CONNECTIONS COMPATIBLES WITH FREDHOLM STRUCTURES ON BANACH MANIFOLDS

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A. Introduction. Vector bundles which admit a reduction to Fredholm group have been much studied last years. We will give here a necessary and sufficient condition of compatibility of a linear connection on a vector bundle with a G-structure on it, and we will apply this result to Fredholm reductions.

So we shall extend holonomy group notion to linear connections on vector bundles with a Banach space as type fibre.

In what follows E, F, G .... will always denote the elements of a real, infinite-dimensional Banach spaces class $\mathfrak{B}$: $L(E, F)$ will be the space of linear continuous maps from E to F with the norm topology. We shall denote $L(E, E)$ by $L(E)$. Let $GL(E)$ be the group of the bijective isomorphisms in $L(E)$.

We will use the following notation and nomenclature (see f.i [13]):

1) $P(E, F)$ — class of perturbations on $\mathfrak{B}$ (see in [6] and [16], p. 32.)

2) $P(E, E) = P(E)$.

Particularly, if $\mathfrak{B}$ is the class of all Banach spaces, then examples of such perturbation classes are:

$C(E, F)$ — compact operators in $L(E, F)$,

$SC(E, F)$ — semicompact operators in $L(E, F)$,

$CSC(E, F)$ — cosemicompact operators in $L(E, F)$,

$\mathfrak{F}(E, F)$ — finite-dimensional operators in $L(E, F)$,

3) $GP(E) = GL(E) \cap (I_E + P(E))$; $GP(E)$ is a normal closed subgroup in $GL(E)$; $GC(E)$ is called Fredholm group of E.

Definition 1. *Let E be a Banach space. A flag in E is a double sequence of closed linear spaces :*

$$E_1 \subset E_2 \subset \dots \subset E_n \dots \subset E : \ E \supset E^{\infty-1} \supset E^{\infty-2} \dots \supset E^{\infty-n} \supset \dots$$

such that :

1) $\dim E_n = n$, $\operatorname{codim} E^{\infty-n} = n$.
2) *We have isomorphisms* $E_n \oplus E^{\infty-n} \simeq E$.
3) $\bigcup_n E_n$ *is dense in* E.

Clearly, a separable Banach space with a Schauder basis admits a flag.

For a Banach space with a flag there are the obvious inclusion maps :

$$GL(E_n) \to GP(E).$$

If we denote by $GL(\infty) = \operatorname{ind}_n \lim GL(E_n)$, it results : [4], [6]

Theorem 1.1. *The natural inclusion map*

$$i : GL(\infty) \to GP(E)$$

is a homotopy equivalence.

Let now M be an infinite-dimensional manifold of class C^r.

Definition 2. *A filtration of M is a sequence of closed submanifolds $M_i \subset M (i \geqslant n_0)$ such that*

1) $M_n \subset M_{n+1}$;
2) $\dim M_n = n$;
3) *if* $M_\infty = \bigcup \{M_n \mid n \geqslant n_0\}$ *is the union of the* M_n*'s with the direct limit topology, the natural map :*

$$i_\infty : M_\infty \to M$$

is a homotopy equivalence.

Example : A flag in E is a filtration of E.

We consider $\pi : B \to M$ a vector bundle of class $C^r (r > 1)$ with Banach space E as type fibre.

Definition 3. *π is a P-vector bundle if it admits a reduction of the structural group to $GP(E)$; a C-vector bundle will be called Fredholm bundle. If a tangent bundle $\tau : TM \to M$ is P-vector bundle, M admits a P-structure.*

Particularly when $P = C$, M is called a Fredholm manifold.

Results of E l w o r t h y and T r o m b a ([4] and [6]) show that the class of manifolds with a P-structure is sufficiently rich to be interesting ; it includes the class of Hilbert manifolds.

