\,m_2}\,f_1(t,s)\,dt\,ds. \tag{2.1}$$

and

$$\Phi_2(t,s) = ts\int_0^\infty\!\!\int_0^\infty e^{-1/2(tu+sv)}\,(tu)^{-\lambda_1-1/2}(sv)^{-\lambda_2-1/2}M^{(tu)}_{k_1+1/2,\,m_1}\times M^{(sv)}_{k_2+1/2,\,m_2}\,f_2(u,v)\,du\,dv. \tag{2.2}$$

Therefore

$$\int_0^\infty\!\!\int_0^\infty \Phi_1(u,v)\,f_2(u,v)\,\frac{du}{u}\,\frac{dv}{v} = \int_0^\infty\!\!\int_0^\infty f_2(u,v)\,du\,dv\;\times$$

$$\times\int_0^\infty\!\!\int_0^\infty e^{-1/2(ut+vs)}(ut)^{-\lambda_1-1/2}(vs)^{-\lambda_2-1/2}M^{(tu)}_{k_1+1/2,\,m_1}\,M^{(vs)}_{k_2+1/2,\,m_2}\,f_1(t,s)\,dt\,ds.$$

Now if

(A) $\mathrm{Re}\,(\delta_1+m_1-\lambda_1+1)>0$ where $f_1(t,s)=O(t^{\delta_1})$ for small t and $\mathrm{Re}\,(\delta_2+m_2-\lambda_2+1)>0$ where $f_1(t,s)=O(s^{\delta_2})$ for small s and

(B) $\mathrm{Re}(\delta_3+m_1-\lambda_1+1)>0$ where $f_2(t,s)=O(t^{\delta_3})$ for small t, and $\mathrm{Re}\,(\delta_4+m_2-\lambda_2+1)>0$ where $f_2\,(t,s)=O(s^{\delta_4})$ for small s, then the change in the order of integration is permitted. Then we shall have

$$\int_0^\infty\!\!\int_0^\infty \Phi_1(u,v)f_2(u,v)\,\frac{du}{u}\,\frac{dv}{v} =$$

$$=\int_0^\infty\!\!\int_0^\infty f_1(t,s)\left[\int_0^\infty\!\!\int_0^\infty e^{-1/2(ut+vs)}\,(ut)^{-\lambda_1-1/2}\,(vs)^{-\lambda_2-1/2}M^{(ut)}_{k_1+1/2,\,m_1}\times\right.$$

$$\left.\times\,M^{(vs)}_{k_2+1/2\,m_2}f_2(u,v)\right]dt\,ds = \int_0^\infty\!\!\int_0^\infty f_1(t,s)\Phi_2(t,s)\,\frac{dt}{t}\,\frac{ds}{s}\,.$$

3. Theorem 2. *If*

$$\Phi_1(p) \xrightarrow[\lambda_1,\,k_1+\frac{1}{2},\,m_1]{M} f_1(x)$$

and

$$\Phi_2(q) \xrightarrow[\lambda_2,\,k_2+\frac{1}{2},\,m_2]{M} f_2(y)$$

then

$$\Phi(p, q) = \Phi_1(p) \cdot \Phi_2(q) \tag{3.1}$$

provided $f_1(x)$ *and* $f_2(y)$ *are bounded and absolutely integrable in* $(0, \infty)$ *and satisfy the conditions of Theorem* 1.

Example. Taking $f(x, y) = x^{\nu_1} y^{\nu_2} \overline{e}^{(\alpha x+\beta y)}$ and using [2, p. 215] we get

$$x^{\nu_1} y^{\nu_2} e^{-(\alpha x+\beta y)} \overset{k_1+\frac{1}{2},\, k_2+\frac{1}{2}}{\underset{\lambda_1,\, \lambda_2,\, m_1,\, m_2}{=\!=\!=\!=}} \times$$

$$\frac{pq(p)^{m_1-\lambda_1} (q)^{m_2-\lambda_2}\, \Gamma(m_1+\nu_1+1-\lambda_1)\, \Gamma(m_2+\nu_2+1-\lambda_2)}{(p+2)^{m_1+\nu_1-\lambda_1+1} (q+\beta)^{m_2+\nu_2-\lambda_2+1}} \times \tag{3.2}$$