The following theorem is given by K. K. M u k h e r j e a [13] :

Theorem 1.2. *Let M be a C^∞, connected, paracompact manifold modelled on E ; we assume that E is a C^∞-smooth, separable Banach space with a flag. If M admits a P-structure, then M admits a filtration with M_∞ dense in M.*

Moreover we can obtain a filtration for which M_n is compact $\forall n$.

We shall denote by E^* the dual Banach space of E and we shall consider the injection :

$$* : L(E) \to L(E^*),$$
$$A \to A^*,$$

A^* being the adjoint operator of A. Then we have

Theorem 1.3. *Let E be a reflexive Banach space and $\pi : B \to M$ a vector bundle with E as type fibre.*

Then :

a) *π is Fredholm bundle if and only if its dual π^* is Fredholm bundle.*
b) *If π is SC-vector bundle, then π^* is CSC-vector bundle ;*
c) *If π is CSC-vector bundle then π^* is SC-vector bundle ;*
d) *If E^* is subprojective space and π is a SC-vector bundle, then π^* is SC-vector bundle ;*
e) *If E is subprojective space and π^* is a CSC-vector bundle then π is CSC-vector bundle :*

Proof. From the hypothesis it results that $*$ is a isomorphism from $GL(E)$ to $GL(E^*)$, so that ([16]) :

a) $* : GC(E) \to GC(E^*)$, d) $* : GSC(E) \to GSC(E^*)$,
b) $* : GSC(E) \to GCSC(E^*)$, e) $* : GCSC(E) \to GCSC(E^*)$
c) $* : GCSC(E) \to GSC(E^*)$,

are isomorphisms. Then the theorem results from the theorems of the construction of the fibre bundles by the cocycle of the transition maps [2].

Corollary. *The conclusions of the theorem 2.1 can be applied to τ and τ^* which are tangent and cotangent bundles of M.*

Let $\pi : B \to M$ be a vector bundle of class C^r as above, M being modelled on F.

Definition 4. *A connection map K for the bundle B is a map*

$$K : TB \to B,$$

such that for any local trivialization (U, φ, Φ) of π, there exists a map

$$\Gamma_\varphi : \varphi(U) \to L(F, E ; E)$$

of class C^{r-1}, which gives the local representation of K, $K\varphi = \Phi \circ K \circ T\Phi^{-1}$ by the formula :

$$K_\varphi(x, \xi, y, \eta) = (x, \eta + \Gamma_\varphi(x).(y, \xi)).$$

We call Γ_φ the local connector for K.

B. Parallel displacement; the holonomy groups. We shall give below some results from the connections theory on vector bundles which we shall use later.

Theorem 2.1. *Let $\pi : B \to M$ be a vector bundle and M admits partition of unity of class C^{r-1}: then:*

a) *there exists a connection map K for π.*

b) *there exists a canonic bijective map from the set of connection maps on B to the set of covariant derivatives on B given by*

$$K \circ T\xi = \nabla\xi, \quad \forall \xi \in \mathfrak{X}_B(M). \tag{1}$$

where $\mathfrak{X}_B(M)$ is the set of the sections in π. The proof is given in [7].

Remarks: 1) $\nabla\xi$ is considered as a section in $L(\tau, \pi) : L(TM, B) \to M$.

2) The implication $K \to \nabla$ is given by (1) and without the hypotheses of existence of the partition of the unity on M.

Theorem 2.2. (See [7]) *If $\tau_1 : TB \to B$ is projection:*

a) *the map $(\tau_1, T\pi, K)$ from the following commutative diagram is an isomorphism of vector bundles over B:*

$$\begin{array}{ccc} TB & \xrightarrow{(\tau_1,\, T\pi\, K)} & \pi^*(TM \oplus B) \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle (\pi^*(\tau \oplus \pi)} \\ B & \longrightarrow & B \end{array}$$

where $\pi^*(TM \oplus B)$ is pull-back bundle over π.

b) Ker $T\pi$ *and* Ker K *are subvector-bundles of $\tau_1 : TB \to B$ and the map i*

$$\begin{array}{ccc} \operatorname{Ker} T\pi \oplus \operatorname{Ker} K & \longrightarrow & TB \\ {\scriptstyle \tau_1|_{\operatorname{Ker} T\pi} \oplus \tau_1|_{\operatorname{Ker} K}}\downarrow & & \downarrow{\scriptstyle \tau_1} \\ B & \longrightarrow & B \end{array}$$

given by $(A, B) \to A + B$ is an isomorphism of vector bundles.

In particular $K : \operatorname{Ker} T \to \pi^*\, B$ and $T\pi : \operatorname{Ker} K \to \pi^*\, TM$ are isomorphisms of vector bundles.

We shall call $\xi_r \in \mathfrak{X}_{\operatorname{Ker} T\pi}(B)$ a vertical element and $\xi_0 \in \mathfrak{X}_{\operatorname{Ker} K}(B)$ a horizontal element.

Theorem 2.3. a) *Let $\pi : B \to M$ be a G-vector bundle with E as type fibre which admits a reduction to G' ($G' \subset G \subset GL(E)$). To a connection $K_{G'}$ on G'-vector bundle π can be associated a connection K_G on the G-vector bundle π and is called the extension of $K_{G'}$.*

b) *Let $\pi : B \to M$ and $\pi' : B' \to M$ be two C^r-vector bundles which admit the connection maps say K_π and $K_{\pi'}$ respectively.*

A connection map on $L(\pi, \pi') : L(B, B') \to M$ is given in a local trivialization by:

$$\Gamma_{L(\pi, \pi')} : \varphi(u) \to L(F, L(E, E') : L(E, E').$$

$$[\Gamma_{L(\pi, \pi')}(x) \,.\, (y, A)]\, \xi = \Gamma_{\pi'}(x) \,.\, (y, A.\xi) - A.\Gamma_\pi(x).(y, \xi).$$

c) *Let M and M' be two manifolds of class C^k modelled on F and F' respectively, $\pi' : B' \to M'$ a C^r-vector bundle $(1 \leqslant r \leqslant k)$ with connection map K' and $h : M \to M'$ a map of class C^r. Then the pull-back $h^*\pi : h^*B' \to M$ is a bundle of class C^r. We define a connection map K_h on $h^*\pi$ by:*

$$i \circ K_h = K' \circ Ti, \tag{2}$$

where $i : h^*B' \to B$ is the natural inclusion.

There follows that the covariant derivative $h^*\nabla'$ is given by:

$$(h^*\nabla')\,(h^*\,\xi') = h^*(\Delta'\xi).\partial h, \qquad \xi \in \mathfrak{X}_{B'}(M'), \tag{3}$$

where ∂h (the tangent map to h) may be considered as a section in $L(\tau, h^*\tau')$ given by $\partial h(v) = Th(v)$.

Proof. a) results from Theorem 2.2 b) and from the fact that a G'-bundle is a G-bundle. b) and c), see [5].

Corollary 2.4. a) *If π is a bundle of class C^r with a C^{r-1} connection there is a C^{r-2} section R in $L(\tau, \tau, \pi; \pi)$ such that for any sections X, Y in τ and ξ of class C^2 in π we have:*

$$\nabla^2\xi.(X, Y) - \nabla^2\xi.(Y, X) = R.(X, Y, \xi). \tag{4}$$

In fact this is a generalization of the classical curvature tensor so we may conserve this name for R

b) *If R is the curvature tensor of K', then the curvature R_h of $h^*\pi$ is given by:*

$$R_h\,.\,(v, u, \xi) = (R' \circ h)\,(\partial h.\, v, \partial h.\, u, \xi). \tag{5}$$

We apply those above to define the notion of parallel displacement along a curve.