$$\times {}_2F_1\left[\begin{matrix} m_1-k_1, m_1+\nu_1-\lambda_1+1 \\ 2m_1+1 \end{matrix}; \frac{p}{p+\alpha}\right] {}_2F_1\left[\begin{matrix} m_2-k_2, m_2+\nu_2-\lambda_2+1 \\ 2m_2+1 \end{matrix}; \frac{q}{q+\beta}\right]$$

for $\mathrm{Re}(\alpha) > \mathrm{Re}(p/2)$, $\mathrm{Re}(\beta) > \mathrm{Re}(q/2)$ and $\mathrm{Re}(\nu_1 + m_1 - \lambda_1 + 1) > 0$ and $\mathrm{Re}(\nu_2 + m_2 - \lambda_2 + 1) > 0$ and $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$.

If in this result we take $\alpha = \beta = 0$ and

$$\lambda_1 = k_1 = m_1 \text{ and } \lambda_2 = k_2 = m_2.$$

we get [3, p. 62]

4. Theorem 3. *If* $\phi(p, q) \doteqdot x^{\nu_1-1} y^{\nu_2-1} \psi(x, y)$

(4.1) *and* $$\psi(p, q) \overset{k_1+\frac{1}{2},\, k_2+\frac{1}{2}}{\underset{\lambda_1,\, \lambda_2,\, m_1,\, m_2}{=\!=\!=\!=}} g(x, y)$$

then

$$\phi(p, q) = pq\, \Gamma(m_1 + \nu_1 + 1 - \lambda_1)\, \Gamma(m_2 + \nu_2 + 1 - \lambda_2) \times$$

$$\int_0^\infty\!\!\int_0^\infty \frac{x^{m_1-\lambda_1} y^{m_2-\lambda_2} g(x, y)}{(p+x)^{m_1+\nu_1-\lambda_1+1} (q+y)^{m_2+\nu_2-\lambda_2+1}} \times \tag{4.2}$$

$$\times {}_2F_1\left[\begin{matrix} m_1-k_1, m_1+\nu_1-\lambda_1+1 \\ 2m_1+1 \end{matrix}; \frac{x}{p+x}\right] {}_2F_1\left[\begin{matrix} m_2-k_2, m_2+\nu_2-\lambda_2+1 \\ 2m_2+1 \end{matrix}; \frac{y}{q+y}\right] dx\,dy$$

provided $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(m_1 + \nu_1 - \lambda_1 + 1) > 0$ and $\mathrm{Re}(m_2 + \nu_2 - \lambda_2 + 1) > 0$ and $x^{m_1-\lambda_1} y^{m_2-\lambda_2} g(x, y)$ is bounded and absolutely integrable in $(0, \infty)$ in x and y.

Proof. Using (3.2) and (4.1) in Theorem (1), and after simplifying we get (4.2).

Corollary. *Taking* $\lambda_1 = k_1 = m_1$ *and* $\lambda_2 = k_2 = m_2$ *we get a Theorem due to* R a t h i e [4, p. 47].

5. Theorem 4. *If* $$\phi(p, q) \overset{k_1+\frac{1}{2},\, k_2+\frac{1}{2}}{\underset{\lambda_1,\, \lambda_2,\, m_1,\, m_2}{=\!=\!=\!=}} f(x, y)$$

and

$$p^{-\nu_1} q^{-\nu_2} f(p, q) \doteqdot g(x, y)$$

hen

$$\phi(p, q) = pq(p)^{m_1-\lambda_1} (q)^{m_2-\lambda_2}\, \Gamma(m_1+\nu_1-\lambda_1+2)\, \Gamma(m_2+\nu_2-\lambda_2+2) \times$$

$$\times \int_0^\infty\!\!\int_0^\infty \frac{g(x, y)}{(p+x)^{m_1+\nu_1-\lambda_1+2} (q+y)^{m_2+\nu_2-\lambda_2+2}} \times \tag{5.1}$$

$$\times {}_2F_1\left[\begin{matrix} m_1-k_1, m_1+\nu_1-\lambda_1+2 \\ 2m_1+1 \end{matrix}; \frac{p}{p+x}\right] \times$$

$$\times {}_2F_1\left[\begin{matrix} m_2-k_2, m_2+\nu_2-\lambda_2+2 \\ 2m_2+1 \end{matrix}; \frac{q}{q+y}\right] dx\,dy$$

provided $\mathrm{Re}(m_1 + \nu_1 - \lambda_1 + 2) > 0$ *and* $\mathrm{Re}(m_2 + \nu_2 - \lambda_2 + 2) > 0$ *and* $f(x, y)$ *and* $g(x, y)$ *are bounded and absolutely integrable in* $(0, \infty)$ *in each of the variables* x *and* y.