Let $c \in C^\infty(I, M)$ be a C^∞-curve in M and K a connection map on $\pi : B \to M$. *Then we consider the pull-back $c^*\pi$,* the connection map K_c given by Theorem 2.3 c) and the covariant derivative $c^*\nabla = \nabla_c$ induced on it.

Definition 5. *The section $\xi_c \in \mathfrak{X}_{c^*\pi}(I)$ is called parallel along c if and only if*

(6) $$\nabla_e \xi_{c^*} e = 0,$$

where e is the basis section of $TR = R \times R$ given by $e(t) = (t, 1)$.

It results from definition that the tangent vector to the curve defined by : $t \to (\xi \circ c)(t)$ is horizontal ; we shall call such curve an horizontal one.

In [7] it is proved the existence and the unicity of a parallel section along $c \in C^\infty(I, M)$ so that $\xi_c(t_0) = \xi_0 \in \pi^{-1}(c(t_0)) = B_{c(t_0)}$ is fixed.

Then we obtain a topological isomorphism

$$T_c|_{[t_0, t_1]} : B_{c(t_0)} \to B_{c(t_1)} \quad \xi_0 \to \xi_c(t_1)$$

called the *parallel displacement along the curve c*, from t_0 to t_1.

The parallel displacement along any piecewise differentiable curve of class C^1 can be defined in an obvious manner as a succession of parallel displacements along the segments of the differentiable curves.

So we obtain a map :

$$T : c \to T_c \in L(B_{c(\alpha)}, B_{c(\beta)}).$$

$\forall c : [\alpha, \beta] - M$, a piecewise differentiable curve.

Generally, the parallel displacement from B_{c_1} to B_{c_2} depends on the curve c which joins the points c_1 and c_2. We can now define the holonomy group as a global measure of curve dependence of the parallel displacement.

For each point p of M we denote by $C(p)$ the loop space at p, that is, the set of all closed curves starting and ending at p ; let $C_0(p)$ be the subset of $C(p)$ consisting of loops which are homotopic to zero.

Definition 6. *We call $\Phi(p) = T(C(p))$, the holonomy group of K with reference point p ; the following group $\Phi_0(p) = T(C_0(p))$ is called the restricted holonomy group of K with reference point p.*

As in the finite-dimensional case, we can realize these groups as subgroups of the structure group G in the following way. Let b be an arbitrarily fixed point of B_p. Each $c \in C(p)$ determines an element say, g, of G such that $T_c(b) = b.g$. The sets of elements $a \in G$ determined by all $c \in C(p)$ and $c \in C_0(p)$ form two subgroups of G, denoted by $\Phi(b)$ and $\Phi_0(b)$ respectively. If M is connected, the holonomy groups $\Phi(b)$, $b \in B$ and the restricted holonomy groups $\Phi_0(b)$, $b \in B$ are all conjugated to each other in G and hence isomorphic with each other. Moreover $\Phi(b)$ and $\Phi(p)$ are isomorphic $\forall b \in B_p$; the same $\Phi_0(b)$ and $\Phi_0(p)$ are isomorphic $\forall b \in B_p$. If M is a connected manifold, we call the holonomy group of M the holonomy group of K on $\pi : TM \to M$.

Theorem 2.5. *Let M be a connected and paracompact manifold, $\pi : B - M$ a C^r-vector bundle with G structure group for which there exists a connection map K. Then :*

a) *$\Phi_0(b)$ is Lie subgroup of G.*

b) *$\Phi(b)$ is Lie subgroup of G.*

c) *$\Phi(b)/\Phi_0(b)$ is countable and $\Phi_0(b)$ is the identity component of $\Phi(b)$.*

d) *the structure group G of π may be reduced to $\Phi(b)$.*

Proof. a) The proof will be given in two steps.

Step 1. We shall show that every element of $\Phi_0(b)$ can be joined to the identity element by a piecewise differentiable curve of class C^k in G which lies in $\Phi_0(b)$. By the factorization lemma [11], which remains true in case of infinite-dimensional manifolds, it is sufficient to show that the element $g \in \Phi_0(b)$ defined by the differentiable loop c can be joined to the identity element in $\Phi_0(b)$ by a differentiable curve of G which lies in $\Phi_0(b)$. If c is locally defined through $t \to c(t)$ we define :

$$f : I \times I \to M, \qquad I = [0, 1].$$

$$(t, s) \to f(t, s) = sa + (1 - s)c(t), \quad (a \text{ fixed in } E).$$

which is a differentiable mapping of class C^k such that $f(t, 0)$ is the curve c and $f(t, 1)$ is the trivial curve. For each fixed s let $g(s)$ be the element of $\Phi_0(b)$ obtained from the loop $f(t, s)$, $0 \leqslant t \leqslant 1$, so that $g(0) = g$ and $g(1) =$ identity. The fact that $g(s)$ is of class C^k in s (as a mapping of I into G) follows from the existence theorem of parallel sections along $c \in C^\infty(I, M)$ and from the fact that the solution of an ordinary linear differential equation $y = A(x, s)y + B(x, s)$ with parameter s is differentiable in s as many times as $A(x, s)y + B(x, s)$ is.

Step 2. We prove for Lie-Banach groups the following result regarding the arcwise connected Lie subgroups. The similar result in finite-dimensional case is given by Yamabe in [18] and Freudenthal in [8].

Lemma. *Let G be a Lie-Banach group and H a subgroup such that every element of H can be joined to the identity by a piecewise differentiable curve of class C^1 which is contained in H. Then H is a Lie-subgroup of G* (in Maissen's sense [12]).

Proof. We identify $T_e(G)$ with Lie algebra g of G, and we consider $h = \{X \in T_e(G)$ so that X is tangent to a differentiable curve of class C^1 contained in $H\}$.

It results that h is a closed Lie subalgebra of g : we shall denote by $\bar{H}$ the arcwise connected Lie subgroup, corresponding to the subalgebra h by [12]. We show that $H = \bar{H}$. The inclusion $H \subset \bar{H}$ is obvious from the definition of $\bar{H}$. We note that the exponential map is a local diffeomorphism from $U \subset h \subset g \simeq T_e G$ onto an open subset $\exp(U)$ of $\bar{H}$ and $e \in \exp(U)$. But from the exponential definition there results that a neighbourhood of e in $\bar{H}$ is contained in H. Since $\bar{H}$ is connected, $\bar{H} \subset H$.

b) Applying the above lemma to the identity component of e in $\bar{\Phi}(b)$ say, $\Phi^0(b)$, it results that it is a Lie subgroup in G and clearly $\Phi_0 \subset \Phi_0$. It follows that Φ is Lie subgroup in G.

c) We define a homomorphism like in the classical case

(7) $$f : \pi^{-1}(M) — \Phi(b)/\Phi_0(b).$$

Since M is connected and paracompact, it satisfies the second axiom of countability. It follows easily that $\pi_1(M)$ is countable. Hence $\Phi(b)/\Phi_0(b)$ is also countable and moreover Φ^0/Φ_0.

We suppose that $\Phi_0 \subset \Phi^0$ in the same manner as for the demonstration of the same result in the finite-dimensional manifolds theory, it results that Φ^0 is a set of first category. This is an absurdity because every infinite-dimensional manifold is a B a i r e space ([2]). It results $\Phi_0 = \Phi^0$

d) results substituting the trivialisation atlas of π by a new atlas $\mathcal{A}' = \{U'_\alpha, \varphi'_\alpha\}$the maps of which are given by the parallel displacement So that $p, p' \in U'$ if and only if $\exists\, b \in B_p$ and $b' \in B_{p'} C^\infty(I, M)$ such that $T_c(b = b)'$.