Proof. Here we have

$$\phi(p, q) = pq \int_0^\infty\!\!\int_0^\infty e^{-(1/2)(px+qt)} (ps)^{-\lambda_1-1/2} (qt)^{-\lambda_2-1/2} M^{(ps)}_{k_1+\frac{1}{2}, m_1} \times \tag{5.2}$$

$$\times M^{(qt)}_{k_2+\frac{1}{2}, m_2} \times f(s, t)\, ds\, dt$$

and

$$f(s, t) = s^{1+\nu_1} t^{1+\nu_2} \int_0^\infty\!\!\int_0^\infty e^{-sx-ty} g(x, y)\, dx\, dy. \tag{5.3}$$

Substituting the value of $f(s, t)$ from (5.3) in (5.2) and changing the order of integration which is permitted under the given conditions and then evaluating the inner integral with the help of (3.2) we get our Theorem.

Corollary. *If in the above Theorem we put*

$$\lambda_1 = k_1 = m_1 \text{ and } \lambda_2 = k_2 = m_2.$$

we get [5, p. 28].

Some more Theorems and the differential and integral properties of this generalized Transform will be given in a subsequent paper.

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ASUPRA UNEI TRANSFORMĂRI GENERALIZATE DE DOUĂ VARIABILE

Rezumat

R. D. Agarwal [1] a dat o generalizare a transformării lui Laplace (1.1) pentru funcţii de o variabilă sub forma (1.2). Procedind asemănător, în această lucrare se defineşte transformata Laplace generalizată pentru funcţii de două variabile prin (1.3), stabilindu-se diferite proprietăţi ale ei prin teoremele 1—4.

SUR LES RAFFINEMENTS D'UN ESPACE FIBRÉ PRINCIPAL DIFFÉRENTIABLE

PAR

DAN I. PAPUC

À la mémoire
de mon Professeur
OCTAVE MAYER

§ 1. Considérations générales. Dans cette Note on étudie une structure notée par $(\xi, \mathfrak{N})$ où : ξ *est un espace fibré principal. différentiable*

$$\xi = (E, B, p, G\,;\, \mathcal{A}), \tag{1}$$

(*E, l'espace total de* ξ et *B, l'espace base de* ξ, sont des variétés paracompactes, différentiables, réelles, à un nombre fini de dimensions, *p, la projection de* ξ, est une application différentiable de E sur B, *G, le groupe structural de* ξ, est un groupe Lie, $\mathcal{A} = \{(U, \varphi)\}$ est *l'atlas maximal de trivialisation de* ξ. Une carte locale de trivialisation $(U, \varphi) \in \mathcal{A}$ est constituée par U, un ouvert de B, appelé *la zone de la carte* (U, φ) et φ, un difféomorphisme $\varphi : U \times G \to p^{-1}(U) \subset E$).

$\mathfrak{N}$ est *une suite de sous-groupes fermés de G.*

$$\mathfrak{N} = (G = H_0 \supset H_1 \supset \ldots \supset H_{q-1} \supset H_q = \{e\}), \tag{2}$$

(e est l'élément unité de G, q est un nombre naturel, $q \geqslant 2$).

Pour abréger, dans ce qui suit nous utiliserons les notations suivantes

$$E_j = E/H_j, \qquad j = 0, 1, \ldots, q. \tag{3}$$

E/H_j est l'espace quotient obtenu de E à l'aide du groupe de transformations H_j de E. Évidemment $E_q = E$, $E_0 = B$

$$H_k^j = H_j/H_k \qquad 0 \leqslant j < k \leqslant q. \tag{4}$$