Corollary 2.6. *Let $\pi : B \to M$ be a vector bundle with the connection map K and Φ the holonomy group of K. Then $\Phi = e$ is a necessary and sufficient condition that π be trivial. If π is $\tau : TM \to M$ we obtain a necessary and sufficient condition for M to be parallelizable.*

Theorem 2.7. *Let $\pi : B \to M$ be a vector bundle as above. The following three conditions are equivalent:*

1) *The parallel displacement depends only on the homotopy classes of curves in M.*

2) $\Phi_0(b) = e, \quad b \in B.$

3) *The curvature tensor R given by (4) is zero.*

Proof. 1) $\to$ 2) is obvious.

1) $\to$ 3) Let $H : I \times I \to M (I = [0, 1])$ be a differential homotopy so that $H(0, 1) = p$ and $H(1, 1) = p'$. Let $v \in B_r$ and $\xi_H \in \mathfrak{X}_{H^*\pi}(I \times I)$ be parallel along the curve $H_1 : t \to H(t, s)$ (where $s \in I$ is fixed) and $\xi_H(0, s) = v, \ \forall s \in I$.

Consequently $\nabla_{H_1}\xi_H(t, s)(e_1) = 0$, $\nabla_{H_2}\xi_H(0, s)(e_2) = 0$, where $H_2 : s \to$ $\to H(t, s)$. It results $\nabla^2\xi_H(t, s).(e_1.e_2) = 0$ and from $R = 0$, it results $\nabla^2\xi_H(t, s)(e_1, e_2) = 0$. So the section $t \to \nabla_{H_2}\xi_H(t, s).e_2$ is parallel along H_1 and as it is zero in $t = 0$, it is zero $\forall t$. In particular $\nabla_{H_2}\xi_H(1, s).e_2 = 0$ and $\xi_H(1, s)$ does not depend on s.

3) $\to$ 1) results getting over the inverse reasoning.

In the assumptions of Theorem 2.7 the homomorphism f by (7) is an isomorphism

$$f : \pi^1(M) \to \Phi.$$

We obtain:

Theorem 2.8. *If $\pi : B \to M$ admits a connection K having $R = 0$ and R is simple connected then π is equivalent to the trivial vector bundle.*

Particularly a simple connected manifold M with a connection K having $R = 0$ is parallelizable.

We will now see how it behaves under pull-backs.

Theorem 2.9. *Let $h : M \to M'$ be the same as in Theorem 2.3. c), $\pi' : B' \to M'$ a C^r-vector bundle with a connection K' and K_h the connection induced on $h^*\pi$. Then:*

a) *h induces a homomorphism:*

$$h_\Phi : \Phi^h \to \Phi;\ h_{\Phi_0} = h_{\Phi/\Phi_0} : \Phi_0^h \to \Phi_0$$

where Φ^h and Φ are the holonomy group of K_h and K.

b) *$\Phi_0^h = e$ if and only if $\Phi_0 = e$.*

c) *if h is a homotopy equivalence and $\Phi_0 = e$, then h_Φ is a isomorphism.*

Proof. a) The map $\alpha \to h \circ \alpha$ defines, according definition 6, the wanted homomorphism.

b) results from Theorem 2.7 noticing that $R_h = 0 \to R' = 0$.

c) is obvious from Theorem 2.7.

Corollary 2.10. *If h is an inclusion map then $\Phi^h \subset \Phi$.*

Finally we shall prove a result which generalizes that one given in [1].

Theorem 2.11. *Let $\pi : B \to M$ be a C^r-vector bundle with M connected which admits a reduction $\pi_G : B \to M$ to the group G. Let K be a connection map on π. Then $\Phi \subset G$ is a necessary and sufficient condition that K is an extension of a connection K_G on π_G.*

Proof. From the unicity of a parallel section along a curve it results that the holonomy group of the two connections are isomorphe and consequently $\Phi \subset G$.

Reciprocally, let K be a connection on π and let us suppose that $\Phi \subset G$. We obtain from Theorem 2.5. d) a G-vector bundle and a G-connection for which K is evidently the extension.

C. Compatible connections with Fredholm structures on Banach manifolds.

We shall firstly prove the following result:

Theorem 3.1. *Let M be a C^∞-manifold modelled on a C^∞-smooth separable with a flag Banach space say E: we assume what M admits a P-structure. Then M_∞ determined by the Theorem 1.2 is a C^∞-manifold on the locally convex metrizable topological vector space $E_\infty = \mathrm{dir}_n \lim E_n$ where $\{E_n\}_{n \geq 1}$ is a flag for E with E_n compact $\forall n$.*

Proof. Let $E' = \bigcup_{n\geq 1} E_n$ be the real vectori subspace of E with finite topology of countable dimension. Then $E' = \mathrm{dir}_n \lim E_n = E_\infty$ (as a topological space) and moreover is a locally convex topological vector space (see [15]) : being paracomcpat space there results that it is metrizable. By our assumptions M admits a filtration $\{M_n\}_{n\geq 1}$ (Theorem 1.2). Let $I_n : M_n \to M_\infty$ and $i_n : E_n \to E_\infty$ be the natural inclusion maps. We show that if $\mathcal{A}^n = (U^n_{\alpha^n}, \varphi^n_{\alpha^n})_{\alpha^n\in\mathcal{Q}^n}$ is an atlas of differential structure on M_n, then

$$\mathcal{A} = \{\widetilde{U}^n_{\alpha^n}, \widetilde{\varphi}^n_{\alpha^n})\}_{\alpha^n\in\mathcal{Q}^n, n\in N}$$

given by $\widetilde{U}^n_{\alpha^n} = I_n(U^n_{\alpha^n})$, $\widetilde{\varphi}^n_{\alpha^n} = i_n \circ \varphi^n_{\alpha^n} \circ I_n^{-1}$ is an atlas of differential structure on M_∞ (in P. Ver Eecke way [3]). In order to prove this, we consider $\widetilde{U}^n_{\alpha^n} \cap \widetilde{U}^{n'}_{\alpha^{n'}} \neq \varnothing$ and hence $U^n_n \cap U^n_{n'} \neq \varnothing$ and we show that the maps $\widetilde{\varphi}^n_{\alpha^n} \circ (\widetilde{\varphi}^{n'}_{\alpha^{n'}})^{-1} = i_n \circ \varphi^n_{\alpha^n} \circ I_n^{-1} \circ I_{n'}(\varphi^{n'}_{\alpha^{n'}})^{-1} \circ i_{n'}^{-1} : \widetilde{\varphi}^{n}_{\alpha^{n'}}(\widetilde{U}^n_{\alpha^n} \cap \widetilde{U}^{n'}_{\alpha^{n'}}) \to \widetilde{\varphi}^n_{\alpha^n}(\widetilde{U}^n_{\alpha^n} \cap \widetilde{U}^{n'}_{\alpha^{n'}})$ is of class C^∞ (in Ver Eecke's sense [3]).

But I_n and $I_{n'}$ coincides on $\widetilde{U}^n_{\alpha^n} \cap \widetilde{U}^{n'}_{\alpha^{n'}}$ and since the maps $\varphi^n_{\alpha^n} \circ (\varphi^{n'}_{\alpha^{n'}})^{-1}$, $i_{n'}$, $i_{n'}^{-1}$ are of class C^∞ the statement results.

Moreover the maps $I_n : M_n \to M_\infty$ are of class C^∞ with respect to the differential structure above introduced.

Corollary 3.2. 1) *M_∞ is orientable manifold if and only if M is completely orientable.*

2) *$\pi : B \to M$ is completely orientable if and only if its dual bundle π^* is completely orientable.*

Proof results using the fact that $i_\infty : M_\infty \to M$ is a homotopy equivalence.

In the assumptions of Theorem 3.1, let $\pi : B \to M$ be a P-vector bundle of class C^r with E as type fibre and K a connection map on π. We denote by K_∞ the connection map induced by K on the pull-back $i^*_\infty \pi$.

Theorem 3.3. a) *If the connection K on π is an extension of a connection K_p on the reduced vector bundle $\pi_{GP(E)} : B \to M$* (K is compatible with a P-structure on π) *then K_∞ is the extension of a K^∞_p-connection on the reduced vector bundle $i^*_\infty \pi_{GP(E)} : i_\infty B \to M_\infty$.*

b) *If K_∞ is compatible with a P—structure on $i^*_\infty \pi$ and its restricted holonomy group say, Φ^∞_0, is $\{e\}$ then K is compatible with a P-structure on π and $\Phi_0 = e$.*

Proof. a) results by Theorem 2.11 and Corollary 2.10. b) results by Theorem 2.9.c).

Remarks. 1) If the connection K is compatible with a $\mathcal{F}$-structure on π then it is compatible with a P-structure on π. The same thing results then for the connection K_∞ on $i^*_\infty \pi$.

2) Because i is a homotopy equivalence it results that it admits a reduction to $GL(\infty)$. Consequently if K_∞ is the extension of a connection on the reduced bundle $i^*_\infty \pi_{GL(\infty)} : i_\infty B \to M_\infty$ it is compatible with every P-structure on $i^*_\infty \pi$. Moreover if we add $\Phi^\infty_0 = e$ it results that K is compatible with every P-structure on π.

We can particularize the above results for $\tau : TM \to M$ with P-structure induced by that one on M (see [6]).

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CONEXIUNI COMPATIBILE CU STRUCTURI FREDHOLM PE VARIETĂŢI MODELATE DE SPAŢII BANACH

Rezumat

Fibratele vectoriale care admit o reducere la grupul Fredholm au fost mult studiate în ultimii ani. Dăm aici o condiţie necesară şi suficientă de compatibilitate a unei conexiuni liniare pe un fibrat vectorial cu o G-structură pe acesta (Teorema 2.11) şi aplicăm acest rezultat la reducerile Fredholm (Teorema 3.3). Pentru aceasta extindem noţiunea de grup de olonomie la conexiuni liniare pe fibrate vectoriale cu fibre spaţii Banach.

GROUPES DE LIE-BANACH ET ESPACES LOCALEMENT SYMÉTRIQUES

PAR

AUREL BEJANCU

Soit M une C^∞-variété différentielle modelée par des espaces de Banach et ∇ une connexion linéaire sur M([2]). On désigne par T et R la torsion et la courbure de ∇.

Définition 1. *La variété M est un espace local symétrique si $T = 0$ et $\nabla_X R = 0$ pour tout $X \in \mathfrak{X}(M)$.*

Soit G un groupe de Lie-Banach de classe C^∞ et ω sa forme linéaire banachique. On désigne par e l'unité de G et par $C(G;\ T_eG)$ le module des applications de classe C^∞ sur G à valeurs dans T_eG. Nous avons démontré dans ([2]) que

$$\nabla_X Y = \omega^{-1}(X < \omega, Y >) \qquad \forall X, Y \in \mathfrak{X}(G) \tag{1}$$

est une connexion linéaire invariante sur G. De plus nous avons démontré les relations :

$$T = \omega^{-1} \circ d\omega\ ; \qquad R = 0. \tag{2}$$

A l'aide de cette connexion nous définissons la connexion $\tilde{\nabla}$ par la formule

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} T(X, Y). \tag{3}$$

Il est aisé de voir que $\tilde{T} = 0$.

Proposition 1. *La torsion T vérifie la relation*

$$T(X, T(Y, Z)) + T(Y, T(Z, X)) + T(Z, T(X, Y)) = 0 \tag{4}$$
$$\forall X, Y, Z \in \mathfrak{X}(G).$$